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Maximizing the Stability Radius of Discrete-Time Linear Positive Systems by Linear Feedbacks

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Abstract. In this paper we shall prove a theorem on the existence of a linear feedback maximizing the real stability radius of a linear positive discrete-time system while preserving positivity of the system.

1. Introduction

Consider a dynamical system described by the following linear difference equation

$$x(t+1) = Ax(t),\ \ t \in \mathbb{N}, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}, x(t) \in \mathbb{R}^n$ and $\mathbb{N} = \{0, 1, 2, \dots\}$. Assume that the system (1) is *Schur stable*, i.e. the spectrum $\sigma(A)$ of A lies in the open unit disk

$$\mathbb{D}^1 = \{s \in \mathbb{C}; |s| < 1\},$$

or, equivalently,

$$\rho(A) < 1,$$

where $\rho(A)$ is the spectral radius of A. Assume that the matrix A is subjected to perturbation of the form

$$A \to A + D\Delta E,$$

where $D \in \mathbb{R}^{n\times l}$ and $E \in \mathbb{R}^{q\times n}$ are given matrices defining the structure of the perturbations and $\Delta \in \mathbb{R}^{l\times q}$ is unknown disturbance matrix. Then the *stability radius* of the matrix A with respect to perturbations of the structure (D, E) is defined by

$$r_{\mathbb{K}}(A) = r_{\mathbb{K}}(A; D, E) = \inf\{\|\Delta\|; \Delta \in \mathbb{K}^{l\times q}, \rho(A + D\Delta E) \geq 1\},$$

$\mathbb{K} = \mathbb{R}, \mathbb{C}$.

Thus, the stability radii measure the *robustness of stability* of the systems under complex and real perturbations, respectively. From practical point of view, the system is "better" if its stability radius is "larger".

It is a well-known fact that, for any given real triple (A, D, E), the *complex stability radius* $r_{\mathbb{C}}(A; D, E)$ is easier to analyze than the *real stability radius* $r_{\mathbb{R}}(A; D, E)$, although, at the first sight, only the stability radii under real perturbations seem to be of practical interest. It has been shown that the computation of the complex stability radius is reduced to a global optimization problem of a Lipschitz function over the real line. Namely, the following formula holds [6]

$$r_{\mathbb{C}}(A; D, E) = \frac{1}{\max_{\omega\in\mathbb{R}} \|E(\imath\omega I - A)^{-1}D\|}. \tag{2}$$

Meanwhile, the problem of deriving the computable formulae for the *real stability radius* is a much more difficult problem and has been solved by Qiu, Bernhardsson, Doyle and others [17], who established an explicit formula for real stability radius involving a global minimization problem over $\mathbb{R}^2$.

The situation looks much simpler for *positive linear systems*. Indeed, we have the following [18]

Theorem 1.1. *Suppose that* $\mathbb{K}^l, \mathbb{K}^q$ *are provided with monotonic norms,* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$ *and* $(A, D, E) \in \mathbb{R}_+^{n\times n} \times \mathbb{R}_+^{n\times l} \times \mathbb{R}_+^{q\times n}$. *If* $\rho(A) < 1$ *, then*

$$r_{\mathbb{C}}(A; D, E) = r_{\mathbb{R}}(A; D, E) = r_{\mathbb{R}_+}(A; D, E) = \|E(I - A)^{-1}D\|^{-1},$$

(*where* $\|\cdot\|$ *is the operator norm*).

The above result has been extended to linear positive systems in infinite-dimentional spaces [4, 19],to systems subjected to perturbations of more general forms [11, 20] and, more recently, to positive retarded systems described by linear functional differential equations [21, 16].

In this paper, for a stabilizable linear system of the form $x(t+1) = Ax(t) + Bu(t), t \in \mathbb{N} = \{0, 1, 2, ...\}$ with non-negative matrices A, B, we look for a linear stabilizing feedback $u = Fx$ such that the closed-loop system $x(t + 1) = (A + BF)x(t), t \in \mathbb{N}$ would have a maximal stability radius while remains a positive one: $A + BF \geq 0$. We remark that the problem of stabilization of positive systems has recently attracted a good deal of attention of researchers in control theory, see e.g. [15].

