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When *M*-Cosingular Modules Are Projective

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Abstract. Let M be an R-module. Talebi and Vanaja investigate the category $\sigma[M]$ such that every M-cosingular module in $\sigma[M]$ is projective in $\sigma[M]$. In the light of this property we call M a COSP-module if every M-cosingular module is projective in $\sigma[M]$. This note is devoted to the investigation of these classes of modules. We prove that every COSP-module is a coatomic module having a semisimple radical. We also characterise COSP-module when every injective module in $\sigma[M]$ is amply supplemented. Finally we obtain that a COSP-module is artinian if and only if every submodule has finite hollow dimension.

1. Introduction

Let R be a ring with identity. All modules are unitary right R-modules. Let M be a module and $A \subseteq M$. Then $A \ll M$ means that A is a small submodule of M. Any submodule A of M is called *coclosed* in M if $A/B \ll M/B$ for any submodule B of M with $B \subseteq A$ implies that $A = B$. $\mathrm{Rad}\,(M)$ denotes the Jacobson radical of M and $\mathrm{Soc}\,(M)$ denotes the socle of M. By $\sigma[M]$ we mean the full subcategory of the category of right modules whose objects are submodules of M-generated modules. A module $N \in \sigma[M]$ is said to be M-*small* if there exists a module $L \in \sigma[M]$ such that $N \ll L$.

Let M be a module. If N and L are submodules of the module M, then N is called a *supplement* of L in M if $M = N + L$ and $N \cap L \ll N$. M is called *supplemented* if every submodule of M has a supplement in M and M is called *amply supplemented* if, for all submodules N and L of M with $M = N + L$, N contains a supplement of L in M.

Let M be a module. In [5], Talebi and Vanaja define $\overline{Z}(N)$ as a dual notion to the M-singular submodule $Z_M(N)$ of $N \in \sigma[M]$ as follows:

$$\overline{Z}(N) = \cap\{\,\mathrm{Ker}\, g \mid g \in \ \mathrm{Hom}\ (L), \quad L \in \mathcal{S}\}$$

where $\mathcal{S}$ denotes the class of all M-small modules. They call N an *M-cosingular* (*non-M-cosingular*) module if $\overline{Z}(N) = 0$ ($\overline{Z}(N) = N$). Clearly every M-small module is M-cosingular. The class of all M-cosingular modules is closed under taking submodules and direct sums by [5, Corollary 2.2] and the class of all non-M-cosingular modules is closed under homomorphic images by [5, Proposition 2.4].

Let M be a module. Talebi and Vanaja investigate the category $\sigma[M]$ that every M-cosingular module is projective in $\sigma[M]$. Inspired by this study we call any module M a *COSP*-module if every M-cosingular module is projective in $\sigma[M]$(for short).

2. Results

First we consider some examples.

Example 2.1. Let p be a prime integer and M denote the $\mathbb{Z}$-module, $\mathbb{Z}/p^k\mathbb{Z}$ with $k \geq 2$. Let $N = p^{(k-1)}\mathbb{Z}/p^k\mathbb{Z}$. It is clear that $N \cong \mathbb{Z}/p\mathbb{Z}$ and $N \cong M/L$ where $L = p\mathbb{Z}/p^k\mathbb{Z}$. Since $N \ll M$, N is M-cosingular. Now N is not M-projective. Otherwise M/L is M projective and $L = 0$ by [4, Lemma 4.30]. Therefore M is not COSP.

Example 2.2. Let S be a simple module. It is clear that every module in $\sigma[S]$ is semisimple. Now if L is an S-small module, then there is $H \in \sigma[S]$ such that $L \ll H$. Since H is semisimple, L is a direct summand of H. Hence $L = 0$. Therefore $\overline{Z}_S(N) = N$ for all $N \in \sigma[S]$ i.e, every $N \in \sigma[S]$ is non-S-cosingular. Thus S is a COSP-module.

