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On the Smoothness of Solutions of the First Initial Boundary Value Problem for Schrödinger Systems in Domains with Conical Points

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Abstract. Some results on the smoothness of generalized solutions of the first initial boundary value problem for strongly Schrödinger systems in domains with conical points on boundary are given.

1. Introduction

Boundary value problems for Schrödinger equations and Schrödinger systems in a finite cylinder $\Omega_T = \Omega \times (0,T)$ have been studied by many authors [4,8,9]. The unique solvability of the first initial boundary value problem for strongly Schrödinger systems in an infinite cylinder $\Omega_\infty = \Omega \times (0,\infty)$ was given in [5]. The aim of this paper is to establish some theorems on the smoothness of generalized solutions of the problem in domains with conical points on boundary.

Let Ω be a bounded domain in $\mathbb{R}^n$. Its boundary $\partial\Omega$ is assumed to be an infinitely differentiable surface everywhere, except for the coordinate origin, in a neighborhood of which Ω coincides with the cone $K = \{x : x/|x| \in G\}$, where G is a smooth domain on the unit sphere S^{n-1}. We introduce some notations: $\Omega_T = \Omega \times (0,T),\ S_T = \partial\Omega \times (0,T),\ \Omega_\infty = \Omega \times (0,\infty),\ S_\infty = \partial\Omega \times (0,\infty), x = (x_1,\dots,x_n) \in \Omega,\ u(x,t) = (u_1(x,t),\dots,u_s(x,t))$ is a vector complex function, $|D^\alpha u|^2 = \sum_{i=1}^{s} |D^\alpha u_i|^2,\ \ u_{t^j} = \big(\partial^j u_1/\partial t^j,\dots,\partial^j u_s/\partial t^j\big),\ |u_{t^j}|^2 = \sum_{i=1}^{s} \big|\partial^j u_i/\partial t^j\big|^2$, $dx = dx_1\dots dx_n,\ \ r = |x| = \sqrt{x_1^2 + \dots + x_n^2}$.

In this paper we use frequently the following functional spaces:

• $H^l_\beta(\Omega)$ - the space of all functions $u(x) = (u_1(x), \dots, u_s(x))$ which have generalized derivatives $D^\alpha u_i$, $|\alpha| \le l$, $1 \le i \le s$, satisfying

$$\|u\|^2_{H^l_\beta(\Omega)} = \sum_{|\alpha|=0}^{l} \int_\Omega r^{2(\beta+|\alpha|-l)} |D^\alpha u|^2 dx < +\infty.$$

• $H^{l,k}(e^{-\gamma t}, \Omega_\infty)$ - the space of all functions $u(x,t)$ which have generalized derivatives $D^\alpha u_i$, $\dfrac{\partial^j u_i}{\partial t^j}$, $|\alpha| \le l$, $1 \le j \le k$, $1 \le i \le s$, satisfying

$$\|u\|^2_{H^{l,k}(e^{-\gamma t},\Omega_\infty)} = \int_{\Omega_\infty} \Big(\sum_{|\alpha|=0}^{l} |D^\alpha u|^2 + \sum_{j=1}^{k} |u_{t^j}|^2 \Big) e^{-2\gamma t} dx dt < +\infty.$$

In particular

$$\|u\|^2_{H^{l,0}(e^{-\gamma t},\Omega_\infty)} = \sum_{|\alpha|=0}^{l} \int_{\Omega_\infty} |D^\alpha u|^2 e^{-2\gamma t} dx dt.$$

• $\overset{\circ}{H}{}^{l,k}(e^{-\gamma t}, \Omega_\infty)$ - the closure in $H^{l,k}(e^{-\gamma t}, \Omega_\infty)$ of the set of all infinitely differentiable in Ω_∞ functions which belong to $H^{l,k}(e^{-\gamma t}, \Omega_\infty)$ and vanish near S_∞.

• $H^{l,k}_\beta(e^{-\gamma t}, \Omega_\infty)$ - the space of all functions $u(x,t)$ which have generalized derivatives $D^\alpha u_i$, $\dfrac{\partial^j u_i}{\partial t^j}$, $|\alpha| \le l$, $1 \le j \le k$, $1 \le i \le s$, satisfying

$$\|u\|^2_{H^{l,k}_\beta(e^{-\gamma t},\Omega_\infty)} = \int_{\Omega_\infty} \Big(\sum_{|\alpha|=0}^{l} r^{2(\beta+|\alpha|-l)} |D^\alpha u|^2 + \sum_{j=1}^{k} |u_{t^j}|^2 \Big) e^{-2\gamma t} dx dt < +\infty.$$

• $H^l_\beta(e^{-\gamma t}, \Omega_\infty)$ - the space of all functions $u(x,t)$ which have generalized derivatives $D^\alpha (u_i)_{t^j}$, $|\alpha| + j \le l$, $1 \le i \le s$, satisfying

$$\|u\|^2_{H^l_\beta(e^{-\gamma t},\Omega_\infty)} = \sum_{|\alpha|+j=0}^{l} \int_{\Omega_\infty} r^{2(\beta+|\alpha|+j-l)} |D^\alpha u_{t^j}|^2 e^{-2\gamma t} dx dt < +\infty.$$

