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Submanifolds with Parallel Mean Curvature Vector Fields and Equal Wirtinger Angles in Sasakian Space Forms*

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Abstract. We study closed submanifolds M of dimension $2n+1$, immersed into a $(4n+1)$-dimensional Sasakian space form (N, ξ, η, φ) with constant φ-sectional curvature c, such that the reeb vector field ξ is tangent to M. Under the assumption that M has equal Wirtinger angles and parallel mean curvature vector fields, we prove that for any positive integer n, M is either an invariant or an anti-invariant submanifold of N if $c > -3$, and the common Wirtinger angle must be constant if $c = -3$. Moreover, without assuming it being closed, we show that such a conclusion also holds for a slant submanifold M (Wirtinger angles are constant along M) in the first case, which is very different from cases in Kähler geometry.

1. Introduction and Main Theorem

Wirtinger angles in contact geometry are something like Kähler angles in complex geometry. Kähler angles of a manifold M immersed into a Kähler manifold N are just some functions that at each point $p \in M$, measure the deviation of the tangent space T_pM of M from a complex subspace of T_pN. This concept was first introduced by Chern and Wolfson [11] for real surfaces immersed into Kähler surfaces N, giving, in this case, a single Kähler angle. Submanifolds of constant Kähler angles (independent of vectors in T_pM and points on M) are

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called slant submanifolds, which was introduced by Chen [7] as a natural generalization of both holomorphic immersions and totally real immersions. Now Wirtinger angles of a Riemannian manifold M immersed into a (or an almost) contact metric manifold $(N, \xi, \eta, \varphi, g)$ are just the Kähler angles defined on the distribution $\Re$ orthogonal to ξ in the tangent bundle TM. The notion of slant submanifolds in contact geometry was introduced by Lotta [16] for submanifolds immersed into an almost contact manifold, which is also a natural generalization of both invariant (the slant angle $= 0$) and anti-invariant (the slant angle $= \pi/2$) submanifolds.

There has been an increasing development of differential geometry of Kähler angles (respectively Wirtinger angles) and slant submanifolds in complex (respectively contact) manifolds in recent years (see for instance [1, 3-5, 7-10, 14-18] and references therein). Examples are given in complex space forms by Chen and Tazawa [9], where they proved that minimal surfaces immersed into CP^2 and CH^2 must be either holomorphic or Lagrangian surfaces. By Hopf's fibration, they [8, 9] also gave the concrete examples of proper slant submanifolds immersed into complex space forms. In [14], the author also studied slant submanifolds satisfying some equalities. J. L. Cabrerizo, A. Carriazo, L. M. Fernández and M. Fernández studied slant and semi-slant submanifolds of a contact manifold [3, 4], and presented existence and uniqueness theorems for slant submanifolds into Sasakian space forms [5], which are similar to that of Chen and Vrancken in complex geometry [10].

Clearly there are obstructions to the existence of slant submanifolds. For instance, there does not exist totally geodesic proper slant submanifolds with slant angle θ $(0 \leq \cos\theta \leq 1)$ in non-trivial complex space forms by Codazzi equations. A natural question is to ask when the submanifold with equal Kähler angles is holomorphic or totally real. When the Kähler angles are not constant on the corresponding submanifold, by making use of the Weitzenböck formula for the Kähler form of N restricted to M, Wolfson [18] studied the minimal, closed, real surfaces immersed into a Kähler surface, and using the Bochner-type technique, Salavessa and Valli [17] studied the same question and generalized Wolfson's theorem [18] to higher dimensions (i.e. $n \geq 2$ with equal Kähler angles). The author [15] generalized the theorem to submanifolds with parallel mean vector fields but restricting the ambient space to a complex space form.

In this article, we consider the above question in contact geometry, i.e., closed submanifolds with equal Wirtinger angles and parallel mean curvature vector fields, immersed into a Sasakian space form, and obtain the following:

Theorem 1.1. *Let $(N, \xi, \eta, \varphi, g)$ be a $(4n+1)$-dimensional Sasakian space form with constant φ-sectional curvature c, and M a real $(2n+1)$-dimensional closed submanifold tangent to ξ with equal Wirtinger angles, immersed into N. If the mean curvature vector field of M is parallel, then*

1. *When $c > -3$, M is either an invariant or an anti-invariant submanifold.*
2. *When $c = -3$, the common Wirtinger angles of M must be constant.*

Remark 1.2. Theorem 1.1 is a Sasakian version of submanifolds in complex space forms but is in fact very different from that of Kähler manifolds because the result in complex geometry depends strictly on the sign of the holomorphic sectional curvatures (see [15]).

