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Infinite-Dimensional Ito Processes with Respect to Gaussian Random Measures and the Ito Formula*

Dang Hung Thang and Nguyen Thinh

Department of Mathematics,
Hanoi National University, 334 Nguyen Trai Str., Hanoi, Vietnam

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Abstract. In this paper, infinite-dimensional Ito processes with respect to a symmetric Gaussian random measure Z taking values in a Banach space are defined. Under some assumptions, it is shown that if X_t is an Ito process with respect to Z and $g(t, x)$ is a C^2-smooth mapping then $Y_t = g(t, X_t)$ is again an Ito process with respect to Z. A general infinite-dimensional Ito formula is established.

1. Introduction

The Ito stochastic integral is essential for the theory of stochastic analysis. Equipped with this notion of stochastic integral one can consider Ito processes and stochastic differential equations. However, the Ito stochastic integral is insufficient for application as well as for mathematical questions. A theory of stochastic integral in which the integrator is a semimartingale has been developed by many authors (see [1, 4, 5] and references therein). The Ito integral with respect to (w.r.t. for short) Levy processes was constructed by Gine and Marcus [3]. In [11, 12], Thang defined the Ito integral of real-valued random function w.r.t. vector symmetric random stable measures with values in a Banach space, including Gaussian random measure.

Let X, Y be separable Banach spaces and Z be an X-valued symmetric

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Gaussian random measure. In this paper, we are concerned with the study of processes X_t of the form

$$X_t = X_0 + \int_0^t a(s,\omega)ds + \int_0^t b(s,\omega)dQ(s) + \int_0^t c(s,\omega)dZ_s \quad (0 \leqslant t \leqslant T), \qquad (1)$$

where $a(s,\omega)$ is an Y-valued adapted random function, $b(t,\omega)$ is an $B(X,X;Y)$-valued adapted random function and $c(s,\omega)$ is an $L(X,Y)$-valued adapted random function on $[0,T]$. Such a X_t is called an Y-valued Ito process with respect to the X-valued symmetric Gaussian random measure Z. Sec. 2 contains the definition and some properties of X-valued symmetric Gaussian random measures which will be used later and can be found in [12]. As a preparation for defining the Y-valued Ito process and establishing the Ito formula, in Secs. 3 and 4 we construct the Ito integral of $L(X,Y)$-valued adapted random functions w.r.t. an X-valued symmetric Gaussian random measure, investigate the quadratic variation of an X-valued symmetric Gaussian random measure and define what the action of a bilinear continuous operator on a nuclear operator is. Theorem 4.3 shows that the quadratic variation of a symmetric Gaussian random measure is its covariance measure. Sec. 5 will be concerned with the definition of Ito process and the establishment of the general Ito formula. The main result of this section is that if X, Y, E are Banach spaces of type 2, X is reflexive, $g(t,x) : [0,T] \times Y \longrightarrow E$ is a function which is continuously twice differentiable in the variable x and continuously differentiable in the variable t and X_t is an Y-valued Ito process w.r.t. Z then the process $Y_t = g(t, X_t)$ is again an E-valued Ito process w.r.t. Z. The differential dY_t is also established (the general infinite-dimensional Ito formula). The result is new even in the case X, Y, E are finite-dimensional spaces.

2. Vector Symmetric Gaussian Random Measure

In this section we recall the notion and some properties of vector symmetric Gaussian random measures, which will be used later and can be found in [12]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, X be a separable Banach space and $(S, \mathcal{A})$ be a measurable space. A mapping $Z : \mathcal{A} \longrightarrow L^2_X(\Omega, \mathcal{F}, \mathbb{P}) = L^2_X(\Omega)$ is called an X-valued symmetric Gaussian random measure on $(S, \mathcal{A})$ if for every sequence (A_n) of disjoint sets from $\mathcal{A}$, the r.v.'s $Z(A_n)$ are Gaussian, symmetric, independent and

$$Z\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} Z(A_n) \quad \text{in } L^2_X(\Omega).$$

For each $A \in \mathcal{A}$, $Q(A)$ stands for the covariance operator of $Z(A)$. The mapping $Q : A \mapsto Q(A)$ is called the covariance measure of Z.

Let $G(X)$ denote the set of covariance operators of X-valued Gaussian symmetric r.v.'s and $N(X', X)$ denote the Banach space of nuclear operators from X' into X. Let $N^+(X', X)$ denote the set of non-negatively definite nuclear operators. It is known that [12] $G(X) \subset N^+(X', X)$ and the equality $G(X) = N^+(X', X)$ holds if and only if X is of type 2.

A characterization of the class of covariance measures of vector symmetric Gaussian random measures is given by following theorem.

Theorem 2.1. [12] *Let Q be a mapping from $\mathcal{A}$ into $G(X)$. The following assertions are equivalent:*

1. *Q is a covariance measure of some X-valued symmetric Gaussian random measure.*
2. *Q is a vector measure with values in Banach space $N(X', X)$ of nuclear operators and non-negatively definite in the sense that:*
   For all sequences $A_1, A_2, \cdots, A_n$ from $\mathcal{A}$ and all sequences $a_1, a_2, \cdots, a_n$ from X' we have
$$\sum_{i=1}^{n}\sum_{j=1}^{n}(Q(A_i \cap A_j)a_i, a_j) \geq 0.$$

Given an operator $R \in G(X)$ and a non-negative measure μ on $(S, \mathcal{A})$, consider the mapping Q from $\mathcal{A}$ into $G(X)$ defined by

$$Q(A) = \mu(A)R.$$

It is easy to check that Q is σ-additive in the nuclear norm and non-negatively definite. By Theorem 2.1 there exists an X-valued symmetric Gaussian random measure W such that for each $A \in \mathcal{A}$ the covariance operator of $W(A)$ is $\mu(A)R$. We call W the *X-valued Wiener random measure* with the parameters (μ, R).

In order to study vector symmetric Gaussian random measures, it is useful to introduce an inner product on $L^2_X(\Omega)$. For $\xi, \eta \in L^2_X(\Omega)$, the inner product $[\xi, \eta]$ is an operator from X' into X defined by

$$a \mapsto [\xi, \eta](a) = \int\limits_{\Omega} \xi(\omega)(\eta(\omega), a)d\mathbb{P}.$$

The inner product have the following properties

Theorem 2.2. [12]

1. *$[\xi, \eta]$ is a nuclear operator and*
$$\|[\xi, \eta]\|_{nuc} \leqslant \|\xi\|_{L_2}\|\eta\|_{L_2}.$$
2. *If the space X is of type 2 then there exists a constant $C > 0$ such that*
$$\|[\xi, \xi]\|_{nuc} \leqslant \|\xi\|^2_{L_2} \leqslant C\|[\xi, \xi]\|_{nuc}.$$
3. *If $\lim \xi_n = \xi$ and $\lim \eta_n = \eta$ in $L^2_X(\Omega)$ then $\lim[\xi_n, \eta_n] = [\xi, \eta]$ in the nuclear norm.*

Let Q be the covariance measure of an X-valued symmetric Gaussian random measure Z. It is easy to see that

$$Q(A) = [Z(A), Z(A)].$$

From Theorem 2.2 we get

Theorem 2.3. [12] *If the space X is of type 2 then there exists a constant $C > 0$ such that for each X-valued symmetric Gaussian random measure Z with the covariance measure Q we have*

$$\mathbb{E}\|Z(A)\|^2 \leqslant C\|Q(A)\| \leqslant C|Q|(A),$$

where $|Q|$ stands for the variation of Q.