• Let X be a Banach space. Denote by $L^\infty(0,\infty;X)$ the space of all measurable functions $u : (0,\infty) \longrightarrow X$, satisfying
$t \longmapsto u(t)$

$$\|u\|_{L^\infty(0,\infty;X)} = ess \sup_{t>0} \big\|u(t)\big\|_X < +\infty.$$

Consider the differential operator of order $2m$

$$L(x,t,D) = \sum_{|p|,|q|=0}^{m} D^p \big(a_{pq}(x,t) D^q \big),$$

where a_{pq} are $s \times s$-matrices of measurable, bounded in $\overline{\Omega_\infty}$, complex functions, $a_{pq} = (-1)^{|p|+|q|} a^*_{qp}$. Suppose that a_{pq} are continuous in $x \in \overline{\Omega}$ uniformly with respect to $t \in [0,\infty)$ if $|p| = |q| = m$, and that for each $t \in [0,\infty)$ the operator $L(x,t,D)$ is uniformly elliptic in $\overline{\Omega}$ with ellipticity constant a_0 independent of time t, i.e., we have

$$\sum_{|p|=|q|=m} a_{pq}(x,t)\xi^p\xi^q\eta\overline{\eta} \geq a_0|\xi|^{2m}|\eta|^2, \tag{1.1}$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$, $\eta \in \mathbb{C}^s \setminus \{0\}$ and $(x,t) \in \overline{\Omega_\infty}$.

Put

$$B(u,u)(t) = \sum_{|p|,|q|=0}^{m} (-1)^{|p|} \int_{\Omega} a_{pq} D^q u \overline{D^p u} dx, \quad u(x,t) \in \overset{\circ}{H}{}^{m,0}(e^{-\gamma t}, \Omega_\infty).$$

For a.e. $t \in [0,\infty)$, the function $x \mapsto u(x,t)$ belongs to $\overset{\circ}{H}{}^m(\Omega)$. On the other hand, since the principal coefficients a_{pq} are continuous in $x \in \overline{\Omega}$ uniformly with respect to $t \in [0,\infty)$ and the constant a_0 in (1.1) is independent of t, by repeating the proof of Garding's inequality [2, p.44], we have

Lemma 1.1. *There exist two constants μ_0 and λ_0 $(\mu_0 > 0, \lambda_0 \geq 0)$ such that*

$$(-1)^m B(u,u)(t) \geq \mu_0 \big\|u(x,t)\big\|^2_{H^m(\Omega)} - \lambda_0 \big\|u(x,t)\big\|^2_{L_2(\Omega)} \tag{1.2}$$

for all $u(x,t) \in \overset{\circ}{H}{}^{m,0}(e^{-\gamma t}, \Omega_\infty)$.

Therefore, using the transformation $u = e^{i\lambda_0 t}v$ if necessary, we can assume that the operator $L(x,t,D)$ satisfies

$$(-1)^m B(u,u)(t) \geq \mu_0 \|u\|^2_{H^m(\Omega)} \tag{1.3}$$

for all $u(x,t) \in \overset{\circ}{H}{}^{m,0}(e^{-\gamma t}, \Omega_\infty)$. This inequality is a basic tool for proving the existence and uniqueness of solutions of a boundary value problem.

2. Main Results

In this paper we consider the following problem: Find a function $u(x,t)$ such that

$$(-1)^{m-1} iL(x,t,D)u - u_t = f(x,t) \ \text{ in } \ \Omega_\infty, \tag{2.1}$$

$$u|_{t=0} = 0, \tag{2.2}$$

$$\frac{\partial^j u}{\partial \nu^j}\Big|_{S_\infty} = 0, \quad j = 0, \dots, m-1, \tag{2.3}$$

where ν is the outer unit normal to S_∞.

A function $u(x,t)$ is called a generalized solution of the problem (2.1) - (2.3) in the space $\overset{\circ}{H}{}^{m,0}(e^{-\gamma t}, \Omega_\infty)$ if and only if $u(x,t)$ belongs to $\overset{\circ}{H}{}^{m,0}(e^{-\gamma t}, \Omega_\infty)$ and for each $T > 0$ the following equality holds

$$(-1)^{m-1}i\sum_{|p|,|q|=0}^{m}(-1)^{|p|}\int_{\Omega_T}a_{pq}D^q u\overline{D^p\eta}dxdt+\int_{\Omega_T}u\overline{\eta}_t dxdt=\int_{\Omega_T}f\overline{\eta}dxdt \tag{2.4}$$

for all test function $\eta \in \overset{\circ}{H}{}^{m,1}(\Omega_T)$ satisfying $\eta(x,T)=0$.

Denote by m^* the number of multi-indices which have order not exceeding m, μ_0 is the constant in (1.3). From Theorems 3.1, 3.2 in [5] and by using induction we obtain the following result.