Proposition 2.3. *Let M be a COSP-module. Then the following statements are true.*
(1) *Every M-small module is semisimple.*
(2) *For every module $N \in \sigma[M]$, $Rad(N) \subseteq Soc(N)$.*

Proof.
(1) Let $N \in \sigma[M]$ and $N \ll K$ for some module $K \in \sigma[M]$. Assume $T \leq N$. Since N and N/T are M-cosingular, $N \oplus N/T$ is M-cosingular. Therefore N/T is N-projective because M is COSP. Thus T is a direct summand of N.
(2) Let $N \in \sigma[M]$. Since $\mathrm{Rad}\,(N) = \sum_{i\in I} N_i$ with $N_i \ll N$, $\mathrm{Rad}\,(N)$ is semisimple by (1). Hence $\mathrm{Rad}\,(N) \subseteq \mathrm{Soc}\,(N)$. ∎

Proposition 2.4. *Let M be a module. Then M is COSP if and only if every module in $\sigma[M]$ is COSP.*

In particular any submodule, homomorphic image and direct sum of COSP-modules are again COSP.

Proof. ($\Longrightarrow$) Let M be a COSP-module and $N \in \sigma[M]$. Assume $A \in \sigma[N]$ is N-cosingular. Note that $A \in \sigma[M]$ and A is M-cosingular. Since M is COSP, A is projective in $\sigma[M]$ and hence projective in $\sigma[N]$.
($\Longleftarrow$) Clear. ■

Example 2.5, Since every simple module is COSP, every semisimple module is also COSP (see Proposition 2.4).

Proposition 2.6. *Let M be a COSP-module. Then every module $N \in \sigma[M]$ has a maximal submodule.*

Proof. Let $N \in \sigma[M]$. By Proposition 2.3, $\mathrm{Rad}\,(N) \subseteq \mathrm{Soc}\,(N)$. If $\mathrm{Soc}\,(N) = N$, then N has a maximal submodule. Assume $\mathrm{Soc}\,(N) \neq N$. Then $\mathrm{Rad}\,(N) \neq N$. This implies that N has a maximal submodule, again. ■

A module M is called *coatomic* if every proper submodule is contained in a maximal submodule.

Theorem 2.7. *Let M be a COSP-module and $N \in \sigma[M]$. Then every nonzero submodule of N is coatomic.*

Proof. Let L be a proper submodule of N. By Proposition 2.6, N/L has a maximal submodule T/L. So T is a maximal submodule of N which contains L. Hence N is coatomic, and the theorem is proved since every submodule of N belongs to $\sigma[M]$. ■

The following example shows that a module for which every submodule is coatomic needs not be COSP.

Example 2.8. In Example 2.1 we show that the $\mathbb{Z}$-module $\mathbb{Z}/p^k\mathbb{Z}$ is not COSP. It is clear that every submodule of M is coatomic.

Corollary 2.9. *Let M be a COSP-module. Then for every module $N \in \sigma[M]$, $Rad\,(N) \ll N$.*

Theorem 2.10. *Let M be a module such that every injective module in $\sigma[M]$ is amply supplemented. If M is a COSP-module then for every module $N \in \sigma[M]$, $N = \overline{Z}(N) + Soc\,(N)$ and $\overline{Z}(N) = \overline{Z}^2(N)$.*

Proof. Let $N \in \sigma[M]$. By [5, Corollary 3.9], $N = A \oplus B$ such that A is non-M-cosingular and B is semisimple. $\overline{Z}(N) = \overline{Z}(A) \oplus \overline{Z}(B) = A \oplus \overline{Z}(B)$ implies that $N = A + \overline{Z}(B) + B = \overline{Z}(N) + \mathrm{Soc}\,(N)$. By the proof of [5, Theorem 3.8(4)], $\overline{Z}^2(N) = \overline{Z}(N)$. ■

Corollary 2.11. *Let M be a module such that every injective module in $\sigma[M]$*

is amply supplemented. Then the following are equivalent.
(1) *M is COSP.*
(2) *for every module $N \in \sigma[M]$, $N = \overline{Z}(N) + Soc\,(N)$.*
(3) *every injective module in $\sigma[M]$ is COSP.*
(4) *every module in $\sigma[M]$ is COSP.*