3. The Ito integral of Operator-Valued Random Functions

Let S be the interval $[0, T]$, $\mathcal{A}$ be the σ-algebra of Borel sets of S and let Z be an X-valued symmetric Gaussian random measure on S with the covariance measure Q.

From now on, we assume that $|Q| \ll \lambda$, where λ is the Lebesgue measure on S. Let $L(X, Y)$ be the space of all continuous linear operators from X into Y. The Ito integral of the form $\int f dZ$, where f is an $L(X, Y)$-valued adapted random function is constructed as follows.

First, we associate to Z a family of increasing σ-algebra $\mathcal{F}_t \subset \mathcal{A}$ as follows: $\mathcal{F}_t$ is the σ-algebra generated by the X-valued r.v.'s $Z(A)$ with $A \in \mathcal{A} \cap [0, t]$.

Let $\mathcal{N}(S, Z, E)$ be the set of E-valued functions $f(t, \omega)$ satisfying the following:

1. $f(t, \omega)$ is adapted w.r.t. Z, i.e. it is jointly measurable and $\mathcal{F}_t$-measurable for each $t \in S$.
2. $\mathbb{E}\int\limits_S \|f(t, \omega)\|^2 d|Q|(t) < \infty$.

Let $\mathcal{M}(S, Z, E)$ be the set of E-valued functions $f(t, \omega)$ such that $f(t, \omega)$ is adapted w.r.t. Z and $\mathbb{P}\left\{\omega : \int\limits_S \|f(t, \omega)\|^2 \, d|Q|(t) < \infty\right\} = 1$ and $\mathcal{S}(S, Z, E)$ be the set of simple functions $f \in \mathcal{N}(S, Z, E)$ of the form

$$f(t, \omega) = \sum_{i=0}^{n} f_i(\omega) 1_{A_i}(t), \tag{2}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_{n+1} = T$, $A_0 = \{0\}$, $A_i = (t_i, t_{i+1}]$ $1 \leqslant i \leqslant n$, f_i is $\mathcal{F}_{t_i}$-measurable.

In this paper, we deal with the spaces $\mathcal{N} := \mathcal{N}(S, Z, L(X, Y))$, $\mathcal{M} := \mathcal{M}(S, Z, L(X, Y))$, $\mathcal{S} := \mathcal{S}(S, Z, L(X, Y))$.

$\mathcal{N}$ is a Banach space with the norm

$$\|f\|^2 := \mathbb{E}\int\limits_T \|f(t, \omega)\|^2 \, d|Q|(t).$$

$\mathcal{M}$ is a Frechet space with the norm

$$\|f\|_s := \mathbb{E}\,\frac{1}{1 + \left(\int \|f\|^2 d|Q|\right)^{1/2}} \left(\int \|f\|^2 d|Q|\right)^{1/2}.$$

$\|f\|_s \to 0$ if and only if $\int \|f(t,\omega\|^2 \, d|Q|(t) \xrightarrow{\mathbb{P}} 0$.

Lemma 3.1.
1. *$\mathcal{S}$ is dense in $\mathcal{N}$ (with norm $\|\cdot\|$).*
2. *$\mathcal{S}$ is dense in $\mathcal{M}$ (with norm $\|\cdot\|_s$).*

Proof. We re-denote spaces $\mathcal{S}, \mathcal{N}, \mathcal{M}$, by $\mathcal{S}(S, \mathcal{F}_t, |Q|, L(X,Y))$, $\mathcal{N}(S, \mathcal{F}_t, |Q|, L(X,Y))$, $\mathcal{M}(S, \mathcal{F}_t, |Q|, L(X,Y))$ respectively.

Put $\alpha(t) = |Q|[0,t], 0 \leqslant t \leqslant T$. Since $0 \leqslant |Q| \ll \lambda$, $\alpha(t)$ is a non-decreasing continuous function. It is easy to check that the mapping

$$\alpha : (S, \mathcal{A}, |Q|) \longrightarrow ([0, \alpha(T)], \Sigma, \lambda)$$

is surjective, measurable and measure-preserving, where Σ is the σ-algebra of Borel sets of $[0, \alpha(T)])$.

Now we prove that α is injective a.s. in the sense that for almost all $x \in [0, \alpha(T)]$, the set $\alpha^{-1}(x)$ consists of only one point.

Indeed, assume x is a number such that the set $\{\, t : \alpha(t) = x \,\}$ consists of more than one point. Because α is continuous and non-decreasing the set $\{t : \alpha(t) = x\}$ is some segment $[a,b]$ with $a < b$. Moreover α is measure-preserving so $|Q|\{t : \alpha(t) = x\} = |Q|[a,b] = \lambda(\{t\}) = 0$. The number of these segments $[a,b]$ on $[0,T]$ must be finite or countable so their $|Q|$-measure is also zero. We conclude that α is bijective a.s. and measure-preserving between the spaces

$$\begin{gathered} \alpha : (S, \mathcal{A}, |Q|) \longrightarrow ([0, \alpha(T)], \Sigma, m), \\ t \mapsto \alpha(t). \end{gathered}$$

We establish the mapping

$$\begin{gathered} f(t,\omega)_{0 \leqslant t \leqslant T} \longleftrightarrow g(s,\omega) = f(\alpha^{-1}s, \omega)_{0 \leqslant s \leqslant \alpha(T)}, \\ (\mathcal{F}_t)_{0 \leqslant t \leqslant T} \longleftrightarrow (\mathcal{G}_s) = (\mathcal{F}_{\alpha^{-1}(s)})_{0 \leqslant s \leqslant \alpha(T)}. \end{gathered}$$

This mapping is one to one between spaces

$$\begin{gathered} \mathcal{S}(S, \mathcal{F}_t, |Q|, L(X,Y)) \longleftrightarrow \mathcal{S}(\Sigma, \mathcal{G}_t, \lambda, L(X,Y)), \\ \mathcal{N}(S, \mathcal{F}_t, |Q|, L(X,Y)) \longleftrightarrow \mathcal{N}(\Sigma, \mathcal{G}_t, \lambda, L(X,Y)), \\ \mathcal{M}(S, \mathcal{F}_t, |Q|, L(X,Y)) \longleftrightarrow \mathcal{M}(\Sigma, \mathcal{G}_t, \lambda, L(X,Y)). \end{gathered}$$

It is not difficult to check that this mapping is norm-preserving.