2. Problem of Optimizing the Stability Radius by Linear Feedbacks

Consider a linear controlled system described by a difference equation of the form

$$x(t+1) = Ax(t) + Bu(t), \qquad t \in \mathbb{N} = \{0, 1, 2, ...\}, \tag{3}$$

where $A \in \mathbb{R}_+^{n\times n}, B \in \mathbb{R}_+^{n\times m}$ are matrices with nonnegative entries. Suppose that the system is stabilizable i.e. there exists a linear state feedback $u = Fx, F \in \mathbb{R}^{m\times n}$ such that

$$\sigma(A - BF) \subset \mathbb{D}^1 := \{s \in \mathbb{C} : |s| < 1\}, \tag{4}$$

or equivalently

$$\rho(A - BF) < 1. \tag{5}$$

Assume that the system is subjected to perturbation of the form

$$A \to A + D\Delta E, \tag{6}$$

where $(D, E) \in \mathbb{R}^{n\times l} \times \mathbb{R}^{q\times n}$ are given structure matrices.
Our aim is to dertermine the supremum of the real stability radius of the system (3) which can be achieved by linear state feedback $u = Fx, F \in \mathbb{R}^{m\times n}$, without destroying positivity of the system.

Problem 2.1.

$$\begin{aligned} \text{maximize} \quad & r_{\mathbb{R}}(A - BF; D, E) \\ \text{subject to} \quad & F \in \mathbb{R}^{m\times n}, A - BF \geq 0, \\ & \sigma(A - BF) \subset \mathbb{D}^1. \end{aligned}$$

Let denote by $\mathcal{F}$ the set of all stabilizing state feedbacks preserving positivity of system (3)

$$\mathcal{F} := \{F \in \mathbb{R}^{m\times n} : A - BF \geq 0, \sigma(A - BF) \subset \mathbb{D}^1\}. \tag{7}$$

In general, $\mathcal{F}$ is not bounded. This is shown by the following example:

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad F_k = \begin{bmatrix} 0 & -k \end{bmatrix}.$$

Then

$$A - BF_k = \begin{bmatrix} 0 & k \\ 0 & 0 \end{bmatrix} \geq 0$$

and $\sigma(A - BF_k) \subset \mathbb{D}^1$. Thus $F_k \in \mathcal{F}$ but $\|F_k\| \to \infty$.

By a known result on robust stability of positive systems, see, e.g. [11, 18], for $F \in \mathcal{F}$,

$$r_{\mathbb{R}}(F) := r_{\mathbb{R}}(A - BF; D, E) = \left[\|E(I - A + BF)^{-1}D\|\right]^{-1} = \|G_F(1)\|^{-1} \tag{8}$$

where

$$G_F(s) := E(sI - A + BF)^{-1}D, s \notin \mathbb{D}^1 \tag{9}$$

is the transfer function of the perturbed closed-loop system. Therefore, Problem 2.1 is equivalent to the following

Problem 2.2.

$$\underset{F\in\mathcal{F}}{\text{minimize}} \quad \|G_F(1)\|,$$

where $G_F(.)$ is defined by (9) and $\mathcal{F}$ is defined by (7).