Theorem 2.1. *Let*

i) $\sup\left\{\left|\frac{\partial a_{pq}}{\partial t}\right| : (x,t)\in\overline{\Omega}_\infty,\quad 0\le |p|,|q|\le m\right\}=\mu<+\infty;$

$\left|\dfrac{\partial^k a_{pq}}{\partial t^k}\right|\le \mu_1,\quad \mu_1=\text{const}>0,\quad \text{for}\quad 2\le k\le h+1;$

ii) $f_{t^k}\in L^\infty(0,\infty;L_2(\Omega))$, *for* $k\le h+1$;

iii) $f_{t^k}(x,0)=0$, *for* $k\le h$.

Then for every $\gamma>\gamma_0=\dfrac{m^*\mu}{2\mu_0}$, *the problem* (2.1) - (2.3) *has exactly one generalized solution* $u(x,t)$ *in the space* $\overset{\circ}{H}{}^{m,0}(e^{-\gamma t},\Omega_\infty)$. *Moreover,* $u(x,t)$ *has derivatives with respect to* t *up to order* h *belonging to* $\overset{\circ}{H}{}^{m,0}(e^{-(2h+1)\gamma t},\Omega_\infty)$ *and the following estimate holds*

$$\|u_{t^h}\|^2_{H^{m,0}(e^{-(2h+1)\gamma t},\Omega_\infty)}\le C\sum_{k=0}^{h+1}\|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(\Omega))},$$

where C *is a positive constant independent of* u *and* f.

From now on for the sake of brevity, we will write γ_h instead of $(2h+1)\gamma$ $(h=1,2,...,)$.

In order to study the smoothness with respect to (x,t) of generalized solutions of the problem (2.1) - (2.3), we assume that coefficients $a_{pq}(x,t)$ of the operator $L(x,t,D)$ are infinitely differentiable in $\overline{\Omega_\infty}$. In addition, we also assume that a_{pq} and its all derivatives are bounded in $\overline{\Omega_\infty}$.

First, we prove the following lemma.

Lemma 2.1. *Let* $f,f_t,f_{tt}\in L^\infty(0,\infty;L_2(K))$ *and* $f(x,0)=f_t(x,0)=0$. *If* $u(x,t)\in \overset{\circ}{H}{}^{m,0}(e^{-\gamma t},\Omega_\infty)$ *is a generalized solution of the problem* (2.1) - (2.3) *in the space* $\overset{\circ}{H}{}^{m,0}(e^{-\gamma t},\Omega_\infty)$ *such that* $u\equiv 0$ *whenever* $|x|>R$, $R=\text{const}$, *then* $u\in H^{2m,1}_m(e^{-\gamma_1 t},K_\infty)$ *and the following estimate holds*

$$\|u\|^2_{H^{2m,1}_m(e^{-\gamma_1 t},K_\infty)}\le C\Big[\|f\|^2_{L^\infty(0,\infty;L_2(K))}+\|f_t\|^2_{L^\infty(0,\infty;L_2(K))}+\|f_{tt}\|^2_{L^\infty(0,\infty;L_2(K))}\Big],$$

where $C=\text{const}$.

Proof. Rewrite the system (2.1) in the following form

$$(-1)^m \sum_{|p|,|q|=0}^{m} D^p\big(a_{pq}(x,t)D^q u\big) = F, \tag{2.5}$$

where $F = i(u_t + f)$. From Theorem 2.1 it follows that $F \in L_2(K)$ for a.e. $t \in [0,\infty)$.

Consider the sequence of domains

$$\Omega^k = \big\{x \in K : 2^{-k} \le |x| \le 2^{-k+1}\big\}, \quad k = 1, 2, ...$$

Choosing a smooth domain $\Omega^{2,0}$ such that $\Omega^2 \subset \Omega^{2,0} \subset \big(\Omega^1 \cup \Omega^2 \cup \Omega^3\big)$. By the theorem on the smoothness of solutions of elliptic problems in a smooth domain [3, Th. 17.2, p. 67], we obtain

$$\int\limits_{\Omega^{2,0}} |D^\alpha u(x,t)|^2 dx \le C \int\limits_{\Omega^{2,0}} \big[\big|F(x,t)\big|^2 + \big|u(x,t)\big|^2\big] dx, |\alpha| \le 2m, C = \text{const}.$$

Hence

$$\int\limits_{\Omega^2} |D^\alpha u(x,t)|^2 dx \le C \int\limits_{\Omega^1\cup\Omega^2\cup\Omega^3} \big[\big|F(x,t)\big|^2 + \big|u(x,t)\big|^2\big] dx, |\alpha| \le 2m, C = \text{const}. \tag{2.6}$$

By substituting $x = \frac{4}{2^{k_1}}x'(k_1 > 2)$ in (2.5) and applying the estimate (2.6), we have

$$\int\limits_{\Omega^2} |D^\alpha_{x'} u(x',t)|^2 dx \le C_1 \int\limits_{\Omega^1\cup\Omega^2\cup\Omega^3} \Big[\big|F(x',t)\big|^2 \Big(\frac{4}{2^{k_1}}\Big)^{4m} + \big|u(x',t)\big|^2\Big] dx', C_1 = \text{const}.$$