Proof.
(1)$\Longleftrightarrow$(3)$\Longleftrightarrow$(4) clear by Proposition 2.4.
(1)$\Longrightarrow$(2) follows from Theorem 2.10.
(2)$\Longrightarrow$(1) Let N be any module in σ. By hypothesis, $N = \overline{Z}(N) + \mathrm{Soc}\,(N)$. Let $L = \overline{Z}(N) \cap \mathrm{Soc}\,(N)$. Since L is a direct summand of $\mathrm{Soc}\,(N)$, there is a submodule T of $\mathrm{Soc}\,(N)$ such that $\mathrm{Soc}\,(N) = L \oplus T$. It is easy to check that $N = \overline{Z}(N) \oplus T$. Thus $\overline{Z}(N) = \overline{Z}^2(N) \oplus \overline{Z}(T)$. So $\overline{Z}(T) \leq \overline{Z}(N) \cap T$. Hence $\overline{Z}(T) = 0$ and $\overline{Z}(N) = \overline{Z}^2(N)$. Now N is a direct sum of the non-M-cosingular module $\overline{Z}(N)$ and a semisimple module T. Thus M is a COSP-module by [5, Corollary 3.9]. ■

Recall that any module M is local if it is hollow and $\mathrm{Rad}\,(M) \neq M$.

Proposition 2.12. *Suppose that R is a local ring and let H be a local R-module such that H is not simple. Then H is not COSP.*

Proof. Let m be the maximal submodule of H and let $S = H/m$. Suppose that H is COSP. By Proposition 2.3, m is semisimple. Since R is local, $m \cong S^{(I)}$ for some set I. Thus H has a submodule $L \cong S$. Since $L \ll H$, L is H-small. Then L is H-cosingular. Therefore L is H-projective. But $L \cong H/m$, then H/m is H-projective. By [4, Lemma 4.30], $m = 0$, contradiction. It follows that H is not COSP. ■

Let N be a module. N is called *lifting* if each of its submodules A contains a direct summand B of N such that $A/B \ll N/B$. N is called *quasi-discrete* if N is lifting and satisfies the following condition:

(D_3) If N_1 and N_2 are direct summands of N with $N = N_1 + N_2$, then $N_1 \cap N_2$ is also a direct summand of N.

Corollary 2.13. *Suppose that the ring R is local. Let M be a module such that every injective module in σ is quasi-discrete. Then the following are equivalent.*
(1) *M is COSP.*
(2) *for every module $N \in \sigma[M]$, $N = \overline{Z}(N) + Soc\,(N)$.*
(3) *every injective module in $\sigma[M]$ is COSP.*
(4) *every module in $\sigma[M]$ is COSP.*
(5) *M is semisimple.*

Proof.
(1)$\Longleftrightarrow$(2)$\Longleftrightarrow$(3)$\Longleftrightarrow$(4) clear by Corollary 2.11.
(3)$\Longrightarrow$(5) Let $\widehat{M}$ be the injective hull of M in $\sigma[M]$. By (3), $\widehat{M}$ is COSP. Since $\widehat{M}$ is quasi-discrete, $\widehat{M}$ has a decomposition $\widehat{M} = \oplus_{i \in I} H_i$ where each H_i is

hollow by [4, Theorem 4.15]. Taking Corollary 2.9 into account, each H_i is a local module. So each H_i is a COSP local module. By Proposition 2.12, each H_i is simple and hence $\widehat{M}$ is semisimple. Therefore M is semisimple.
(5)$\Longrightarrow$(1) Clear by Example 2.5. ∎

Suppose that the ring R is commutative and noetherian. Let Ω be the set of all maximal ideals of R. If $m \in \Omega$, M an R-module, we denote as [7, p. 53] by $K_m(M) = \{x \in M \mid x = 0$ or the only maximal ideal over $Ann(x)$ is $m\}$ as the m-local component of M. We call M m-local if $K_m(M) = M$. In this case M is an R_m-module by the following operation: $(r/s)x = rx'$ with $x = sx'$ ($r \in R$, $s \in R - m$). The submodules of M over R and over R_m are identical.

For $K(M) = \{x \in M \mid Rx$ is supplemented$\}$ we always have a decomposition $K(M) = \oplus_{m\in\Omega} K_m(M)$ and for a supplemented module M we have $M = K(M)$ [7, Propositions 2.3 and 2.5].