Proposition 2.3. *The map*

$$r_{\mathbb{R}}(\cdot) : \mathbb{R}^{m\times n} \to \mathbb{R}_+$$

is continuous and monotone on $\mathcal{F}$, i.e.

$$F_1 \le F_2 \;\Rightarrow\; r_{\mathbb{R}}(F_1) \le r_{\mathbb{R}}(F_2). \tag{10}$$

Proof. Since , for each $F \in \mathcal{F}$, $A - BF \ge 0$ and $\sigma(A - BF) \subset \mathbb{D}^1$, it follows from [10] that $r_{\mathbb{R}}(F) = r_{\mathbb{C}}(F)$. Therefore, by Proposition 2.4 in [9], $r_{\mathbb{R}}$ is continuous on $\mathcal{F}$. To prove the monotonicity, note first that, for any matrix $M \in \mathbb{R}^{p\times p}$ and any $s \notin \sigma(M)$,

$$(sI - M)^{-1} = \frac{1}{s}(I + \frac{1}{s}M + \frac{1}{s^2}M^2 + \cdots).$$

Therefore, if $M_1, M_2 \in R_+^{p\times p}$ are Schur stable and $M_1 \le M_2$ then $(I - M_1)^{-1} \le (I - M_2)^{-1}$. Since all matrices under consideration are nonnegative, from the above we deduce that, if $F_1, F_2 \in \mathcal{F}$ and $F_1 \le F_2$, then

$$0 \le G_{F_1}(1) = E(I - A + BF_2)^{-1}D \le E(I - A + BF_1)^{-1}D = G_{F_2}(1)$$

and hence

$$\|G_{F_2}(1)\| \le \|G_{F_1}(1)\|.$$

Therefore, by (8), we get the desired inequality (10) for $r_{\mathbb{R}}$. ■

From the above proof, it follows that for $F_1, F_2 \in \mathcal{F}$

$$BF_1 \le BF_2 \qquad \Rightarrow \qquad r_{\mathbb{R}}(F_1) \le r_{\mathbb{R}}(F_2). \tag{11}$$

We shall make use of monotoncity (11) rather than (10) in proving the main result below.

Since $F_0 = 0 \in \mathcal{F}$ (by the assumtion (4)) it follows from (10) that every *nonnegative feedback* $F \ge 0$ satisfying $A - BF \ge 0$ yields at least a better robustness for stability of A : $r_{\mathbb{R}}(A - BF; D, E) \ge r_{\mathbb{R}}(A; D, E)$. In this context, it is of interest to consider the problem of maximizing the stability radius of positive systems by nonnegative feedbacks, that is

Problem 2.4.

$$\underset{F\in\mathcal{F}_+}{\text{maximize}} \quad r_r(A - BF; D, E),$$

where

$$\mathcal{F}_+ := \{F \in \mathbb{R}_+^{m\times n} : A - BF \ge 0, \sigma(A - BF) \subset \mathbb{D}^1\}. \tag{12}$$

In the above definition of the set of nonnegative stabilizing feedbacks of the positive systems (3) the condition $\sigma(A-BF)\subset\mathbb{D}^1$ can be dropped. Indeed, if $F\in\mathbb{R}_+^{m\times n}$ satisfies $A-BF\geq 0$ then, since $A\geq A-BF\geq 0$, it readily follows, by monotonicity of the spectral radius (for nonnegative matrices) and by (4), that $\rho(A-BF)\leq\rho(A)<1$. So $\sigma(A-BF)\subset\mathbb{D}^1$.

Throughout this paper, we shall assume that matrix B is of full rank, i.e.

$$\text{rank}\ \ B=m. \tag{13}$$

In order to deal with Problems 2.1, 2.4 above, we first give a geometric description of the sets $\mathcal{F},\mathcal{F}_+$. Let denote by b^i the i-th row of matrix B and by a_j and f_j the j-th columns of matrix A and matrix F, respectively. Then for $(A,B,F)\in\mathbb{R}_+^{n\times n}\times\mathbb{R}^{n\times m}\times\mathbb{R}^{m\times n}$ we can write

$$A=[a_1\cdots a_n],\ B=\begin{bmatrix}b^1\\ \vdots\\ b^n\end{bmatrix},\ F=[f_1\cdots f_n], \tag{14}$$

where $a_j\in\mathbb{R}_+^n\ ,b^{i\top}\in\mathbb{R}^m,\ f_j\in\mathbb{R}^m$.