By a proof similar to that in [6] we obtain $\mathcal{S}(\Sigma, \mathcal{G}_t, \lambda, L(X,Y))$ is dense in $\mathcal{N}(\Sigma, \mathcal{G}_t, \lambda, L(X,Y))$ and $\mathcal{S}(\Sigma, \mathcal{G}_t, \lambda, L(X,Y))$ is dense in $\mathcal{M}(\Sigma, \mathcal{G}_t, \lambda, L(X,Y))$ so the lemma is proved. ∎

From now on, if $f \in L(X,Y), x \in X$ then we write fx for $f(x)$ for brevity.

If $f \in \mathcal{S}$ is a simple function of the form (2), we define

$$\int_S f dZ = \sum_{i=1}^{n} f_i Z(A_i).$$

Lemma 3.2. *Let X, Y be Banach spaces of type 2. Then there exists a constant $K > 0$ such that for every $f \in \mathcal{S}$:*

$$\mathbb{E} \| \int f\, dZ \|^2 \leqslant K \int \mathbb{E}\, \|f\|^2 d|Q|.$$

Proof. Assume that f is of the form (2). Put $Z_i = Z(A_i)$, $\mathcal{F}_i = \mathcal{F}_{t_i}$.

Since Y is of type 2, by Theorem 2.2, there exists a constant C_1 such that

$$\begin{aligned}\mathbb{E} \Big\| \sum_{i=0}^{n} f_i Z_i \Big\|^2 &\leqslant C_1 \Big\| \Big[\sum_{i=0}^{n} f_i Z_i, \sum_{j=0}^{n} f_j Z_j \Big] \Big\|_{nuc} \\ &\leqslant C_1 \sum_{i=0}^{n} \sum_{j=0}^{n} \big\| [f_i Z_i, f_j Z_j] \big\|_{nuc} \\ &= C_1 \sum_{i=1}^{n} \big\| [f_i Z_i, f_i Z_i] \big\|_{nuc} + 2C_1 \sum_{j>i} \big\| [f_i Z_i, f_j Z_j] \big\|_{nuc}. \end{aligned} \tag{3}$$

If $j > i$ then $f_i \in \mathcal{F}_j, f_j \in \mathcal{F}_j, Z_i \in \mathcal{F}_j$. Let $a \in X'$ be arbitrary. We have $\langle f_i Z_i, a \rangle \in \mathcal{F}_j$ and

$$[f_i Z_i, f_j Z_j](a) = \mathbb{E}\big(\langle f_i Z_i, a \rangle (f_j Z_j)\big) = \mathbb{E}\mathbb{E}\big(\langle f_i Z_i, a \rangle (f_j Z_j) | \mathcal{F}_j\big).$$

$$\mathbb{E}\big(\langle f_i Z_i, a \rangle (f_j Z_j) | \mathcal{F}_j\big) = \langle f_i Z_i, a \rangle \mathbb{E}(f_j Z_j | \mathcal{F}_j) = \langle f_i Z_i, a \rangle f_j \mathbb{E}(Z_j | \mathcal{F}_j).$$

Because Z_j is independent of $\mathcal{F}_j$ then $\mathbb{E}(Z_j|\mathcal{F}_j) = 0$. It follows that

$$[f_i Z_i, f_j Z_j](a) = 0 \quad , \forall a \in X'.$$

That is

$$[f_i Z_i, f_j Z_j] = 0,$$

which implies the sencond term in (3) is zero.

If $j = i$, we have

$$\begin{aligned}\sum_{i=1}^{n} \big\| [f_i Z_i, f_i Z_i] \big\|_{nuc} &\leqslant \sum_{i=1}^{n} \mathbb{E} \|f_i Z_i\|^2 \\ \leqslant \sum_{i=1}^{n} \mathbb{E}\big(\|f_i\|^2 \|Z_i\|^2\big) &= \sum_{i=1}^{n} \mathbb{E}\|f_i\|^2 \mathbb{E}\|Z_i\|^2.\end{aligned}$$

Since X is of type 2, by Theorem 2.3, there exists a constant C_2 such that

$$\mathbb{E}\|Z_i\|^2 \leqslant C_2 |Q|(A_i).$$

Hence, we obtain

$$\mathbb{E}\Big\| \sum_{i=0}^{n} f_i Z_i \Big\|^2 \leqslant C_1 C_2 \sum_{i=0}^{n} \mathbb{E}\|f_i\|^2 |Q|(A_i)$$
$$= K \int \mathbb{E}\|f\|^2 \, d|Q| \quad (\text{where } K = C_1 C_2).$$

■

From Lemmas 3.1 and 3.2 we get

Theorem 3.3. *Let X, Y be Banach spaces of type 2. Then there exists a unique linear continuous mapping $f \mapsto \int\limits_S f dZ = \int\limits_0^T f(t,\omega) dZ(t)$ from $\mathcal{N}$ into $L^2_Y(\Omega)$ such that for each simple function $f \in \mathcal{S}$ given by (2) we have*

$$\int\limits_0^T f(t,\omega) dZ(t) = \int\limits_S f dZ = \sum_{i=1}^{n} f_i Z(A_i).$$

By using technique similar to the proof of Lemma 3.2 and the Ito's method in [6] we can define the random integral $\int f dZ$ for random functions $f \in \mathcal{M}$.

Theorem 3.4. *Let X, Y be Banach spaces of type 2. Then there exists a unique linear continuous mapping $f \mapsto \int\limits_S f dZ$ from $\mathcal{M}$ into $L^0_Y(\Omega)$ such that for each simple function $f \in \mathcal{S}$ given by (2) we have:*

$$\int\limits_S f dZ = \sum_{i=1}^{n} f_i Z(A_i).$$

Put $Q_t = Q[0,t]$. By Theorem 2.3, there exists a constant C such that $\mathbb{E}\|Z(A)\|^2 \leqslant C|Q|(A)$. From this inequality together with the assumption that $|Q| \ll \lambda$, it follows that the process Q_t has a continuous modification (see [13]). Hence, from now on, we may assume without loss of generality that the process Q_t is continuous.

By a standard argument as in the proof of Lemma 3.2 and the Ito's method we can prove the following

Theorem 3.5. (Continuous modification) *Let X, Y be Banach spaces of type 2. Put*

$$X_t = \int\limits_0^t f(s,\omega) dZ(s) = \int\limits_0^T f(s,\omega) 1_{[0,t]} dZ(s),$$

where $f \in \mathcal{M}$. Then X_t has a continuous modification.