Returning to variables $x_1, \dots, x_n$, we obtain

$$\int\limits_{\Omega^{k_1}} |D^\alpha u(x,t)|^2 r^{2(|\alpha|-m)} dx$$
$$\le C_2 \int\limits_{\Omega^{k_1-1}\cup\Omega^{k_1}\cup\Omega^{k_1+1}} \big[\big|F(x,t)\big|^2 r^{2m} + r^{-2m}\big|u(x,t)\big|^2\big] dx, C_2 = \text{const}.$$

Summing these inequalities for all $k_1 > 2$ we obtain

$$\int\limits_{\sum_{k>2}\Omega^k} \big|D^\alpha u(x,t)\big|^2 r^{2(|\alpha|-m)} dx$$
$$\le C_3 \int\limits_{\sum_{k>1}\Omega^k} \big[\big|F(x,t)\big|^2 r^{2m} + r^{-2m}\big|u(x,t)\big|^2\big] dx, \quad C_3 = \text{const}. \tag{2.7}$$

Since the solution is equal to 0 outside a neighborhood of the conical point, from (2.7) we have

$$\int_K \left|D^\alpha u(x,t)\right|^2 r^{2(|\alpha|-m)}dx \le C_4 \int_K \left[\left|F(x,t)\right|^2 r^{2m} + r^{-2m}\left|u(x,t)\right|^2\right]dx, \ \ C_4 = \text{const}. \tag{2.8}$$

From conditions $\frac{\partial^j u}{\partial \nu^j}\Big|_{S_\infty} = 0, \quad j = 0, \dots, m-1$, we have

$$\int_K r^{-2m}\left|u(x,t)\right|^2 dx \le C_5 \sum_{|\beta|=m} \int_K \left|D^\beta u\right|^2 dx, \ \ C_5 = \text{const}.$$

Hence

$$\int_K r^{2(|\alpha|-m)}\left|D^\alpha u(x,t)\right|^2 dx$$
$$\le C_6 \int_K \left[|f|^2 + |u_t|^2 + \sum_{|\beta|=m} |D^\beta u|^2\right]dx, \quad C_6 = \text{const},$$

Integrating this inequality with respect to t from 0 to ∞ after multiplying its both sides by $e^{-6\gamma t}$ and applying Theorem 2.1, we have the statement. Lemma 2.1 is proved. ■

Now let ω be a local coordinate system on S^{n-1}. The principal part of the operator $L(x,t,D)$ at origin 0 can be written in the form

$$L_0(0,t,D) = r^{-2m}Q(\omega, t, rD_r, D_\omega), \ \ D_r = \frac{i\partial}{\partial r},$$

where Q is a linear operator with smooth coefficients. From now on the following spectral problem will play an important role

$$Q(\omega, t, \lambda, D_\omega)v(\omega) = 0, \quad \omega \in G, \tag{2.9}$$
$$D^j_\omega v(\omega) = 0, \quad \omega \in \partial G, \quad j = 0, \dots, m-1. \tag{2.10}$$

It is well known [1, Th.7, p.39] that for every $t \in [0,\infty)$ its spectrum is discrete.

Propostion 2.1. *Let $u(x,t)$ be a generalized solution of the problem* (2.1) - (2.3) *in the space $\overset{\circ}{H}{}^{m,0}(e^{-\gamma t}, \Omega_\infty)$ such that $u \equiv 0$ whenever $|x| > R$, $R =$ const, and let $f_{t^k} \in L^\infty(0,\infty; L_2(K))$ for $k \le 2m+1$, $f_{t^k}(x,0) = 0$ for $k \le 2m$. In addition suppose that the strip*

$$m - \frac{n}{2} \le \ \ \operatorname{Im}\lambda \le 2m - \frac{n}{2}$$

does not contain points of spectrum of the problem (2.9) - (2.10) *for every $t \in [0,\infty)$. Then $u(x,t) \in H^{2m}_0(e^{-\gamma_{2m}t}, K_\infty)$ and the following estimate holds*

$$\left\|u\right\|^2_{H^{2m}_0(e^{-\gamma_{2m}t}, K_\infty)} \le C \sum_{k=0}^{2m+1} \left\|f_{t^k}\right\|^2_{L^\infty(0,\infty;L_2(K))},$$

where $C = \text{const} > 0$.

Proof. First, we prove that

$$\|u_{t^s}\|^2_{H_0^{2m,0}(e^{-\gamma_{s+1}t},K_\infty)} \le C \sum_{k=0}^{2m+1} \|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(K))}, \tag{2.11}$$

where $C = \text{const}$, $s \le 2m-1$.