For each $j\in\underline{N}:=\{1,\cdots,n\}$ we define a closed polyhedron in $\mathbb{R}^m$

$$V^j=\{f\in\mathbb{R}^m;Bf\leq a_j\}. \tag{15}$$

Then the set $\mathcal{F}$ (7) can be described as

$$\mathcal{F}=\big\{F=[f_1\cdots f_n]\in\mathbb{R}^{m\times n};\quad f_j\in V^j\ \text{ for }\ j\in\underline{N},\ \ \sigma(A-BF)\subset\mathbb{D}^1\big\}.$$

The set of nonnegative stabilizing feedbacks is described similarly, but in view of the above remark, we can write

$$\mathcal{F}_+=\big\{F=[f_1\cdots f_n]\in\mathbb{R}^{m\times n};\quad f_j\in V_+^j\ \text{ for }\ j\in\underline{N}\big\}=\prod_{j=1}^n V_+^j,$$

where

$$V_+^j:=\{f\in\mathbb{R}_+^m;Bf\leq a_j\}. \tag{16}$$

It turns out that Problem (2.4) always admits an optimal solution, as shown by the following

Proposition 2.5. *Suppose* $(A,B,D,E)\in\mathbb{R}_+^{n\times n}\times\mathbb{R}_+^{n\times m}\times\mathbb{R}_+^{n\times l}\times\mathbb{R}_+^{q\times n}$ *and* B *is of full rank. Then there always exists a nonnegative stabilizing feedback* $F^*\in\mathcal{F}_+$ *so that*

$$r_{\mathbb{R}}(F^*)=\max_{F\in\mathcal{F}_+}r_{\mathbb{R}}(F). \tag{17}$$

Proof. Since $\mathcal{F}_+$ is closed, it suffices to show, in view of Propostion 2.3, that $\mathcal{F}_+$ is bounded. Assume to the contrary that $V_+^{j_0}$ is unbounded for some $1\leq j_0\leq n$. It follows that there exists a sequence $\{f_{j_0}^{(\nu)}\}\in V_+^{j_0},\nu=1,2,\cdots$ such that

$f^{(\nu)}_{k_0 j_0} \to \infty$ when $\nu \to \infty$ for some $k_0,\ 1 \le k_0 \le m$. Then, for each $i \in \{1, \cdots, n\}$, by definition of (16), we have

$$a_{ij_0} \ge \left\langle b^{i\top}, f^{(\nu)}_{j_0} \right\rangle = \sum_{k=1}^{m} b_{ik} f^{(\nu)}_{kj_0} \ge b_{ik_0} f^{(\nu)}_{k_0 j_0},$$

which implies readily that $b_{ik_0} = 0$. Thus the k_0-th column of B is equal to zero. This conflicts, however, the assumption that B is of full rank. ∎

We now turn to Problem 2.1. It is well-known that for an arbitrary stable complex matrix $A \in \mathbb{C}^{n\times n}$, there does not exist in general a feedback matrix maximizing the complex stability radius $r_{\mathbb{C}}$ and the supremum value

$$\gamma^* := \sup_F r_{\mathbb{C}}(F)$$

can often be reachable only by high-gain feedbacks, that is, there only exists a sequence of complex feedbacks F_k such that $r_{\mathbb{C}}(F_k) \to \gamma^*$ and $\|F_k\| \to \infty$ as $k \to \infty$.

We shall show that the situation is simpler for positive systems. We first prove the following.