In Sec. 2, we recall some known facts on Sasakian manifolds and list some basic formulae that will be used later. The main theorem's proof is given in Sec. 3, together with an important corollary.

2. Preliminaries and Formulas

In this section, we collect some basic formulas and results for later use. We begin with some basic facts on Sasakian spaces. We refer to [2] for more detailed treatment.

An odd-dimensional differentiable manifold N has an almost contact structure if it admits a global vector field ξ, a one-form η and a (1,1)-tensor field φ satisfying

$$\eta(\xi) = 1, \quad \text{and} \quad \varphi^2 = -id + \eta \otimes \xi. \tag{2.1}$$

In that case, one can always find a compatible Riemannian metric g, i.e., such that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \tag{2.2}$$

for all vector fields X and Y on N.

$(N, \xi, \eta, \varphi, g)$ is an almost contact metric manifold. If the additional property $d\eta(X, Y) = g(\varphi X, Y)$ holds, then $(N, \xi, \eta, \varphi, g)$ is called a contact metric manifold. As a consequence, the characteristic curves (i.e. the integral curves of the characteristic vector field ξ) are geodesics.

If $\overline{\nabla}$ is the Riemannian connection of g on N, then

$$\overline{\nabla}_X \xi = \varphi X. \tag{2.3}$$

A contact metric manifold, $(N, \xi, \eta, \varphi, g)$, for which ξ is a Killing vector field is called a K-contact manifold. Finally, if the Riemannian curvature tensor of N satisfies

$$\overline{R}(X, Y)\xi = \eta(Y)X - \eta(X)Y \tag{2.4}$$

for all vector fields X and Y, then the contact metric manifold is Sasakian. In that case, ξ is a Killing field, and

$$(\overline{\nabla}_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X = \overline{R}(X, \xi)Y. \tag{2.5}$$

If it has constant φ-sectional curvature c, then the Sasakian manifold is called a Sasakian space form. At this time its curvature tensor is given by (cf. [19])

$$\begin{aligned}\overline{R}(X, Y)Z = \frac{1}{4}(c+3)\{g(Y, Z)X - g(X, Z)Y\} - \frac{1}{4}(c-1)\{&\eta(Y)\eta(Z)X \\ &- \eta(X)\eta(Z)Y + g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi \\ &- g(\varphi Y, Z)\varphi X + g(\varphi X, Z)\varphi Y + 2g(\varphi X, Y)\varphi Z\}.\end{aligned} \tag{2.6}$$

Now let $(N, \xi, \eta, \varphi, g)$ be a Sasakian space of dimension $4n+1$, and $x : M \longrightarrow N$ be an immersed submanifold M of real dimension $2n+1$. Denote by $\langle , \rangle$ the Riemannian metric g of N as well as the induced metric of M from N. We denote by ∇, $\nabla^{\perp}$, A, and B the induced Levi-Civita connection, the induced normal connection from N, the Weingarten operator and the second fundamental form of submanifold M, respectively. As usual, TM and $T^{\perp}M$ are the tangent and normal bundles of M in N, respectively.

For any $X, Y, Z \in TM$, the Codazzi equation is given by (cf. [6])

$$(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = (\overline{R}(X, Y)Z)^{\perp}, \tag{2.7}$$

where $\nabla_X B$ is defined by

$$(\nabla_X B)(Y, Z) = \nabla_X^{\perp}(B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z).$$

The Weingarten form A and the second fundamental form B are related by

$$\langle A_v X, Y\rangle = \langle B(X, Y), v\rangle, \quad v \in T^{\perp}M.$$

For any $X \in TM$, and $v \in T^{\perp}M$, we write

$$\varphi X = PX + NX, \quad \varphi v = tv + fv,$$

where PX (resp. tv) and NX (resp. fv) denote the tangent and the normal components of φX (resp. φv), respectively.

A submanifold is said to be invariant (resp. anti-invariant) if N (resp. P) is identically zero. In [16], Lotta proved the following theorem which generalizes a well-known result of Yano and Kon (cf. [19])

Lemma 2.1. [16] *Let M be a submanifold of a contact metric manifold N. If ξ is orthogonal to M, then M is anti-invariant.*

So from now on, we suppose that the characteristic vector field ξ is tangent to M.

Denote by $\Re$ the orthogonal distribution to ξ in TM. For any X tangent to M at p, such that X is not proportional to ξ_p, the Wirtinger angle $\theta(X)$ of X is defined to be the angle between φX and T_pM. In fact since $\varphi\xi = 0$, $\theta(X)$ agrees with the angle between $\varphi(X)$ and $\Re_p$. If $\theta(X)$ is independent of the choice of $X \in T_pM - \langle\xi_p\rangle$, we say M is a submanifold with equal Wirtinger angles. M is said to be slant [16] if the angle θ is a constant on M. Clearly invariant and anti-invariant immersions are slant immersions with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$ respectively.