Theorem 3.6. *Suppose f_n, f are random functions such that $f_n \to f$ in the space $\mathcal{M} = \mathcal{M}(S, Z, L(X,Y))$, i.e*

$$\int_S \|f_n - f\|^2 \, d|Q| \to 0 \quad \textit{in probability.}$$

Then we have

$$\sup_{0\leqslant t\leqslant T} \Big\| \int_0^t f_n \, dZ - \int_0^t f \, dZ \Big\| \to 0 \ \ \textit{in probability.}$$

4. Quadratic Variation of X-Valued Symmetric Gaussian Random Measures

First, let us recall some notions and properties of tensor product of Banach spaces which can be found in [2]. Let $X \otimes Y$ be the algebraic tensor product of X and Y. Then $X \otimes Y$ become a normed space under the greatest reasonable crossnorm γ given by

$$\gamma(u) = \inf\Big\{ \sum_{i=1}^n \|x_i\|\|y_i\| : x_i \in X, y_i \in Y, u = \sum_{i=1}^n x_i \otimes y_i \Big\}.$$

The completion of $X \otimes Y$ under γ is denoted by $X\widehat{\otimes}Y$ and call the projective tensor product of X and Y. Thus, $u \in X\widehat{\otimes}Y$ if and only if there exists sequences $(x_n) \in X, (y_n) \in Y$ such that $\sum_{i=1}^n \|x_n\|\|y_n\| < \infty$ and $u = \sum_{n=1}^\infty x_i \otimes y_i$ in γ-norm.

Let $B(X, Y; E)$ be the Banach space of continuous bilinear operators from $X \times Y$ into E and $L(X\widehat{\otimes}Y, E)$ be the Banach space of linear continuous operators from $X\widehat{\otimes}Y$ into E. Then we have

Theorem 4.1. [2, p. 230] *$B(X, Y; E)$ is isometrically isomorphic to $L(X\widehat{\otimes}Y, E)$. In particular, $(X\widehat{\otimes}Y)'$ is isometrically isomorphic to $L(X, Y')$.*

Suppose that X is reflexive. For each $u \in X\widehat{\otimes}X$, let $J(u)$ be an operator from X' into X given by

$$J(u)(a) = \sum_{i=n}^\infty (x_n, a) y_n$$

if $u = \sum_{n=1}^\infty x_i \otimes y_i$.

It is plain that $J(u)$ is well-defined, $J(u) \in N(X', X)$ and $J : X\widehat{\otimes}X \to N(X', X)$ is surjective. The following theorem shows that J is injective.

Theorem 4.2. *The correspondence $u \mapsto J(u)$ is injective.*

Proof. Suppose that $u = \sum\limits_{n=1}^\infty x_i \otimes y_i$ and $J(u) = 0$. Let $b \in L(X, X')$ be arbitrary. By Theorem 4.1, $L(X, X')$ is the dual of $X\widehat{\otimes}X$ with $(u, b) = \sum_{n=1}^\infty (y_n, bx_n)$ so it is sufficient to show that $\sum_{n=1}^\infty (y_n, bx_n) = 0$. Indeed, for each $x \in X$, we have $\sum_{n=1}^\infty (x_n, b^* x) y_n = 0$ or $\sum_{n=1}^\infty (x, bx_n) y_n = 0$. Because X is reflexive, by

Grothendieck's conjecture proved by Figiel ([2, p. 260]), X has the approximation property. Because $\sum_{n=1}^{\infty} \|bx_n\|\|y_n\| < \infty$, by applying Theorem 4 ([2, p. 239]), we obtain $\sum_{n=1}^{\infty}(y_n, bx_n) = 0$ as desired. ■

Note that if $\xi, \eta \in L^2_X(\Omega)$ then $\xi \otimes \eta$ is a random variable taking values in $X \otimes X$ and the inner product $[\xi, \eta] = \mathbb{E}(\xi \otimes \eta)$.

From now on, assume that X is reflexive. For brevity, for each $T \in N(X', X)$ and $\phi \in B(X, X; Y) \simeq L(X \widehat{\otimes} X, Y)$, the action of ϕ on T is understood as $\phi(J^{-1}T)$ and is denoted by ϕT, which is an element of Y.

Before stating a new theorem we recall some integrable criteria for vector -valued functions with respect to vector-measures with finite variation, which we use in this paper

Suppose that f is an $B(X, X; Y)$-valued deterministic function on $[0, T]$. Then the following assertions are equivalent

1. f is Q-integrable (i.e. there exists integral $\int_0^T f dQ$).
2. f is $|Q|$-integrable (Bochner-integrable).
3. $\|f\|$ is $|Q|$-integrable.

Let Δ be a partition of $S = [0, T] : 0 = t_0 < t_1 < \cdots < t_{n+1} = T$, $A_0 = \{0\}$, $A_i = (t_i, t_{i+1}]$. For brevity, we write Z_i for $Z(A_i)$. The following theorem is essential for establishing the infinite-dimensional Ito formula.

Theorem 4.3. *Suppose that X is reflexive, X, Y are of type 2 and Z is an X-valued symmetric Gaussian random measure on $[0, T]$ with the covariance measure Q. Let $f(t, \omega)$ be a $B(X, X; Y)$-valued random function adapted w.r.t. Z satisfying*

$$\mathbb{E} \int_S \|f(t, \omega)\|^2 d|Q|(t) < \infty.$$

Then we have

$$\sum_{i=1}^{n} f(t_i)(Z_i \otimes Z_i) \longrightarrow \int_0^T f(t) dQ(t) \quad \text{in} \quad L^2_Y(\Omega)$$

as the gauge $|\Delta| = \max_i |Q|(A_i)$ tends to 0.

Theorem 4.3 can be expressed formally by the formula

$$dZ \otimes dZ = dQ.$$

We call $\int_0^T f(t) dQ(t)$ the value of quadratic variation of Z at $f(t)$.

Proof. Put $f_i = f(t_i)$, $\mathcal{F}_i = \mathcal{F}_{t_i}$, $Z_i^2 = Z_i \otimes Z_i$, $Q_i = Q(A_i)$, $|Q|_i = |Q|(A_i)$. Because Y is of type 2 there exists a constant C_1 such that

$$
\begin{aligned}
&\mathbb{E}\Big\|\sum_{i=1}^{n} f(t_i)(Z_i\otimes Z_i)-\int\limits_0^T f(t)dQ(t)\Big\|^2\\
=&\,\mathbb{E}\Big\|\sum_{i=1}^{n} f_i Z_i^2-\sum_{i=1}^{n} f_i Q_i\Big\|^2=\mathbb{E}\Big\|\sum_{i=1}^{n} f_i(Z_i^2-Q_i)\Big\|^2\\
\leqslant&\, C_1\Big\|\Big[\sum_{i=1}^{n} f_i(Z_i^2-Q_i),\sum_{j=1}^{n} f_j(Z_j^2-Q_j)\Big]\Big\|_{nuc}
\end{aligned}
$$

$$
\begin{aligned}
&=C_1\Big\|\mathbb{E}\Big(\sum_{i=1}^{n} f_i(Z_i^2-Q_i)\otimes\sum_{j=1}^{n} f_j(Z_j^2-Q_j)\Big)\Big\|\\
&\leqslant C_1\sum_{i,j=1}^{n}\Big\|\mathbb{E}\big(f_i(Z_i^2-Q_iR)\otimes f_j(Z_j^2-Q_j)\big)\Big\|.
\end{aligned}
$$