Rewrite the system (2.1) in the form

$$(-1)^m L_0(0,t,D)u = F(x,t),$$

where $F(x,t) = i(u_t + f) + (-1)^m\big[L_0(0,t,D) - L(x,t,D)\big]u$. Since $a_{pq}(x,t)$ are infinitely differentiable in $\overline{\Omega_\infty}$ and $u(x,t)$ has generalized derivatives with respect to x up to order $2m$ (see Lemma 2.1), we have

$$L(x,t,D)u = \sum_{|\alpha|=0}^{2m} a_\alpha(x,t) D^\alpha u.$$

Since $u \in H_m^{2m,0}(e^{-\gamma_1 t}, K_\infty)$ and $|a_\alpha(x,t) - a_\alpha(0,t)| \le C|x|$, $C = \text{const}$, one can see that

$$\big[L_0(0,t,D) - L(x,t,D)\big]u \in H^{0,0}_{m-1}(e^{-\gamma_1 t}, K_\infty).$$

(To verify this statement it suffices to consider the case $r = |x| \le 1$). It follows from Theorem 2.1 and Lemma 2.1 that $F(x,t) \in H^{0,0}_{m-1}(e^{-\gamma_1 t}, K_\infty)$. Therefore $F \in H^0_{m-1}(K)$ for a.e. $t \in [0,\infty)$. On the other hand, in the strip $m - \frac{n}{2} \le \operatorname{Im}\lambda \le m + 1 - \frac{n}{2}$, there are no points of spectrum of the problem (2.9) - (2.10) for every $t \in [0,\infty)$. From results of elliptic problems [7, Th.6.4, p.139] it follows that $u \in H^{2m}_{m-1}(K)$ for a.e. $t \in [0,\infty)$ and

$$\|u\|^2_{H^{2m}_{m-1}(K)} \le C\big[\|f\|^2_{L_2(K)} + \|u_t\|^2_{L_2(K)} + \|u\|^2_{H^{2m}_m(K)}\big],$$

where C =const. Repeating the above argument, we obtain

$$\|u\|^2_{H^{2m}_0(K)} \le C\big[\|f\|^2_{L_2(K)} + \|u_t\|^2_{L_2(K)} + \|u\|^2_{H^{2m}_m(K)}\big].$$

Hence

$$\|u\|^2_{H_0^{2m,0}(e^{-\gamma_1 t},K_\infty)} \le C\big[\|f\|^2_{L^\infty(0,\infty;L_2(K))} + \|u_t e^{-\gamma_1 t}\|^2_{L_2(K_\infty)} + \|u\|^2_{H_m^{2m,0}(e^{-\gamma_1 t},K_\infty)}\big].$$

From Theorem 2.1 and Lemma 2.1 it follows that

$$\|u\|^2_{H_0^{2m,0}(e^{-\gamma_1 t},K_\infty)} \le C \sum_{k=0}^{2} \|f_{t^k}\|^2_{L^\infty(0,\infty;L_2(K))},$$

i.e., (2.11) is proved for $s = 0$.

Now assume that (2.11) is true for $s-1$. By differentiating the system (2.1) s times with respect to t and by putting $v = u_{t^s}$, we obtain

$$(-1)^{m-1}Lv = -i(v_t + f_{t^s}) + (-1)^m \sum_{k=1}^{s} \binom{s}{k} L_{t^k} u_{t^{s-k}},$$

where

$$L_{t^k} = \sum_{|p|,|q|=0}^{m} D^p\Big(\frac{\partial^k a_{pq}}{\partial t^k} D^q\Big).$$

By Theorem 2.1 the function $v = u_{t^s}$ still satisfies the boundary conditions. Therefore from inductive hypothesis and by repeating arguments of the proof in the case $s = 0$, we obtain (2.11).

Since

$$\|u\|^2_{H_0^{2m}(e^{-\gamma_{2m}t},K_\infty)} \le \sum_{s=0}^{2m-1} \|u_{t^s}\|^2_{H_0^{2m,0}(e^{-\gamma_{s+1}t},K_\infty)} + \|u_{t^{2m}}\|^2_{H_0^{0,0}(e^{-\gamma_{2m}t},K_\infty)},$$

from (2.11) and Theorem 2.1, the statement follows. Proposition 2.1 is proved. ∎

We consider now the following Dirichlet problem

$$\begin{cases} (-1)^m L_0(0,t,D)u = F(x,t), \quad x \in K, \\ \dfrac{\partial^j u}{\partial \nu^j}\Big|_{\partial K} = 0, \quad j = 0,\dots,m-1. \end{cases} \tag{2.12}$$

Lemma 2.2. *Let $u(x,t)$ be a generalized solution of the Dirichlet problem (2.12) for a.e $t \in [0,\infty)$ such that $u \equiv 0$ whenever $|x| > R$, R =const, and $u(x,t) \in H^{2m+l-1,0}_{\beta-1}(e^{-\gamma t},K_\infty)$. Let $F \in H^{l,0}_\beta(e^{-\gamma t},K_\infty)$. Then $u(x,t) \in H^{2m+l,0}_\beta(e^{-\gamma t},K_\infty)$ and*

$$\|u\|^2_{H^{2m+l,0}_\beta(e^{-\gamma t},K_\infty)} \le C\big[\|F\|^2_{H^{l,0}_\beta(e^{-\gamma t},K_\infty)} + \|u\|^2_{H^{2m+l-1,0}_{\beta-1}(e^{-\gamma t},K_\infty)}\big],$$

where $C =$ const.