By (2.1) and (2.2), the following are known facts (cf. [19])

$$P^2 = -I - tN + \eta \otimes \xi, \quad NP + fN = 0, \tag{2.8}$$

$$Pt + tf = 0, \quad f^2 = -I - Nt. \tag{2.9}$$

By (2.3) and (2.5), by differentiating φ along a tangent vector field $X \in TM$ and comparing the tangent and normal components, we have (cf. [19])

$$\nabla_X \xi = PX, \quad NX = B(X, \xi). \tag{2.10}$$
$$(\nabla_X P)Y = A_{NY}X + tB(X,Y) - g(X,Y)\xi + \eta(Y)X, \tag{2.11}$$
$$(\nabla_X N)Y = -B(X, PY) + fB(X,Y), \tag{2.12}$$

where $\nabla_X P$ and $\nabla_X N$ are the covariant derivatives of P and N, respectively defined by

$$(\nabla_X P)Y = \nabla_X(PY) - P\nabla_X Y, \tag{2.13}$$
$$(\nabla_X N)Y = \nabla_X^{\perp}(NY) - N\nabla_X Y. \tag{2.14}$$

Now let us assume that $x : M \longrightarrow N$ is an immersion with equal Wirtinger angle θ, then (cf. [7. 8])

$$\langle PX, PY\rangle = \cos^2\theta\langle X, Y\rangle, \quad X, Y \in TM,$$

with $\cos\theta$ a locally Lipschitz function on M, smooth on the open set where it does not vanish. On an open set without invariant and anti-invariant points, we can choose a locally orthonormal frame $\{e_0, e_1, \cdots, e_{2n}\}$ of TM, such that

$$e_0 = \xi, \quad Pe_i = \cos\theta e_{n+i}, \quad Pe_{n+i} = -\cos\theta e_i, \quad i = 1, \cdots, n,$$

and a local orthonormal frame $\{e_{2n+1}, \cdots, e_{4n}\}$ of $T^{\perp}M$ such that

$$Ne_i = \sin\theta e_{2n+i}, \;\; Ne_{n+i} = \sin\theta e_{3n+i}, \;\; i = 1, \cdots, n.$$

Obviously by the definition of t and f, using (2.8) and (2.9), we have

$$te_{2n+i} = -\sin\theta e_i, \quad te_{3n+i} = -\sin\theta e_{n+i},$$
$$fe_{2n+i} = -\cos\theta e_{3n+i}, \quad fe_{3n+i} = \cos\theta e_{2n+i}.$$

By (2.11), for any $X, Y, Z \in TM$

$$\langle(\nabla_X P)Y, Z\rangle = \langle A_{NY}X + tB(X,Y), Z\rangle - \langle X, Y\rangle\eta(Z) + \langle X, Z\rangle\eta(Y). \tag{2.15}$$

The following index convention will be used: $\alpha, \beta, \gamma, \cdots, \in \{1, \cdots, 2n\}$ and $i, j, k, \cdots, \in \{1, \cdots, n\}$. The component of the second fundamental form is denoted by $B^{2n+\gamma}_{\alpha\beta}$. Let $X = e_\alpha, Y = e_k$ and $Z = e_{n+l}$ in (2.15), making use of (2.13), by a direct calculation we have

$$B^{2n+k}_{\alpha,n+l} - B^{3n+l}_{\alpha k} = \operatorname{ctg}\theta(\Gamma^{n+l}_{\alpha,n+k} - \Gamma^{l}_{\alpha k}) + \frac{e_\alpha(\cos\theta)}{\sin\theta}\delta_{kl}, \tag{2.16}$$

where $\Gamma^{\gamma}_{\alpha\beta}$ is the connection coefficient of M, defined by

$$\nabla_{e_\alpha} e_\beta = \Gamma^{\gamma}_{\alpha\beta} e_\gamma, \quad \Gamma^{\gamma}_{\alpha\beta} = -\Gamma^{\beta}_{\alpha\gamma}.$$

3. Proof of Theorem 1.1

Let $\mathcal{L} = \{p \in M \,|\, \cos\theta(p) = 0\}$, and let $\mathcal{L}^0$ denotes the largest open set contained in $\mathcal{L}$.