Lemma 2.14. *Suppose that the ring R is commutative noetherian. Let m be a maximal ideal of R and M an m-local R-module. The following are equivalent.*
(1) *M is COSP over R.*
(2) *M is COSP over R_m.*

Proof. It is easily seen that $\sigma[M_R] = \sigma[M_{R_m}]$ and every $N \in \sigma[M_R]$ is m-local. Hence if $N \in \sigma[M_R]$, then the submodules of N over R and over R_m are identical. Therefore a module $N \in \sigma[M_R]$ is M_R-small if and only if it is M_{R_m}-small. Moreover, since M is m-local, every mapping $f : N \longrightarrow L$ of N into L where N and L are in $\sigma[M_R]$ is an R-homomorphism if and only it is an R_m-homomorphism. In fact, if $f : N \longrightarrow L$ is an R-homomorphism, $x \in N$, $r \in R$ and $s \in R - m$, then there is $x' \in N$ such that $x = sx'$ (because $Ann(x) + Rs = R$). Thus $f[(r/s)x] = f(rx') = rf(x')$. But $f(x) = sf(x')$. So $rf(x') = (r/s)f(x)$. This gives that f is an R_m-homomorphism. It follows that a module $N \in \sigma[M_R]$ is M-cosingular over R if and only if it is M-cosingular over R_m and N is projective in $\sigma[M_R]$ if and only if N is projective in $\sigma[M_{R_m}]$, and the proof is complete. ∎

An R-module M is called *locally noetherian* (*locally artinian*) if every finitely generated submodule of M is noetherian (artinian).

Theorem 2.15. *Suppose that the ring R is commutative noetherian. Let M be a module such that every injective module in $\sigma[M]$ is lifting. Then the following are equivalent.*
(1) *M is COSP.*
(2) *for every module $N \in \sigma[M]$, $N = \overline{Z}(N) + Soc(N)$.*
(3) *every injective module in $\sigma[M]$ is COSP.*
(4) *every module in $\sigma[M]$ is COSP.*
(5) *M is semisimple.*

Proof.
(1)$\Longleftrightarrow$(2)$\Longleftrightarrow$(3)$\Longleftrightarrow$(4) clear by Corollary 2.11 and [4, Proposition 4.8].

(3)$\Longrightarrow$(5) Let $\widehat{M}$ be the injective hull of M in σ. By (3), $\widehat{M}$ is COSP. Since R is notherian, $\widehat{M}$ is locally noetherian. From [6, Theorem 27.4] it follows that $\widehat{M} = \oplus_{i\in I} H_i$ is a direct sum of indecomposable modules H_i. By [4, Lemma 4.7, Corollary 4.9], each H_i is hollow. Therefore each H_i is local by Corollary 2.9. Let $i \in I$. Since H_i is an indecomposable supplemented module, H_i is m-local for some maximal ideal m of R. Thus H_i is an R_m-module and it is a local module over R_m. By Proposition 2.4, H_i is a COSP R-module. So H_i is a COSP R_m-module (see Lemma 2.14). We conclude from Proposition 2.12 that H_i is a simple R_m-module. Thus H_i is a simple R-module. Consequently, $\widehat{M}$ is a semisimple R-module. Hence M is a semisimple R-module.
(5)$\Longrightarrow$(1) Clear by Example 2.5. ∎

Let M_1 and M_2 be modules. M_1 is called *small M_2-projective* if every homomorphism $f : M_1 \longrightarrow M_2/A$, where A is a submodule of M_2 and $\Im(f) \ll M_2/A$, can be lifted to a homomorphism $g : M_1 \longrightarrow M_2$.