Proof. By repeating the proof of the inequality (2.6), we have

$$\int_{\Omega^2} |D^\mu u(x,t)|^2 dx \le C \int_{\Omega^1\cup\Omega^2\cup\Omega^3} \Big[\sum_{|\alpha|\le l} |D^\alpha F(x,t)|^2 + |u(x,t)|^2\Big] dx, \quad |\mu| = 2m+l,$$

where $\Omega^1, \Omega^2, \Omega^3$ are defined as in Lemma 2.1, $C =$ const. From this inequality and by arguments which are analogous to the proof of the inequality (2.7), we obtain

$$\int\limits_{K} r^{2\beta}\big|D^{\mu}u(x,t)\big|^{2}dx \le C\int\limits_{K}\Big[\sum_{|\alpha|\le l} r^{2(\beta+|\alpha|-l)}|D^{\alpha}F(x,t)|^{2} + r^{2(\beta-2m-l)}|u(x,t)|^{2}\Big]dx.$$

Integrating this inequality with respect to t from 0 to ∞ after multiplying its both sides by $e^{-2\gamma t}$, we have

$$\int\limits_{K_\infty} r^{2\beta}\big|D^{\alpha}u(x,t)\big|^{2}e^{-2\gamma t}dxdt$$
$$\le C\big[\|F\|^2_{H^{l,0}_{\beta}(e^{-\gamma t},K_\infty)} + \int\limits_{K_\infty} r^{2(\beta-2m-l)}|u(x,t)|^{2}e^{-2\gamma t}dxdt\big]$$
$$\le C\big[\|F\|^2_{H^{l,0}_{\beta}(e^{-\gamma t},K_\infty)} + \|u\|^2_{H^{2m+l-1,0}_{\beta-1}(e^{-\gamma t},K_\infty)}\big]. \tag{2.13}$$

We have

$$\|u\|^2_{H^{2m+l,0}_{\beta}(e^{-\gamma t},K_\infty)} = \sum_{|\mu|=2m+l}\int\limits_{K_\infty} r^{2\beta}|D^{\mu}u(x,t)|^{2}e^{-2\gamma t}dxdt + \|u\|^2_{H^{2m+l-1,0}_{\beta-1}(e^{-\gamma t},K_\infty)}.$$

Hence and from (2.13) the statement follows. Lemma 2.2 is proved. ■

Proposition 2.2. *Let $f_{t^k} \in L^{\infty}(0,\infty;H^{l}_{0}(K))$ for $k \le 2m+l+1$, $f_{t^k}(x,0)=0$ for $k \le 2m+l$ and let $u(x,t)$ be a generalized solution of the problem* (2.1) - (2.3) *in the space $\overset{\circ}{H}{}^{m,0}(e^{-\gamma t},\Omega_\infty)$ such that $u \equiv 0$ whenever $|x|>R$, $R = \text{const}$. In addition suppose, that the strip*

$$m-\frac{n}{2} \le Im\,\lambda \le 2m+l-\frac{n}{2}$$

does not contain points of spectrum of the problem (3.9) - (3.10) *for every $t \in [0,\infty)$. Then $u(x,t) \in H^{2m+l}_{0}(e^{-\gamma_{2m+l}t},K_\infty)$ and the following estimate holds*

$$\|u\|^2_{H^{2m+l}_{0}(e^{-\gamma_{2m+l}t},K_\infty)} \le C\sum_{k=0}^{2m+l+1}\|f_{t^k}\|^2_{L^{\infty}(0,\infty;H^{l}_{0}(K))},$$

where $C = \text{const}$.

Proof. We will use induction on l. If $l=0$, the statement follows from Proposition 2.1. Let the statement be true for $l-1$.

We prove the inequality

$$\|u\|^2_{H^{2m+l-s}_{0}(e^{-\gamma_{2m+l-s}t},K_\infty)} \le C\sum_{k=0}^{2m+l+1}\|f_{t^k}\|^2_{L^{\infty}(0,\infty;H^{l}_{0}(K))}, \tag{2.14}$$

for $s = l, l-1, \dots, 0$, where $C = \text{const}$.

Since $f_{t^k} \in L^\infty(0,\infty; H_0^l(K))$ for $k \le 2m+l+1$, $f_{t^k}(x,0) = 0$ for $k \le 2m+l$, so from Theorem 2.1 we obtain $u_{t^{l+1}} \in H_0^{m,0}(e^{-\gamma_{l+1}t}, K_\infty)$. From this and from arguments, which are analogous to the proof of Proposition 2.1, we obtain the inequality (2.14) for $s = l$.