If $j>i$ then $f_i, Z_i^2-Q_i, f_j$ are $\mathcal{F}_j$-measurable, Z_j^2 is independent of $\mathcal{F}_j$, which implies

$$
\begin{aligned}
&\mathbb{E}\big(f_i(Z_i^2-Q_i)\otimes f_j(Z_j^2-Q_j)|\mathcal{F}_j\big)\\
&=f_i(Z_i^2-Q_i)\otimes\mathbb{E}\big(f_j(Z_j^2-Q_j)|\mathcal{F}_j\big)\\
&=\big(f_i(Z_i^2-Q_i)\big)\otimes\big(f_j\mathbb{E}(Z_j^2-Q_j|\mathcal{F}_j)\big)\\
&=\big(f_i(Z_i^2-Q_i)\big)\otimes\big(f_j\mathbb{E}(Z_j^2-Q_j)\big)=0.
\end{aligned}
$$

If $i=j$ then

$$
\begin{aligned}
&\Big\|\mathbb{E}\big(f_i(Z_i^2-Q_i)\otimes f_i(Z_i^2-Q_i)\big)\Big\|\\
\leqslant&\,\mathbb{E}\|f_i(Z_i^2-Q_i)\|^2\leqslant\mathbb{E}\big(\|f_i\|^2\|Z_i^2-Q_i\|^2\big)\\
=&\,\mathbb{E}\|f_i\|^2\mathbb{E}\|Z_i^2-Q_i\|^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbb{E}\|Z_i^2-Q_i\|^2&\leqslant\mathbb{E}(\|Z_i^2\|+\|Q_i\|)^2\\
&\leqslant\mathbb{E}\|Z_i\|^4+2|Q|_i\mathbb{E}\|Z_i\|^2+|Q|_i^2.
\end{aligned}
$$

Because Z_i is an X-valued Gaussian random variable, there exists a constant C_2 such that

$$\mathbb{E}\|Z_i\|^4\leqslant C_2\big(\mathbb{E}\|Z_i\|^2\big)^2.$$

Moreover, $\mathbb{E}\|Z_i\|^2\leqslant C_1|Q|_i$. Consequently,

$$\begin{aligned}
&\mathbb{E}\Big\| \sum_{i=1}^{n} f(t_i)(Q_i \otimes Q_i) - \int_0^T f(t)dQ(t)\Big\|^2 \\
&\leqslant C_1 \sum_{i=1}^{n} \mathbb{E}\|f_i\|^2 \big(\mathbb{E}\|Z_i\|^4 + 2|Q|_i \mathbb{E}\|Z_i\|^2 + |Q|_i^2\big) \\
&\leqslant \sum_{i=1}^{n} C_1(C_1^2 C_2 + 2C_1 + 1)\|f_i\|^2 |Q|_i^2 = K \sum_{i=1}^{n} \|f_i\|^2 |Q|_i^2 \\
&\leqslant K \max_i |Q|_i \sum_{i=1}^{n} \|f_i\|^2 |Q|_i,
\end{aligned}$$

which tends to $K \cdot 0 \cdot \int \mathbb{E}\|f\|^2 d|Q| = 0$ when $|\Delta| \to 0$. ■

5. Ito Processes and Ito Formula

Definition. *Let X, Y be separable Banach spaces, Z is an X-valued symmetric Gaussian random measure on $[0,T]$ with the covariance measure Q. An Y-valued random process X_t is called an Y-valued Ito process w.r.t Z if it is of the form*

$$X_t = X_0 + \int_0^t a(s,\omega)ds + \int_0^t b(s,\omega)dQ(s) + \int_0^t c(s,\omega)dZ_t \quad (0 \leqslant t \leqslant T),$$

where $a(s,\omega)$ is an Y-valued adapted random function, $b(t,\omega)$ is an $B(X,X;Y)$-valued adapted random funtion and $c(s,\omega)$ is an $L(X,Y)$-valued adapted random function w.r.t. Z satisfying

$$\mathbb{P}\Big\{\omega : \int_0^T \|a(t,\omega)\|\, dt < \infty\Big\} = 1,$$

$$\mathbb{P}\Big\{\omega : \int_0^T \|b(t,\omega)\|\, d|Q|(t) < \infty\Big\} = 1,$$

$$\mathbb{P}\Big\{\omega : \int_0^T \|c(t,\omega)\|^2\, d|Q|(t) < \infty\Big\} = 1.$$

In this case, we say that X_t has the Ito differential dX_t given by

$$dX_t = adt + bdQ_t + cdZ_t.$$

Theorem 5.2. (The general infinite-dimensional Ito formula) *Assume that X, Y, E are separable Banach spaces of type 2, X is reflexive, Z is an X-valued* *symmetric Gaussian random measure on $[0,T]$ with the covariance measure Q and X_t is an Y-valued Ito process w.r.t Z*

$$dX_t = adt + bdQ_t + cdZ_t.$$

Let $g : [0,\infty) \times Y \to E$ be a function which is continuously differentiable in the first variable and continuously twice differentiable in the second variable (strongly differentiable).

Put $Y_t := g(t, X_t)$. Then Y_t is again an E-valued Ito process and

$$dY_t = \Big[\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x}a\Big]\,dt + \Big[\frac{\partial g}{\partial x}\circ b + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}\circ c^2\Big]\,dQ_t + \frac{\partial g}{\partial x}\circ c\,dZ_t.$$

where $c \in L(X,Y)$, c^2 stands for the mapping from $X \times X$ into $Y \times Y$ defined by $c^2(x,y) = (cx, cy)$ and $u \circ v$ denotes the composition of mappings u and v.

Proof. We have to prove

$$\begin{aligned} g(t,X_t) = &\int_0^t \Big[\frac{\partial g}{\partial t}(s,X_s) + \frac{\partial g}{\partial x}(s,X_s)a(s)\Big]\,ds \\ &+ \Big[\frac{\partial g}{\partial x}(s,X_s)\circ b + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(s,X_s)\circ c^2(s)\Big]dQ \\ &+ \int \frac{\partial g}{\partial x}(s,X_s)\circ c(s)\,dQ_s \quad \text{a.s.} \end{aligned} \tag{4}$$

We divide the proof into 2 step.

Step 1. We consider the case where g, $\frac{\partial g}{\partial t}$, $\frac{\partial g}{\partial x}$, $\frac{\partial^2 g}{\partial x^2}$ are bounded.