Theorem 3.1. *Let N be a $(4n+1)$-dimensional Sasakian space form with constant φ-sectional curvature c, and let M be a $(2n+1)$-dimensional submanifold immersed into N with equal Wirtinger angle θ. If the mean curvature vector field of M in N is parallel, then on $\mathcal{L}^0 \bigcup (M - \mathcal{L})$ we have*

$$\triangle \cos\theta \leq -\Big\{\frac{3}{2}(c-1) + 6\Big\}\sin^2\theta\cos\theta. \tag{3.1}$$

Proof. First we assume $0 < \cos\theta < 1$, so that we can choose the orthonormal frame fields given in Sec. 2. Define a function F by

$$F = \sum_{k=1}^{n} \langle Pe_k, Pe_k\rangle = n\cos^2\theta. \tag{3.2}$$

By (2.10) and (2.11) we see that (for details see (3.3) below)

$$\xi(n\cos^2\theta) = \xi F = 0.$$

Thus the Laplacian of F can be given as follows

$$\triangle F = tr(\nabla dF) = \sum_{\alpha=1}^{2n}(e_\alpha^2 F - dF(\nabla_{e_\alpha} e_\alpha)).$$

By (2.10)$\sim$(2.14), we can do the following calculations

$$\begin{aligned}
e_\beta e_\alpha F &= e_\beta \sum_k e_\alpha \langle Pe_k, Pe_k\rangle \\
&= \nabla_{e_\beta} \sum_k 2\langle \nabla_{e_\alpha}(Pe_k), Pe_k\rangle \\
&= \nabla_{e_\beta} \sum_k 2\langle (\nabla_{e_\alpha} P)e_k + P\nabla_{e_\alpha} e_k, Pe_k\rangle \\
&= \nabla_{e_\beta} \sum_k 2\big\{\langle A_{Ne_k} e_\alpha + tB(e_\alpha, e_k) - \langle e_\alpha, e_k\rangle\xi + \eta(e_k)e_\alpha + P\nabla_{e_\alpha} e_k, Pe_k\rangle\big\} \\
&= \nabla_{e_\beta} \sum_k 2\big\{\langle A_{Ne_k} e_\alpha + tB(e_\alpha, e_k), Pe_k\rangle + \langle P\nabla_{e_\alpha} e_k, Pe_k\rangle\big\} \\
&= \nabla_{e_\beta} \sum_k 2\big\{\sin\theta\langle A_{e_{2n+k}} e_\alpha, Pe_k\rangle - \langle B(e_\alpha, e_k), NPe_k\rangle\big\} \\
&= \nabla_{e_\beta} \sum_k 2\sin\theta\cos\theta(B^{2n+k}_{\alpha,n+k} - B^{3n+k}_{\alpha k}) \\
&= \sum_k 2e_\beta(\sin\theta\cos\theta)(B^{2n+k}_{\alpha,n+k} - B^{3n+k}_{\alpha k}) \\
&\quad + \sum_k 2\sin\theta\cos\theta\{\nabla_{e_\beta} B^{2n+k}_{\alpha,n+k} - \nabla_{e_\beta} B^{3n+k}_{\alpha k}\}.
\end{aligned} \tag{3.3}$$

Similar calculation as above gives

$$dF(\nabla_{e_\beta} e_\alpha) = \sum_k 2\sin\theta\cos\theta\big\{\langle B(\nabla_{e_\beta} e_\alpha, e_{n+k}), e_{2n+k}\rangle - \langle B(\nabla_{e_\beta} e_\alpha, e_k), e_{3n+k}\rangle\big\}. \tag{3.4}$$