Proposition 2.6. *Suppose* $(A, B, D, E) \in \mathbb{R}^{n\times n}_+ \times \mathbb{R}^{n\times m}_+ \times \mathbb{R}^{n\times l}_+ \times \mathbb{R}^{q\times n}_+$ *and* B *is of full rank. Then there exists a sequence of bounded stabilizing feedbacks* $F_0 = 0, F^b_k \in \mathcal{F}, k = 1, 2, \cdots$ *such that*

$$\begin{aligned} &0 \le r_{\mathbb{R}}(A; D, E) = r_{\mathbb{R}}(0) \le r_{\mathbb{R}}(F^b_k) \le r_{\mathbb{R}}(F^b_{k+1}) \le \cdots, \\ &\lim_{k\to\infty} r_{\mathbb{R}}(F^b_k) = \gamma^* = \sup_{F\in\mathcal{F}} r_{\mathbb{C}}(F). \end{aligned} \tag{18}$$

Proof. For each $j \in \{1, \cdots, n\}$ the convex polyhedron (15) can be written in the form of intersection of n halfspaces:

$$V^j = \bigcap_{i=1}^{n} \left\{ f \in \mathbb{R}^m;\ \ \langle b^{i\ \top}, f\rangle \le a_{ij} \right\}. \tag{19}$$

Then the recession cone of V^j (the set of all infinity directions of V^j) is given by

$$V^j_0 = \bigcap_{i=1}^{n} \left\{ f \in \mathbb{R}^m;\ \ \langle b^{i\ \top}, f\rangle \le 0 \right\}. \tag{20}$$

By the Finite Basis Theorem (see [23, p. 46]), there exists a finite subset $\{v^j_1, v^j_2, \cdots, v^j_{p_j}\} \subset V^j$ such that

$$V^j = V^j_0 + V^j_b, \qquad V^j_b = \mathrm{co}\,\{v^j_1, v^j_2, \cdots, v^j_{p_j}\}.$$

Every $f \in V^j$ can be represented in the form

$$f = f^b + f^0, \qquad f^b \in V^j_b, \quad f^0 \in V^j_0.$$

By definition of V^j_0, it follows that for all $f \in V^j$,

$$Bf \leq Bf^b. \tag{21}$$

Moreover, since $f^b \in V_b^j \subset V^j$, we have, by the definition of (15)

$$Bf^b \leq a_j. \tag{22}$$

Since the above reasoning holds for each $j \in \{1, \cdots, n\}$, we get a convex bounded set (a polytope)

$$V_b = V_b^1 \times V_b^2 \times \cdots \times V_b^n \subset \mathbb{R}^{m \times n},$$

such that for each stabilizing feedback $F \in \mathcal{F}$, there exists

$$F^b = [f_1^b \cdots f_n^b] \in V^b$$

such that (by (21)-(22)),

$$BF \leq BF^b \leq A. \tag{23}$$

Therefore, we obtain $A - BF \geq A - BF^b \geq 0$, which, by monotonicity of the spectral radius, implies $\rho(A - BF^b) \leq \rho(A - BF) < 1$ or equivalently $\sigma(A - BF^b) \subset \mathbb{D}^1$. Thus $F^b \in \mathcal{F}$. Moreover, by Proposition 2.3,

$$r_{\mathbb{R}}(F^b) \geq r_{\mathbb{R}}(F).$$

In other words, we have shown that for each stabilizing feedback F such that $F \in V := \{F : A - BF \geq 0\}$, we always can find a stabilizing feedback F^b in a compact basis $V^b \subset V$ which yields a better stability robustness for the closed-loop system.

Now suppose $\{F_0 = 0, F_k, k = 1, 2, \cdots\} \subset \mathcal{F}$ is a maximizing sequence, i.e.

$$\begin{aligned} &0 \leq r_{\mathbb{R}}(A; D, E) = r_{\mathbb{R}}(0) \leq r_{\mathbb{R}}(F_k) \leq r_{\mathbb{R}}(F_{k+1}) \leq \cdots, \\ &\lim_{k \to \infty} r_{\mathbb{R}}(F_k) = \gamma^* = \sup_{F \in \mathcal{F}} r_{\mathbb{C}}(F). \end{aligned}$$

Then, by the above argument, there exists a sequence of bounded stabilizing feedbacks $\{F_k^b\} \subset V_b \cap \mathcal{F}$ satisfying the conditions (18). This proves the assertion. ■

Corollary 2.7. *Let the assumptions of Proposition* 2.6 *be satisfied. Then Problem* 2.1 *admits an optimal solution if and only if the following maximization problem over a bounded polytope*

$$\begin{aligned} \text{maximize} \quad & r_{\mathbb{R}}(A - BF; D, E) \\ \text{subject to} \quad & F \in V^b \cap \mathcal{F} \end{aligned}$$

has an optimal solution. Moreover, the set of optimal solutions of the two problems coinside.