First, we prove (4) for the simple functions a, b, c. Clearly, it suffices to prove for functions a, b, c of the form $a(s,\omega) \equiv a(\omega)$, $b(s,\omega) \equiv b(\omega)$, $c(s,\omega) \equiv c(\omega)$.

Let $\{t_j\}$ be a partition of $[0,t]$. By Taylor formula we have

$$g(t,X_t) = g(0,X_0) + \sum_j \Delta g(t_j,X_j) = g(0,X_0) + \sum_j \frac{\partial g}{\partial t}\Delta t_j + \sum_j \frac{\partial g}{\partial x}\Delta X_j \tag{5}$$

$$+ \frac{1}{2}\sum_j \frac{\partial^2 g}{\partial t^2}(\Delta t_j)^2 + \frac{1}{2}\sum_j \frac{\partial^2 g}{\partial t \partial x}(\Delta t_j)(\Delta X_j) + \frac{1}{2}\sum_j \frac{\partial^2 g}{\partial x^2}(\Delta X_j)^2 + \sum_j R_j, \tag{6}$$

where $\frac{\partial g}{\partial t}, \frac{\partial g}{\partial x}, \frac{\partial^2 g}{\partial t \partial x}, \frac{\partial^2 g}{\partial x^2}$ are values of these maps at (t_j, X_{t_j}) and

$$\begin{aligned} \Delta t_j &= t_{j+1} - t_j, \\ \Delta X_j &= X_{t_{j+1}} - X_{t_j}, \\ \Delta g(t_j,X_j) &= g(t_{j+1},X_{t_{j+1}}) - g(t_j,X_{t_j}), \\ R_j &= 0\Big(|\Delta t|^2 + |\Delta X_j|^2\Big), \quad \forall j. \end{aligned}$$

Put $\Delta Z_j = Z[t_i,t_{i+1}]$, $\Delta Q_j = Q[t_i,t_{i+1}]$. We have $\Delta X_j = a\Delta t_j + b\Delta Q_j + c\Delta Z_j$. When $\max|\Delta t_j| \to 0$ then

$$\sum_j \frac{\partial g}{\partial t}\Delta t_j \longrightarrow \int_0^t \frac{\partial g}{\partial t}(s, X_s)ds.$$

$$\sum_j \frac{\partial g}{\partial x}\Delta X_j = \sum_j \frac{\partial g}{\partial x}\, a\,\Delta t_j \;+\; \sum_j \big(\frac{\partial g}{\partial x}\circ b\big)\Delta Q_j \;+\; \sum_j \big(\frac{\partial g}{\partial x}\circ c\big)\Delta Z_j$$

$$\longrightarrow \int_0^t \frac{\partial g}{\partial x}(s, X_s)\, a\, ds + \int_0^t \big(\frac{\partial g}{\partial x}(s, Q_s)\circ b\big)dQ_s + \int_0^t \big(\frac{\partial g}{\partial x}(s, Q_s)\circ c\big)dZ_s.$$

$$\begin{aligned}\sum_j \frac{\partial^2 g}{\partial x^2}(\Delta X_j)^2 &= \sum_j \frac{\partial^2 g}{\partial x^2}(a\Delta t_j)(a\Delta t_j) + \sum_j \frac{\partial^2 g}{\partial x^2}(b\Delta Q_j)(b\Delta Q_j)\\ &\quad + 2\sum_j \frac{\partial^2 g}{\partial x^2}(a\Delta t_j)(b\Delta Q_j) + 2\sum_j \frac{\partial^2 g}{\partial x^2}(a\Delta t_j)(c\Delta Z_j)\\ &\quad + 2\sum_j \frac{\partial^2 g}{\partial x^2}(b\Delta Q_j)(c\Delta Z_j)\\ &\quad + \sum_j \frac{\partial^2 g}{\partial x^2}(c\Delta Z_j)(c\Delta Z_j). \end{aligned} \tag{7}$$

We shall show that all the terms in the right-hand side of (7), except the last term, converge to 0. Indeed, for example, for the term $\sum_j \frac{\partial^2 g}{\partial x^2}(b\Delta Q_j)(c\Delta Z_j)$, we have

$$\begin{aligned}\Big\|\sum_j \frac{\partial^2 g}{\partial x^2}(b\Delta Q_j)(c\Delta Z_j)\Big\| &\leqslant \sum_j \Big\|\frac{\partial^2 g}{\partial x^2}\Big\|\,\|b\|\,\|c\|\|\Delta Q_j\|\,\|\Delta Z_j\|\\ \leqslant \sum_j \Big\| \sup_{\substack{0\leqslant s\leqslant t\\ \|x\|\leqslant \sup_s X_s(\omega)}} &\frac{\partial^2 g}{\partial x^2}(s,x)\Big\|\,\|b(\omega)\|\,\|c(\omega)\|\sup_j \|\Delta Z_j\|\,|Q|(A_j)\\ = |Q|([0,t])\,\Big\| \sup_{\substack{0\leqslant s\leqslant t\\ \|x\|\leqslant \sup_s X_s(\omega)}} &\frac{\partial^2 g}{\partial x^2}(s,x)\Big\|\,\|b(\omega)\|\,\|c(\omega)\|\sup_j \|\Delta Z_j\|,\end{aligned}$$

which tends to 0 when $\max|\Delta t_j| \to 0$, (because $\frac{\partial^2 g}{\partial x^2}$ is bounded and Z_s is uniformly continuous on $[0,t]$).

Similarly, the terms $\sum_j \frac{\partial^2 g}{\partial t^2}(\Delta t_j)^2$, $\sum_j \frac{\partial^2 g}{\partial t \partial x}(\Delta t_j)(\Delta X_j)$ and $\sum_j R_j$ in the right-hand side of (5) converge to 0.

By Theorem 4.3, the third term

$$\sum_j \frac{\partial^2 g}{\partial x^2}(c\Delta Z_j)(c\Delta Z_j) = \sum_j \sum_j \big(\frac{\partial^2 g}{\partial x^2}\circ c^2\big)(Z_i \otimes Z_i)$$

converges in probability to $\int\limits_0^t \Big(\dfrac{\partial^2 g}{\partial x^2}(s, X_s) \circ c^2(s)\Big) dQ$ when $\max|\Delta t_j| \to 0$. Hence, (4) holds for a, b, c being simple functions.