Let us denote the last two terms in (3.3) by A_1 and B_1. For A_1 we get

$$\begin{aligned}
A_1 &= \nabla_{e_\beta} B^{2n+k}_{\alpha,n+k} = \nabla_{e_\beta}\langle B(e_\alpha, e_{n+k}), e_{2n+k}\rangle \\
&= \langle \nabla^\perp_{e_\beta}(B(e_\alpha, e_{n+k}), e_{2n+k}\rangle + \langle B(e_\alpha, e_{n+k}), \nabla^\perp_{e_\beta} e_{2n+k}\rangle \\
&= \langle (\nabla_{e_\beta} B)(e_\alpha, e_{n+k}) + B(\nabla_{e_\beta} e_\alpha, e_{n+k}) + B(e_\alpha, \nabla_{e_\beta} e_{n+k}), e_{2n+k}\rangle \\
&\quad + \langle B(e_\alpha, e_{n+k}), \nabla^\perp_{e_\beta} e_{2n+k}\rangle \\
&= \langle (\nabla_{e_{n+k}} B)(e_\alpha, e_\beta) + (\overline{R}(e_\beta, e_{n+k})e_\alpha)^\perp + B(\nabla_{e_\beta} e_\alpha, e_{n+k}) \\
&\quad + B(e_\alpha, \nabla_{e_\beta} e_{n+k}), e_{2n+k}\rangle + \langle B(e_\alpha, e_{n+k}), \nabla^\perp_{e_\beta} e_{2n+k}\rangle \\
&= \langle \nabla^\perp_{e_{n+k}}(B(e_\alpha, e_\beta)) + (\overline{R}(e_\beta, e_{n+k})e_\alpha)^\perp - B(\nabla_{e_{n+k}} e_\alpha, e_\beta) - B(e_\alpha, \nabla_{e_{n+k}} e_\beta) \\
&\quad + B(\nabla_{e_\beta} e_\alpha, e_{n+k}) + B(e_\alpha, \nabla_{e_\beta} e_{n+k}), e_{2n+k}\rangle + \langle B(e_\alpha, e_{n+k}), \nabla^\perp_{e_\beta} e_{2n+k}\rangle \\
&= \langle \nabla^\perp_{e_{n+k}}(B(e_\alpha, e_\beta)) + (\overline{R}(e_\beta, e_{n+k})e_\alpha)^\perp, e_{2n+k}\rangle + \langle B(\nabla_{e_\beta} e_\alpha, e_{n+k}), e_{2n+k}\rangle \\
&\quad + \langle B(e_\alpha, \nabla_{e_\beta} e_{n+k}) - B(\nabla_{e_{n+k}} e_\alpha, e_\beta) - B(e_\alpha, \nabla_{e_{n+k}} e_\beta), e_{2n+k}\rangle \\
&\quad + \langle B(e_\alpha, e_{n+k}), \nabla^\perp_{e_\beta} e_{2n+k}\rangle.
\end{aligned} \tag{3.5}$$

Here we have used the Codazzi equation (2.7) in the fifth equality. By a similar calculation we also have

$$\begin{aligned}
B_1 &= \nabla_{e_\beta} B^{3n+k}_{\alpha k} = \nabla_{e_\beta}\langle B(e_\alpha, e_k), e_{3n+k}\rangle \\
&= \langle \nabla^\perp_{e_k}(B(e_\alpha, e_\beta)) + (\overline{R}(e_\beta, e_k)e_\alpha)^\perp, e_{3n+k}\rangle + \langle B(\nabla_{e_\beta} e_\alpha, e_k), e_{3n+k}\rangle \\
&\quad + \langle B(e_\alpha, \nabla_{e_\beta} e_k) - B(\nabla_{e_k} e_\alpha, e_\beta) - B(e_\alpha, \nabla_{e_k} e_\beta), e_{3n+k}\rangle \\
&\quad + \langle B(e_\alpha, e_k), \nabla^\perp_{e_\beta} e_{3n+k}\rangle.
\end{aligned} \tag{3.6}$$

Combining (3.3), (3.4), (3.5) and (3.6), and using the expression of $\triangle F$ we have

$$\begin{aligned}
\triangle F = &\sum_{\alpha,k} 2e_\alpha(\sin\theta\cos\theta)(B^{2n+k}_{\alpha,n+k} - B^{3n+k}_{\alpha k}) \\
&+ \sum_{\alpha,k} 2\sin\theta\cos\theta\big\{\langle \nabla^\perp_{e_{n+k}}(B(e_\alpha, e_\alpha)) + (\overline{R}(e_\alpha, e_{n+k})e_\alpha)^\perp, e_{2n+k}\rangle \\
&\qquad - \langle \nabla^\perp_{e_k}(B(e_\alpha, e_\alpha)) + (\overline{R}(e_\alpha, e_k)e_\alpha)^\perp, e_{3n+k}\rangle\big\} \\
&+ \sum_{\alpha,k} 2\sin\theta\cos\theta\big\{\langle B(e_\alpha, \nabla_{e_\alpha} e_{n+k}) - 2B(\nabla_{e_{n+k}} e_\alpha, e_\alpha), e_{2n+k}\rangle \\
&\qquad - \langle B(e_\alpha, \nabla_{e_\alpha} e_k) - 2B(\nabla_{e_k} e_\alpha, e_\alpha), e_{3n+k}\rangle\big\} \\
&+ \sum_{\alpha,k} 2\sin\theta\cos\theta\big\{\langle B(e_\alpha, e_{n+k}), \nabla^\perp_{e_\alpha} e_{2n+k}\rangle - \langle B(e_\alpha, e_k), \nabla^\perp_{e_\alpha} e_{3n+k}\rangle\big\}.
\end{aligned} \tag{3.7}$$