Assume that (2.14) is true for $s = l, l-1, \dots, j+1$. Put $v = u_{t^j}$. From (2.11) it follows that

$$(-1)^{m-1} L v = F_j,$$

where

$$F_j = -i(v_t + f_{t^j}) + (-1)^m \sum_{k=1}^{j} \binom{j}{k} L_{t^k} u_{t^{j-k}}, \ \ L_{t^k} = \sum_{|p|,|q|=0}^{m} D^p \Big(\frac{\partial^k a_{pq}}{\partial t^k} D^q \Big).$$

By inductive hypothesis on l, we obtain

$$\sum_{k=1}^{j} \binom{j}{k} L_{t^k} u_{t^{j-k}} \in H_0^{l-j}(e^{-\gamma_{l-j}t}, K_\infty).$$

On the other hand, by inductive hypothesis on s we have $v_t \in H_0^{l-j}(e^{-\gamma_{l-j}t}, K_\infty)$. Therefore $F_j \in H_0^{l-j}(e^{-\gamma_{l-j}t}, K_\infty)$. Since

$$H_0^{l-j}\big(e^{-\gamma_{l-j}t}, K_\infty\big) \subset H_{-1}^{l-j-1,0}(e^{-\gamma t}, K_\infty)$$

so $F_j \in H_{-1}^{l-j-1,0}(e^{-\gamma t}, K_\infty)$.

By repeating arguments which are analogous to the proof of Proposition 2.1, we obtain $v \in H_{-1}^{2m+l-j-1,0}(e^{-\gamma t}, K_\infty)$. Hence and from Lemma 2.2 it follows that $u_{t^j} = v \in H_0^{2m+l-j,0}(e^{-\gamma t}, K_\infty)$ and

$$\|u_{t^j}\|^2_{H_0^{2m+l-j,0}(e^{-\gamma t}, K_\infty)} \le C \sum_{k=0}^{2m+l+1} \|f_{t^k}\|^2_{L^\infty(0,\infty; H_0^l(K))}, \tag{2.15}$$

where $C = \text{const}$.

We have

$$\|u_{t^j}\|^2_{H_0^{2m+l-j}(e^{-\gamma_{2m+l-j}t}, K_\infty)} \le \|u_{t^{j+1}}\|^2_{H_0^{2m+l-j-1}(e^{-\gamma_{2m+l-j-1}t}, K_\infty)} + \|u_{t^j}\|^2_{H_0^{2m+l-j,0}(e^{-\gamma t}, K_\infty)}. \tag{2.16}$$

By inductive hypothesis on s, from (2.14) we obtain

$$\|u_{t^{j+1}}\|^2_{H_0^{2m+l-j-1}(e^{-\gamma_{2m+l-j-1}t}, K_\infty)} \le C \sum_{k=0}^{2m+l+1} \|f_{t^k}\|^2_{L^\infty(0,\infty; H_0^l(K))}, \ C = \text{const}.$$

Hence from this and (2.15), (2.16) it follows that

$$\|u_{t^j}\|^2_{H_0^{2m+l-j}(e^{-\gamma_{2m+l-j}t}, K_\infty)} \le C \sum_{k=0}^{2m+l+1} \|f_{t^k}\|^2_{L^\infty(0,\infty; H_0^l(K))}, \quad C = \text{ const}.$$

For $j = 0$ we obtain the statement. Propostion 2.2 is proved. ∎

We can now state our theorem on the smoothness of generalized solutions of the problem (2.1) - (2.3) in the whole domain.

Theorem 2.2. *Let $u(x,t)$ be a generalized solution of the problem* (2.1) - (2.3) *in the space $\overset{\circ}{H}{}^{m,0}(e^{-\gamma t},\Omega_\infty)$ and let $f_{t^k} \in L^\infty(0,\infty; H_0^l(\Omega))$ for $k \le 2m+l+1$, $f_{t^k}(x,0) = 0$ for $k \le 2m+l$. In addition, supppose that the strip*

$$m - \frac{n}{2} \le Im\lambda \le 2m + l - \frac{n}{2}$$

does not contain points of spectrum of the problem (2.9) - (2.10) *for every $t \in [0,\infty)$. Then $u(x,t) \in H_0^{2m+l}(e^{-\gamma_{2m+l} t},\Omega_\infty)$ and the following estimate holds*

$$\|u\|^2_{H_0^{2m+l}(e^{-\gamma_{2m+l}t},\Omega_\infty)} \le C \sum_{k=0}^{2m+l+1} \|f_{t^k}\|^2_{L^\infty(0,\infty;H_0^l(\Omega))},$$

where $C =$ const.