Choose a sequence of X-valued simple random functions $a^{(n)}(s,\omega)$ satisfying

$$\int\limits_0^t \|a^{(n)}(s,\omega) - a(s,\omega)\|\, ds \to 0 \ \text{ for almost } \omega,$$

a sequence of $B(X,X;Y)$-valued simple random functions $b^{(n)}(s,\omega)$ satisfying

$$\int\limits_0^t \|b^{(n)}(s,\omega) - b(s,\omega)\|\, d|Q| \to 0 \ \text{ for almost } \omega,$$

and a sequence of $L(X,Y)$-valued simple random functions $c^{(n)}(s,\omega)$ converging to $c(s,\omega)$ in the space $\mathcal{M}$, i.e.

$$\int\limits_0^t \|c^{(n)}(s) - c(s)\|^2\, d|Q| \to 0 \ \text{ in probability.} \tag{8}$$

Put $X_t^{(n)} = \int\limits_0^t a^{(n)}(s)\, ds + \int\limits_0^t b^{(n)}(s)\, dQ_s + \int\limits_0^t c^{(n)}(s)\, dZ_s$.

Clearly, X_t is continuous. From (8) and Theorem 3.6 we get that $X_s^{(n)}$, $0 \leqslant s \leqslant t$ uniformly converges to X_s in probability, i.e

$$\sup_{0\leqslant s\leqslant t} \|X_s^{(n)} - X_s\| \to 0 \quad \text{in probability.} \tag{9}$$

Because (4) holds for simple functions a, b, c we get

$$\begin{aligned} g(t, X_t^{(n)}) = &\int\limits_0^t \Big[\, \frac{\partial g}{\partial t}(s, X_s^{(n)})\, ds + \frac{\partial g}{\partial x}(s, X_s) a^{(n)}(s) \Big]\, ds \\ &+ \int\limits_0^t \Big[\, \frac{\partial g}{\partial x}(s, X_s^{(n)}) \circ b^{(n)} + \frac{1}{2}\, \frac{\partial^2 g}{\partial x^2}(s, X_s^{(n)}) \circ (c^{(n)})^2(s) \Big]\, dQ \\ &+ \int\limits_0^t \frac{\partial g}{\partial x}(s, X_s) \circ c^{(n)}(s)\, dZ_s. \end{aligned} \tag{10}$$

The boundedness of $\dfrac{\partial g}{\partial x}$ together with (9) imply that the left-hand side of (10) converges to the left-hand side of (4) in probability.

From (8) and (9), we can choose a sequence $n_k \to \infty$ such that

$$\int\limits_0^t \|c^{(n_k)}(s) - c(s)\|^2\, d|Q|(s) \to 0 \quad \text{a.s.}$$

$$\sup_{0\leqslant s\leqslant t} \|X_s^{(n_k)} - X_s\| \to 0 \quad \text{a.s.}$$

Moreover,

$$\int_0^t \|a^{(n)}(s) - a(s)\|\, ds \to 0 \quad \text{a.s.}$$

$$\int_0^t \|b^{(n)}(s,\omega) - b(s,\omega)\|\, d|Q| \to 0 \quad \text{a.s.}$$

Consequently, the first and second integral on the right-hand side of (10) converges a.s., so converges, in probability to the first and second integral on the right-hand side of (4), respectively (when n_k tends to ∞). Now we shall show that the third integral on the right-hand side of (10) also converges in probability to the third integral on the right-hand side of (4). Indeed,

$$\begin{aligned}
&\int_0^T \Big\| \frac{\partial g}{\partial x}(s, X_s^{(n)}) c^{(n)}(s) - \frac{\partial g}{\partial x}(s, X_s) c(s) \Big\|^2 d|Q|(s) \\
&\leqslant \int_0^T \Big\| \frac{\partial g}{\partial x}(s, X_s^{(n)}) \Big\|^2 \|c^{(n)}(s) - c(s)\|^2\, d|Q|(s) \\
&\quad + \int_0^T \Big\| \frac{\partial g}{\partial x}(s, X_s) - \frac{\partial g}{\partial x}(s, X_s^{(n)}) \Big\|^2 \|c(s)\|^2\, d|Q|(s). \qquad (11)
\end{aligned}$$

From (8) together with the boundedness of $\dfrac{\partial g}{\partial x}$ it follows that the first integral on the right-hand side of (11) converges in probability to 0.

From (9) together with the boundedness of $\dfrac{\partial^2 g}{\partial x^2}$ it follows that $\Big\| \dfrac{\partial g}{\partial x}(s, X_s^{(n)}) - \dfrac{\partial g}{\partial x}(s, X_s) \Big\|^2$ converges uniformly on $[0,t]$ to 0 in probability. Moreover, $\int_0^T \|c(s)\|^2\, d|Q|(s) < \infty$ then the second integral on the right-hand side of (11) converges in probability to 0 and then the left-hand side of (11) converges to 0 in probability. Thus, by Theorem 3.6, the third integral on the right-hand side of (10) converges in probability to the third integral on the right-hand side of (4). Both sides of (10) converge to both sides of (4) respectively, so (4) holds in case g, $\dfrac{\partial g}{\partial t}$, $\dfrac{\partial g}{\partial x}$, $\dfrac{\partial^2 g}{\partial x^2}$ are bounded.

Step 2. g is an arbitrary function satisfying the conditions of the theorem. For each N, we choose the function $g_N(t,x)$ such that g_N is identical with g on $\|x\| \leqslant N$, $0 \leqslant t \leqslant T$ and g_N, $\dfrac{\partial g_N}{\partial t}$, $\dfrac{\partial g_N}{\partial x}$, $\dfrac{\partial^2 g_N}{\partial x^2}$ are bounded. From the proof in the step 1, the equation (4) holds for g_N.

Put $A_N = \{\omega : \sup_{0\leqslant s\leqslant t} \|X_s(\omega)\| \leqslant N\}$. Note that on A_N, the functions g_N, $\frac{\partial g_N}{\partial t}$, $\frac{\partial g_N}{\partial x}$, $\frac{\partial^2 g_N}{\partial x^2}(s, X_s)$ are identical with g, $\frac{\partial g}{\partial t}$, $\frac{\partial g}{\partial x}$, $\frac{\partial^2 g}{\partial x^2}(s, X_s)$ respectively. Hence (4) holds for almost all $\omega \in A_N$.

On the other hand, $\mathbb{P}\{\cup_{N=1}^{\infty} A_N\} = 1$ then (4) holds for almost all $\omega \in \Omega$.