We denote the last two terms by A_2 and B_2, respectively, in the above equation. For A_2,

$$\begin{aligned}
A_2 = & \sum_{\alpha,k} 2\sin\theta\cos\theta\langle B(e_\alpha, e_{n+k}), \nabla^\perp_{e_\alpha} e_{2n+k}\rangle \\
= & \sum_{\alpha,k} 2\sin\theta\cos\theta\Big\langle B(e_\alpha, e_{n+k}), \nabla^\perp_{e_\alpha}\Big(\frac{1}{\sin\theta}Ne_k\Big)\Big\rangle \\
= & \sum_{\alpha,k} 2\sin\theta\cos\theta\Big\{\Big\langle B(e_\alpha, e_{n+k}), \frac{-e_\alpha(\sin\theta)}{\sin^2\theta}Ne_k + \frac{1}{\sin\theta}((\nabla_{e_\alpha}N)e_k + N\nabla_{e_\alpha}e_k)\Big\rangle\Big\} \\
= & \sum_{\alpha,k} 2\sin\theta\cos\theta\Big\{\Big\langle B(e_\alpha, e_{n+k}), \frac{-e_\alpha(\sin\theta)}{\sin\theta}e_{2n+k}\Big\rangle \\
& \qquad + \frac{1}{\sin\theta}\Big\langle B(e_\alpha, e_{n+k}), -B(e_\alpha, Pe_k) + fB(e_\alpha, e_k) + N\nabla_{e_\alpha}e_k\Big\rangle\Big\} \\
= & \sum_{\alpha,k}\big\{-2\cos\theta e_\alpha(\sin\theta)B^{2n+k}_{\alpha,n+k} - 2\cos^2\theta\langle B(e_\alpha, e_{n+k}), B(e_\alpha, e_{n+k})\rangle \\
& + 2\cos\theta\langle B(e_\alpha, e_{n+k}), fB(e_\alpha, e_k) + N\nabla_{e_\alpha}e_k\rangle\big\}.
\end{aligned}$$

In the above calculation, we have used (2.12), (2.14) and the property of the chosen frame. Similarly,

$$\begin{aligned}
B_2 = & \sum_{\alpha,k} 2\sin\theta\cos\theta\langle B(e_\alpha, e_k), \nabla^\perp_{e_\alpha} e_{3n+k}\rangle \\
= & \sum_{\alpha,k}\big\{-2\cos\theta e_\alpha(\sin\theta)B^{3n+k}_{\alpha k} + 2\cos^2\theta\langle B(e_\alpha, e_k), B(e_\alpha, e_k)\rangle \\
& + 2\cos\theta\langle B(e_\alpha, e_k), fB(e_\alpha, e_{n+k}) + N\nabla_{e_\alpha}e_{n+k}\rangle\big\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
A_2 - B_2 = & \sum_{\alpha,k}\{-2\cos\theta e_\alpha(\sin\theta)(B^{2n+k}_{\alpha,n+k} - B^{3n+k}_{\alpha k})\} - 2\cos^2\theta\sum_{\alpha,\beta}\|B(e_\alpha, e_\beta)\|^2 \\
& + \sum_{\alpha,k} 2\cos\theta\big\{\langle B(e_\alpha, e_{n+k}), fB(e_\alpha, e_k)\rangle - \langle B(e_\alpha, e_k), fB(e_\alpha, e_{n+k})\rangle\big\} \\
& + \sum_{\alpha,k} 2\cos\theta\big\{\langle B(e_\alpha, e_{n+k}), N\nabla_{e_\alpha}e_k\rangle - \langle B(e_\alpha, e_k), N\nabla_{e_\alpha}e_{n+k}\rangle\big\}.
\end{aligned} \tag{3.8}$$

As f is skew-symmetric, inserting (3.8) into (3.7), and using (2.10) we have

$$\begin{aligned}\triangle F = &\sum_{\alpha,k} 2\sin\theta e_\alpha(\cos\theta)(B^{2n+k}_{\alpha,n+k} - B^{3n+k}_{\alpha k}) \qquad (3.9)\\
&+\sum_{\alpha,k} 2\sin\theta\cos\theta\big\{\langle\nabla^\perp_{e_{n+k}}\vec{H} + (\overline{R}(e_\alpha,e_{n+k})e_\alpha)^\perp, e_{2n+k}\rangle\\
&\qquad - \langle\nabla^\perp_{e_k}\vec{H} + (\overline{R}(e_\alpha,e_k)e_\alpha)^\perp, e_{3n+k}\rangle\big\}\\
&+\sum_{\alpha,k} 2\sin\theta\cos\theta\big\{\langle B(e_\alpha,\nabla_{e_\alpha}e_{n+k}) - 2B(\nabla_{e_{n+k}}e_\alpha,e_\alpha), e_{2n+k}\rangle\\
&\qquad - \langle B(e_\alpha,\nabla_{e_\alpha}e_k) - 2B(\nabla_{e_k}e_\alpha,e_\alpha), e_{3n+k}\rangle\big\} \qquad (3.10)\\
&+\sum_{\alpha,k} 4\cos\theta\big\{\langle B(e_\alpha,e_{n+k}), fB(e_\alpha,e_k)\rangle\big\} - 2\cos^2\theta\sum_{\alpha,\beta}\|B(e_\alpha,e_\beta)\|^2 \qquad (3.11)\\
&+\sum_{\alpha,k} 2\cos\theta\big\{\langle B(e_\alpha,e_{n+k}), N\nabla_{e_\alpha}e_k\rangle - \langle B(e_\alpha,e_k), N\nabla_{e_\alpha}e_{n+k}\rangle\big\}. \qquad (3.12)\end{aligned}$$