Proof. Surrounding the point 0 by a neighborhood U_0 with small diameter so that the intersection of Ω and U_0 coincides with K. Consider a function $u_0 = \varphi_0 u$, where $\varphi_0 \in \overset{\circ}{C}{}^\infty(U_0)$ and $\varphi_0 \equiv 1$ in some neighborhood of 0. The function u_0 satisfies the system

$$(-1)^{m-1} i L(x,t,D) u_0 - (u_0)_t = \varphi_0 f + L'(x,t,D)u,$$

where $L'(x,t,D)$ is a linear differential operator having order less than $2m$. The coefficients of this operator depend on the choice of the function φ_0 and equal to 0 outside U_0. This and the arguments analogous to the proof of Proposition 2.2 show that

$$\|\varphi_0 u\|^2_{H_0^{2m+l}(e^{-\gamma_{2m+l}t},\Omega_\infty)} \le C \sum_{k=0}^{2m+l+1} \|f_{t^k}\|^2_{L^\infty(0,\infty;H_0^l(\Omega))}. \tag{2.17}$$

The function $\varphi_1 u = (1-\varphi_0)u$ equals 0 in some neighborhood of the conical point. We can apply theorems on the smoothness of solutions of elliptic problems in a smooth domain to this function and obtain $\varphi_1 u \in H_0^{2m+l}(\Omega)$ for a.e. $t \in [0,\infty)$. Hence by Theorem 2.1 we have $\varphi_1 u \in H_0^{2m+l}(e^{-\gamma_{2m+l}t},\Omega_\infty)$ and

$$\|\varphi_1 u\|^2_{H_0^{2m+l}(e^{-\gamma_{2m+l}t},\Omega_\infty)} \le C \sum_{k=0}^{2m+l+1} \|f_{t^k}\|^2_{L^\infty(0,\infty;H_0^l(\Omega))}. \tag{2.18}$$

Since $u = \varphi_0 u + \varphi_1 u$ so from (2.17) and (2.18) we obtain

$$\|u\|^2_{H_0^{2m+l}(e^{-\gamma_{2m+l}t},\Omega_\infty)} \le C \sum_{k=0}^{2m+l+1} \|f_{t^k}\|^2_{L^\infty(0,\infty;H_0^l(\Omega))}.$$

Theorem 2.2 is proved. ∎

Finally, we give an example.

Example. Consider the following problem

$$i\triangle u - u_t = f \text{ in } \Omega_\infty, \tag{2.19}$$

$$u|_{t=0} = 0, \tag{2.20}$$

$$u|_{S_\infty} = 0. \tag{2.21}$$

The Laplacian in polar coordinate (r, ω) in $\mathbb{R}^n$ is given by

$$(\triangle u)(r,\omega) = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\Big(r^{n-1}\frac{\partial}{\partial r}\Big)u(r,\omega) + \frac{1}{r^2}\triangle_\omega u(r,\omega),$$

where $\triangle_\omega$ is the Laplace - Beltrami operator on the unit sphere S^{n-1}. It follows that the spectral problem has the form

$$\triangle_\omega v + [(i\lambda)^2 + i(2-n)\lambda]v = 0, \omega \in G, \tag{2.22}$$

$$v|_{\partial G} = 0. \tag{2.23}$$

Let $u(x,t)$ be a generalized solution of the problem (2.19) - (2.21) in the space $\overset{\circ}{H}{}^{1,0}(e^{-\gamma t}, \Omega_\infty)$. We consider the following cases:

1. Case $n = 2$. Assume that in a neighborhood of the coordinate origin, $\partial\Omega$ coincides with a rectilinear angle having measure β. Then the problem (2.22) - (2.23) becomes

$$v_{\omega\omega} - \lambda^2 v = 0, 0 < \omega < \beta, v(0) = v(\beta) = 0. \tag{2.24}$$

Upon some computations we find that eigenvalues of the problem (2.24) are $\lambda_k = \pm\dfrac{ik\pi}{\beta}, k \in \mathbb{N}^*$. Therefore, if $\beta < \dfrac{\pi}{l+1}$ then the strip $0 \le \mathrm{Im}\lambda \le 1 + l$ does not contain eigenvalues of the problem (2.24). By Theorem 2.2, we obtain that $u(x,t) \in H_0^{2+l}(e^{-\gamma_{2+l}t}, \Omega_\infty)$ if $f_{t^k} \in L^\infty(0,\infty; H_0^l(\Omega))$ for $k \le 2 + l + 1$, $f_{t^k}(x,0) = 0$ for $k \le 2 + l$.

2. Case $n = 3$. It is known [6, p. 290] that if Ω is a convex domain, the strip $-\dfrac{1}{2} \le \mathrm{Im}\lambda < 1$ does not contain eigenvalues of the problem (2.22) - (2.23). Thus, if Ω is convex, from Theorem 2.2 we have $u(x,t) \in H_0^2(e^{-\gamma_2 t}, \Omega_\infty)$ if $f, f_t, f_{tt}, f_{ttt} \in L^\infty(0,\infty; L_2(\Omega))$, $f(x,0) = f_t(x,0) = f_{tt}(x,0) = 0$.

3. Case $n > 3$. In this case, the strip

$$1 - \frac{n}{2} \le \mathrm{Im}\lambda \le 2 - \frac{n}{2}$$

does not contain eigenvalues of the problem (2.22) - (2.23) (see [6; p.289]). By Theorem 2.2, we obtain that $u \in H_0^2(e^{-\gamma_2 t}, \Omega_\infty)$ if $f, f_t, f_{tt}, f_{ttt} \in L^\infty(0,\infty; L_2(\Omega))$, $f(x,0) = f_t(x,0) = f_{tt}(x,0) = 0$.

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