That completes the proof of the Ito formula. ∎

Let us now specialize Theorem 5.2 to the case when the symmetric Gaussian random measure Z is the X-valued Wiener random measure W with the parameter (λ, R) (λ is the Lebesgue measure). In this case $dQ = Rdt$, and

$$dX_t = adt + bRdt + cdW_t = (a + bR)dt + cdW_t,$$

so the Ito process X_t with respect to the X-valued Wiener random measure W is of the form

$$dX_t = adt + bdW_t,$$

where $a(s, \omega)$ is an Y-valued adapted random function and $b(s, \omega)$ is an $L(X, Y)$-valued adapted random function with respect to W on $[0, T]$. From Theorem 5.2 we get

Theorem 5.3. *Assume that X, Y, E are separable Banach spaces of type 2, X is reflexive, W is the X-valued Wiener random measure with parameter (λ, R) and X_t is an Y-valued Ito process*

$$dX_t = adt + bdW_t.$$

Let $g : [0, \infty) \times Y \to E$ be a function which is continuously differentiable in the first variable and continuously twice differentiable in the second variable (strongly differentiable). Put $Y_t := g(t, X_t)$. Then Y_t is again an E-valued Ito process and

$$dY_t = \Big[\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x}a + \frac{1}{2}\big(\frac{\partial^2 g}{\partial x^2} \circ b^2\big) \cdot R\Big]\, dt + \Big(\frac{\partial g}{\partial x} \circ b\Big) dW_t.$$

Now we go on to specialize Theorem 5.3 to the case X, Y, E are finite dimensional spaces.

Suppose $X = \mathbb{R}^n$, $Y = \mathbb{R}^d$, $E = \mathbb{R}^k$, $R = (r_{i,j})$ is a non-negatively definite $n \times n$ matrix. W is an X-valued Wiener random measure with the parameters (λ, R).

Let X_t be a d-dimensional Ito process given by

$$dX_t = adt + bdW_t,$$

where $a = (a_i(t))$ is a d-dimensional random function, $b = (b_{i,j}(t))$ is a $d \times n$ random matrix satisfying

$$\mathbb{P}\{\omega : \int_0^T |a_i(t,\omega)|\, dt < \infty\} = 1,$$

$$\mathbb{P}\{\omega : \int_0^T |b_{i,j}(t,\omega)|^2\, dt < \infty\} = 1.$$

Then we have

Theorem 5.4. (The multi-dimensional Ito formula) *Suppose that $g(t,x) : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}^k$ is a function satisfying the conditions of Theorem* 5.2 *and put $Y_t = g(t, X_t)$. Then Y_t is an Ito process and we have*

$$dY_t = \Big[\,\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} \times a + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d}\frac{\partial^2 g}{\partial x_i \partial x_j}(b \times R' \times b')_{i,j}\,\Big]\, dt + \frac{\partial g}{\partial x} \times b \times dW_t. \quad (12)$$

Proof. R is an operator in $N(X', X) = X \otimes X$, whose action is given by

$$R : (\mathbb{R}^n)' \longrightarrow \mathbb{R}^n$$
$$x = (x_1, x_2, \cdots, x_n) \mapsto R\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{k=1}^{n} \langle r_k, x\rangle e_k,$$

where $r_k = (r_{k,i})_{i=1,n} \in \mathbb{R}^n$ (the k-th row vector of matrix R). Thus, $R = \sum_{k=1}^{n} r_k \otimes e_k$ and so $fR = \sum_{k=1}^{n} f(r_k, e_k)$, for any $f \in B(\mathbb{R}^n, \mathbb{R}^n; E)$. We have

$$\Big(\frac{\partial^2 g}{\partial x^2} \circ b^2\Big) R = \sum_{k=1}^{n} \frac{\partial^2 g}{\partial x^2} \circ b^2 (r_k, e_k) = \sum_{k=1}^{n} \frac{\partial^2 g}{\partial x^2}(br_k, be_k)$$
$$= \sum_{k=1}^{n}\sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\partial^2 g}{\partial x_i \partial x_j}\big((br_k)_i, (be_k)_j\big) \ \ ((br_k)_i \text{ is } i\text{-th element of vector } br_k)$$
$$= \sum_{i=1}^{d}\sum_{j=1}^{d}\sum_{k=1}^{n} \frac{\partial^2 g}{\partial x_i \partial x_j}(b_i r_k, b_{j,k}) \ \ (b_i \text{ is } i\text{-th row vector of matrix } b)$$
$$= \sum_{i=1}^{d}\sum_{j=1}^{d} \frac{\partial^2 g}{\partial x_i \partial x_j}(b \times R' \times b')_{i,j},$$
$$\frac{\partial g}{\partial x} a = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} a_i = \frac{\partial g}{\partial x} \times a,$$
$$\Big(\frac{\partial g}{\partial x} \circ b\Big) x = \frac{\partial g}{\partial x} \times b \times x, \quad \forall x \in \mathbb{R}^n,$$

where $(\times)$ denotes the product of matrices.

Hence, in this case the Ito formula has the form

$$dY_t = \Big[\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} \times a + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2 g}{\partial x_i \partial x_j} (b \times R' \times b')_{i,j} \Big] dt + \frac{\partial g}{\partial x} \times b \times dW_t.$$

∎

In particular, if W is the n-dimensional Wiener process with independent components then the matrix R is the unit matrix. In this case, the multi-dimensional Ito formula (12) becomes

$$dY_t = \Big[\frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} \times a + \frac{1}{2} \sum_{i=1}^{d} \sum_{j=1}^{d} \frac{\partial^2 g}{\partial x_i \partial x_j} (b \times b')_{i,j} \Big] dt + \frac{\partial g}{\partial x} \times b \times dW_t.$$

This is the well-known Ito formula.

References

1. K. Bichteler, Stochastic integration and L^p-theory of semimartingales, *Ann. Prob.* **9** (1981) 49–89.
2. J. Diestel and J. J. Uhl, *Vector measures*, American Mathematical Society, 1977.
3. E. Gine and M. B.Marcus, The central limit theorem for stochastic integrals with respect to Levy process, *Ann. Prob.* **11** (1983) 54–77.
4. N. Ikeda and S. Watanabe, *Stochastic Diffirential Equation and Diffusion Process*, North Holland, 1981.
5. H. Kunita, Stochastic integral bases on Martingales taking values in Hilbert spaces, *Nogoya Math.* **38** (1970) 41–52.
6. K. Ito, *Lectures on Stochastic Processes*, Tate Institute, Bombay, 1961.
7. K. Ito, Stochastic integrals, *Proc. Imp. Acad. Tokyo* **20** (1944) 519–524.
8. M. Ledoux and M. Talagrand, *Probability in Banach spaces*, Springer-Verlag, 1991.
9. W. Linde, *Infinitely Divisible and Stable Measures on Banach Spaces*, Teubner-Texte zur Mathematik, Bd. **58**. Leipzig, 1983.
10. D. H. Thang, Vector symmetric random measures and random integrals, *Theor. Proba. Appl.* **37** (1992) 526–533.
11. D. H. Thang, On Ito stochastic integral with respect to vector stable random measures, *Acta Math. Vietnam.* **21** (1996) 171–181.
12. D. H. Thang, Vector random stable measures and random integrals, *Acta Math Vietnam.* **26** (2001) 205–218.
13. D. H. Thang, Random mapping on infinite dimensional spaces, *Stochastics and stochastics Report* **88** (1997) 51–73.
14. D. H. Thang, From random series to random integral and random mapping, *Vietnam J. Math.* **30** (2002) 305–327.