Here $\vec{H}$ is the mean curvature vector field of M, defined by $\vec{H} = trB$. Since

$$\begin{aligned}\sum_{\alpha,k}\langle B(\nabla_{e_{n+k}}e_\alpha,e_\alpha), e_{2n+k}\rangle &= \sum_{\alpha,\beta,k}\Gamma^\beta_{n+k,\alpha}B^{2n+k}_{\beta\alpha} + \sum_{\alpha,k}\Gamma^\xi_{n+k,\alpha}B^{2n+k}_{\xi\alpha}\\
&= -\sum_{\alpha,\beta,k}\Gamma^\alpha_{n+k,\beta}B^{2n+k}_{\beta\alpha} + \sum_k \sin\theta\Gamma^\xi_{n+k,k},\end{aligned}$$

where we have used (2.10), and $B^{2n+k}_{\xi\alpha}$ means $\langle B(\xi,e_\alpha), e_{2n+k}\rangle$, after rearranging indices in the last term and using (2.10) again we see that this term is equal to $n\sin\theta\cos\theta$. Similarly

$$\sum_{\alpha,k}\langle B(\nabla_{e_k}e_\alpha,e_\alpha), e_{3n+k}\rangle = -n\sin\theta\cos\theta.$$

Therefore

$$\begin{aligned}(3.10) &= 2\sin\theta\cos\theta\sum_{\alpha,k}\{\langle B(e_\alpha,\nabla_{e_\alpha}e_{n+k}), e_{2n+k}\rangle - \langle B(e_\alpha,\nabla_{e_\alpha}e_k), e_{3n+k}\rangle\}\\
&\quad - 8n\sin^2\theta\cos^2\theta\\
&= 2\sin\theta\cos\theta\sum_{\alpha,\beta,k}\{\Gamma^\beta_{\alpha,n+k}B^{2n+k}_{\alpha\beta} - \Gamma^\beta_{\alpha k}B^{3n+k}_{\alpha\beta}\} - 12n\sin^2\theta\cos^2\theta.\end{aligned}$$

By the property of the chosen frame fields and using the basic inequality, we have

$$\begin{aligned}(3.11) &= 4\cos^2\theta\sum_{\alpha,k,l}(B^{2n+l}_{\alpha,n+k}B^{3n+l}_{\alpha k} - B^{3n+l}_{\alpha,n+k}B^{2n+l}_{\alpha k}) - 2\cos^2\theta\sum_{\alpha,\beta}\|B(e_\alpha,e_\beta)\|^2\\
&\le 2\cos^2\theta\sum_{\alpha,k,l}\{(B^{2n+l}_{\alpha,n+k})^2 + (B^{3n+l}_{\alpha k})^2 + (B^{3n+l}_{\alpha,n+k})^2 + (B^{2n+l}_{\alpha k})^2\}\\
&\qquad - 2\cos^2\theta\sum_{\alpha,\beta}\|B(e_\alpha,e_\beta)\|^2 \le 0. \qquad (3.13)\end{aligned}$$

By the property of the chosen frame fields again

$$(3.12) = 2\sin\theta\cos\theta \sum_{\alpha,\beta,k} \big(B^{2n+\beta}_{\alpha,n+k}\Gamma^{\beta}_{\alpha k} - B^{2n+\beta}_{\alpha k}\Gamma^{\beta}_{\alpha,n+k}\big).$$

Let $k = l$ in (2.16) we see that

$$B^{2n+k}_{\alpha,n+k} - B^{3n+k}_{\alpha k} = \frac{e_\alpha(\cos\theta)}{\sin\theta}.$$

This leads to

$$(3.9) = 2\sum_{\alpha,k} \big\{e_\alpha(\cos\theta)\big\}^2 = 2n\|\nabla\cos\theta\|^2,$$

since $\xi(\cos\theta) = 0$.