The following assertion gives a sufficient condition for the existence of optimal solutions.

Corollary 2.8. *Under the assumptions of Proposition* 2.6*, if the bounded polytope of nonnegative matrices*

$$\mathcal{P} := \{A - BF : F \in V^b\}$$

is robust Schur stable, then Problem 2.1 *admits an optimal solution.*

In view of the above result, it is of interest to consider conditions of robust stability of a polytope of matrices. It is well known that, in general, the stability of all vertex matrices is not sufficient for robust stability of the whole polytope, that is a Kharitonov like result does not hold for matrices. Note, however, that here we are dealing with a polytope of nonnegative matrices and hence the situation may be simpler.

In the particular case of single imput systems, Proposition 2.6 yields a complete solution of Problem 2.1. Indeed, we prove first the following

Proposition 2.9. *Suppose* $(A, B, D, E) \in \mathbb{R}_+^{n\times n} \times \mathbb{R}_+^{n\times m} \times \mathbb{R}_+^{n\times l} \times \mathbb{R}_+^{q\times n}, \rho(A) < 1$*. If the compact basis* V^b *of the polyhedron* $V = \{F : A - BF \geq 0\}$ *consists of only one point* F^* *then either* $F^0 = 0$ *or* F^* *is the unique optimal solution for Problem* 2.1*.*

Proof. If $F^* = 0$ then, by definition, $V = V_0 := \{F : BF \leq 0\}$. Therefore, by (11), for any $F \in \mathcal{F}, r_{\mathbb{R}}(F) \leq r_{\mathbb{R}}(0) := r_{\mathbb{R}}(A; D, E)$, thus $F^0 = 0$ is the optimal solution. Suppose now that $F^* \neq 0$ and there exists a stabilizing feedback $F \in V \setminus V_0$ such that $r_{\mathbb{R}}(F) > r_{\mathbb{R}}(0)$. Then by the Finite Basis Theorem, F can be represented in the form $F = F^* + F_0$ where $F_0 \in V_0$. It follows that $BF \leq BF^*$, which again by (11) implies $r_{\mathbb{R}}(F) \leq r_{\mathbb{R}}(F^*)$. Since this holds for any feedback in $\mathcal{F}$ (which yields a strictly better robustness of stability) we conclude that F^* is an optimal solution. ∎

Corollary 2.10. *Suppose* $(A, B, D, E) \in \mathbb{R}_+^{n\times n} \times \mathbb{R}_+^{n\times 1} \times \mathbb{R}_+^{n\times l} \times \mathbb{R}_+^{q\times n}, \rho(A) < 1$*. The maximization problem of the stability radius*

$$\begin{aligned} \text{maximize} \quad & r_{\mathbb{R}}(A - BF; D, E) \\ \text{subject to} \quad & F \in \mathbb{R}^{1\times n}, A - BF \geq 0, \\ & \sigma(A - BF) \subset \mathbb{D}^1. \end{aligned}$$

admits a unique optimal solution F^**, which is given by*

$$\begin{aligned} &F^* = [f_1^* \; f_2^* \; \cdots \; f_n^*], \\ &f_i^* = \min\{a_{ij}/b_i \, ; \, 1 \leq i \leq n, b_i \neq 0\}. \end{aligned} \tag{24}$$