Lemma 2.16. *Let M be any module such that every simple module in σ is small M-projective. If M is non-M-cosingular and every M-cosingular module is semisimple, then M is COSP.*

Proof. Assume $\overline{Z}(M) = M$ and every M-cosingular module is semisimple. Let $S \in \sigma[M]$ be M-cosingular simple. Let $f : S \longrightarrow M/T$ be any nonzero homomorphism with $T \leq M$. Assume $\Im(f) = K/T$ with $K \leq M$. Note that $S \cong K/T$. Let $L/T \leq M/T$ and $M/T = K/T + L/T$. Then either $K/T \cap L/T = 0$ or $K/T \cap L/T \neq 0$. If $K/T \cap L/T = 0$, then $M/T = K/T \oplus L/T$. Now K/T is non-M-cosingular since M is non-M-cosingular. Therefore S is non-M-cosingular. So $S = 0$, a contradiction. Thus $K/T \cap L/T \neq 0$. Then $K/T \cap L/T = K/T$ and hence $K \subseteq L$. Therefore $M/T = L/T$. Thus $\Im(f) \ll M/T$. Since S is small M-projective, f lifts to a homomorphism $g : S \longrightarrow M$. Therefore S is projective in $\sigma[M]$ and hence every M-cosingular module is projective in $\sigma[M]$. ∎

Lemma 2.17. *Let M be a locally artinian COSP-module. Then every injective module in $\sigma[M]$ is non-M-cosingular.*

Proof. Let $N \in \sigma$ be injective. By the proof of [5, Theorem 3.8(4)], $N = A \oplus B$ such that A is non-M-cosingular and B is M-cosingular. By [5, Corollary 2.9], $B = 0$. Therefore N is non-M-cosingular. ∎

Proposition 2.18. *Let M be a module such that every injective module in σ is amply supplemented. If M is a COSP-module, then every M-cosingular module is semisimple.*

Proof. By [5, Corollary 3.9]. ∎

Theorem 2.19. *Let M be an injective locally artinian module in $\sigma[M]$ such that every injective in $\sigma[M]$ is amply supplemented. Assume that S is small M-projective for every simple module S in $\sigma[M]$. Then the following are equivalent.*

(1) *M is a COSP-module.*
(2) *M is non-M-cosingular and every M-cosingular module is semisimple.*

Proof. Clear by Lemma 2.17, Proposition 2.18 and Lemma 2.6. ■

Let M be a module. M is called *finitely cogenerated* if $\mathrm{Soc}\,(M)$ is finitely generated and $\mathrm{Soc}\,(M)$ is essential in M (see [3, Proposition 19.1]). Any module M is said to have *finite hollow dimension* if there exists an epimorphism from M to a finite direct sum of hollow modules with small kernel. Every artinian module has finite hollow dimension and every factor module of any module with finite hollow dimension has finite hollow dimension again. Many important results on modules with finite hollow dimension are collected in [2]. So for details see [2].

Theorem 2.20. *For a COSP-module M the following conditions are equivalent.*
(1) *M has dcc on small submodules.*
(2) *$\mathrm{Rad}\,(M)$ is artinian.*
(3) *Every small submodule of M is (semesimple) finitely generated.*

Proof.
(1)$\Longleftrightarrow$(2) This is shown in [1, Theorem 5] for arbitrary modules.
(2)$\Longrightarrow$(3) Let $K \ll M$. Then $K \subseteq \mathrm{Rad}\,(M)$ and hence K is artinian. Since M is COSP, $\mathrm{Rad}\,(M)$ is semisimple by Proposition 2.3. Hence K is semisimple and finitely generated.
(3)$\Longrightarrow$(2) Let $K \ll M$. Then K is semisimple by Proposition 2.3. Since K is finitely generated, K is artinian. By [1, Theorem 5], $\mathrm{Rad}\,(M)$ is artinian.
■

Corollary 2.21. *Let M be a COSP-module. Then the following are equivalent.*
(1) *M is artinian.*
(2) *every submodule of M has finite hollow dimension.*
(3) *for every submodule N of M, $N/\,\mathrm{Rad}\,(N)$ is finitely cogenerated.*

Proof.
(2)$\Longleftrightarrow$(3) By [2, (3.5.6)] and Corollary 2.9.
(1)$\Longrightarrow$(2) Clear since every artinian module has finite hollow dimension.
(2)$\Longrightarrow$(1) By Proposition 2.3, every small submodule of M is semisimple. By (2), every small submodule of M is finitely generated. Then $\mathrm{Rad}\,(M)$ is artinian by Theorem 2.20. Since M has finite hollow dimension, M is artinian by [2, (3.5.14)].

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