Inserting (3.9)$\sim$(3.13) into $\triangle F$ and using (3.2) we get

$$\begin{aligned}
&\triangle(n\cos^2\theta) \le 2\sin\theta\cos\theta\sum_k\{\langle\nabla^\perp_{e_{n+k}}\vec{H}, e_{2n+k}\rangle - \langle\nabla^\perp_{e_k}\vec{H}, e_{3n+k}\rangle\}\\
&+ 2\sin\theta\cos\theta\sum_{\alpha,k}\{\langle(\overline{R}(e_\alpha, e_{n+k})e_\alpha)^\perp, e_{2n+k}\rangle - \langle(\overline{R}(e_\alpha, e_k)e_\alpha)^\perp, e_{3n+k}\rangle\}\\
&+ 2n\|\nabla\cos\theta\|^2 - 12n\sin^2\theta\cos^2\theta + F_1,
\end{aligned} \tag{3.14}$$

where

$$F_1 = 2\sin\theta\cos\theta\sum_{\alpha,\beta,k}\big\{\Gamma^{\beta}_{\alpha,n+k}B^{2n+k}_{\alpha\beta} - \Gamma^{\beta}_{\alpha k}B^{3n+k}_{\alpha\beta} + B^{2n+\beta}_{\alpha,n+k}\Gamma^{\beta}_{\alpha k} - B^{2n+\beta}_{\alpha k}\Gamma^{\beta}_{\alpha,n+k}\big\}.$$

Now we assume N is a Sasakian space form with constant φ-sectional curvature c. By (2.6) we have

$$\begin{aligned}
\sum_{\alpha,k}\langle(\overline{R}(e_\alpha, e_{n+k})e_\alpha)^\perp, e_{2n+k}\rangle &= -\frac{3}{4}n(c-1)\sin\theta\cos\theta,\\
\sum_{\alpha,k}\langle(\overline{R}(e_\alpha, e_k)e_\alpha)^\perp, e_{3n+k}\rangle &= \frac{3}{4}n(c-1)\sin\theta\cos\theta.
\end{aligned}$$

Therefore, when the mean curvature vector field $\vec{H}$ is parallel, (3.14) reads

$$\triangle(n\cos^2\theta) \le 2n\|\nabla\cos\theta\|^2 - 3n(c-1)\sin^2\theta\cos^2\theta - 12n\sin^2\theta\cos^2\theta + F_1. \tag{3.15}$$

It is easy to see that formula (3.15) is independent of the chosen local frame fields. Thus for any $p \in M$, we can choose a local normal coordinate for M at p such that $\Gamma^{\gamma}_{\alpha\beta}(p) = 0$, and so $F_1 = 0$. An easy calculation shows that

$$\triangle(n\cos^2\theta) = 2n\|\nabla\cos\theta\|^2 + 2n\cos\theta\triangle\cos\theta.$$

Plugging these into (3.15) we get

$$\triangle\cos\theta \le -\Big\{\frac{3}{2}(c-1) + 6\Big\}\sin^2\theta\cos\theta.$$

Generally, $\cos\theta$ is only locally Lipschitz on M, but smooth on the open set of anti-invariant points. Obviously the last formula also holds on invariant points and the largest open set of anti-invariant points, Therefore it surely holds on $\mathcal{L}^0 \bigcup (M - \mathcal{L})$. ∎

Proof of Theorem 1.1. From Theorem 3.1 we see that (3.1) is valid on all M but the set of anti-invariant points with no interior. Now M is closed and so $\cos\theta$ extends smoothly on all M, i.e. (3.1) holds on the whole submanifold M.
1. When $c > -3$, integrating (3.1) over M and noting that $\cos\theta \geq 0$, we have

$$-\int_M \left\{\frac{3}{2}(c-1)+6\right\} \sin^2\theta \cos\theta d \operatorname{Vol} M \geq 0,$$

which implies either $\sin\theta = 0$, or $\cos\theta = 0$, that is M is either an invariant or an anti-invariant submanifold.
2. When $c = -3$, (3.1) takes the form

$$\triangle \cos\theta \leq 0.$$

By the maximum principle of Hopf, we see that $\cos\theta$ is constant, and therefore M has constant Wirtinger angles. ∎

When M has constant Wirtinger angles, as an immediate result of Theorem 3.1, we have the following corollary

Corollary 3.2. *Let M^{2n+1} be a slant submanifold with parallel mean curvature vector field, immersed into a Sasakian space form with constant φ-sectional curvature $c > -3$. Then M is either an invariant or an anti-invariant submanifold.*

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