Proof. By the definition of the set $V^j, j = 1, \cdots, n$ (see (19)) we have

$$V^j = \bigcap_{i=1}^{n} \{f \in \mathbb{R}^1; \;\; b_i f \leq a_{ij}\} = \big\{f \in \mathbb{R}^1; \;\; f \leq \min_{1\leq i\leq n, b_i\neq 0} a_{ij}/b_i\big\}.$$

Therefore, the compact basis V_b of the polyhedron $V := \{F \in \mathbb{R}^{1\times n}; BF \leq A\}$ consists of the only point F^* defined by (24). This proves, by Proposition 2.9, the assertion. ∎

By Corollary 2.7 Problem 2.1 is reduced to a maximazition problem on the bounded polytope $V^b \cap \mathcal{F}$. Note, however, that the set $V^b \cap \mathcal{F}$ is, in general, not closed and this situation may cause difficulties or phenomena in solving the above problem. This is shown by the following example. Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad D = I_3, E = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Clearly, $\rho(A) = 0$ and, by a direct calculation,

$$G(1) = E(I - A)^{-1}D = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Therefore, with respect to the ∞-norm,

$$r_{\mathbb{R}}(0) := r_{\mathbb{R}}(A; D, E) = \|G_0(1)\|_\infty^{-1} = \frac{1}{2}.$$

We now try to maximize the stability radius of A by stabilizing feedbacks which do not destroy the nonnegativity, i.e., to find $F \in \mathbb{R}^{2\times 3}$ such that $A - BF \geq 0$ and $r_{\mathbb{R}}(F) := r_{\mathbb{R}}(A - BF; D, E) = \max$. First, by (19), we have that $V^2 = V^3 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$ and

$$V^1 = \left\{ f = \begin{bmatrix} cf^1 \\ f^2 \end{bmatrix} \in \mathbb{R}^2 : f^1 \leq 1, f^2 \leq 0, f^1 + f^2 \leq 0 \right\}.$$

Therefore, the polyhedron $V := \{F \in \mathbb{R}^{2\times 3}; BF \leq A\}$ has the following compact basis

$$V^b = \text{ co} \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} \times \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \times \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

It is easy to verify that

$$V^b \cap \mathcal{F} = V^b \setminus \{F^1\}; \quad F^1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Consequently, by the Proposition 2.6, the maximization problem for the stability radius under consideration is equivalent to the problem

$$\max_{0 \leq \varepsilon < 1} r_{\mathbb{R}}(F^\varepsilon), \quad \text{where} \quad r_{\mathbb{R}}(F^\varepsilon) = r_{\mathbb{R}}(A - BF^\varepsilon; D, E).$$

and

$$F^\varepsilon = \begin{bmatrix} \varepsilon & 0 & 0 \\ -\varepsilon & 0 & 0 \end{bmatrix}.$$

On the other hand, it is easy to verify that for each $\varepsilon \in [0, 1)$

$$G_{F^\varepsilon}(1) = E(I - A + BF^\varepsilon)^{-1}D = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix};$$

it follows that

$$r_{\mathbb{R}}(F^\varepsilon) = \|G_{F^\varepsilon}(1)\|_\infty^{-1} = \frac{1}{2}, \quad \forall \varepsilon \in [0, 1].$$

Thus, by the above assertion, all $F^\varepsilon, 0 \le \varepsilon < 1$ are optimal solutions of the maximization problem. In the meantime, F^1 is destabilizing and

$$\|G_{F^1}\|_\infty = 1.$$

The following proposition gives easily verifiable sufficient conditions for existence of the optimal solution for Problem 2.1.

Proposition 2.11. *Suppose* $(A, B, D, E) \in \mathbb{R}_+^{n\times n} \times \mathbb{R}_+^{n\times m} \times \mathbb{R}_+^{n\times l} \times \mathbb{R}_+^{q\times n}$ *and* B *is of full rank. Then Problem* 2.1 *admits optimal solutions if the matrix* E *has no zero column and the matrix* D *has no zero row.*

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