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Further Applications of the KKM-Maps Principle in Hyperconvex Metric Spaces

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Abstract. In this paper we establish some further applications of the KKM-maps principle in hyperconvex metric spaces such as Ky Fan inequality, fixed point theorems, minimax theorems,... using the notion of sub-admissible sets instead of admissible sets as usual.

1. Introduction

After Khamsi's paper [8] on the KKM-maps principle in hyperconvex metric spaces, several authors have established main applications of this principle such as the Ky Fan minimax inequality, fixed point theorems, minimax theorems, equilibrium point theorems,... in such spaces, see for instance [4, 9, 10, 16, 17]. The authors have used there the notion of admissible sets instead of convex sets in vector spaces.

In Sec. 3 of this paper, we show that in all these results, the notion of admissible sets can be replaced by a weaker one: sub-admissible sets introduced in [16].

Moreover, in Sec. 4 we establish a version of the KKM-maps principle for mappings with open values and apply it to obtain fixed point theorems for upper semicontinuous mappings in hyperconvex metric spaces. This is a continuation of our earlier work [4].

2. Preliminaries

The notion of hyperconvex metric spaces was introduced by Aronszajn and Panitchpakdi in [1]. For the convenience of the readers we recall some definitions.

Definition 1. *A metric space X is said to be hyperconvex if for any collection of points $\{x_\alpha : \alpha \in I\}$ of X and any collection of nonnegative reals $\{r_\alpha : \alpha \in I\}$ such that: $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$ for all $\alpha, \beta \in I$, then $\bigcap_{\alpha \in I} B(x_\alpha, r_\alpha) \neq \emptyset$.*
Here $B(x, r)$ denotes the closed ball centered at x with radius r.

The classical hyperconvex spaces are the real line with the usual distance, $(\mathbb{R}^n, \|.\|_\infty)$, $l^\infty(I)$ for any index set I. Note that the spaces $(\mathbb{R}^n, \|.\|)$ with the euclidean norm are not hyperconvex (Consider three discs in a plane, which are pairwise tangent). In addition, a hyperconvex subset of $(\mathbb{R}^2, \|.\|_\infty)$ need not be convex.

Example. The set $\{(x, y) \in \mathbb{R}^2 : y = x, 0 \leq x \leq 1\} \bigcup \{(x, y) \in \mathbb{R}^2 : y = 2 - x, 1 \leq x \leq 2\}$ is hyperconvex, but not convex.

Definition 2. *A set in a metric space is said to be admissible if it is an intersection of some closed balls. The collection of all admissible sets in a metric space X is denoted by $\mathcal{A}(X)$.*

It is known that if $C \in \mathcal{A}(X)$ then for each $r > 0$, the set $\{x \in X : d(x, C) \leq r\}$, denoted by $N_r(C)$ and called the closed r-neighborhood of C, belongs also to $\mathcal{A}(X)$.

Definition 3. *The admissible hull of a set A in a metric space, denoted by $ad(A)$, is the smallest admissible set containing A.*

In [8], Khamsi established the following KKM-maps principle in hyperconvex metric spaces.

Theorem 2.1. (KKM-maps principle). *Let X be a hyperconvex metric space, C be an arbitrary subset of X, and $F : C \to 2^X$ be a KKM-map such that $F(x)$ is closed for every $x \in C$. Then the family $\{Fx : x \in C\}$ has the finite intersection property.*

In [16] Wu et al. introduced the following notion.

Definition 4. *A set A in a metric space is said to be sub-admissible if for each finite subset D of A we have $ad(D) \subset A$.*

Clearly, each admissible set is sub-admissible and it is not difficult to show that every compact sub-admissible set is admissible. The collection of all sub-admissible sets in a metric space X is denoted by $\mathcal{B}(X)$.

Definition 5. *Let A be an admissible set in a metric space. A function $f : A \to \mathbb{R}$ is said to be quasi-convex (or quasi-concave) if for each $r \in \mathbb{R}$, the set $\{x \in A : f(x) < r\}$ (respectively, $\{x \in A : f(x) > r\}$) is sub-admissible.*

Note that this definition is slightly different from that introduced in [10, 17] where $<$ ($>$) is replaced by $\leq$ ($\geq$) and the set is admissible instead of sub-admissible.

Definition 6. *A multivalued mapping T from a topological space X into a topological space Y is said to be upper semicontinuous (usc) at a point x_0 of X if for any open set U containing Tx_0 there exists a neighborhood V of x_0 such that $T(V) \subset U$.*

T is said to be usc if it is usc at each point of X.

It is known that if T is usc and Y is compact then the graph of T is closed, where graph $T = \{(x, y) \in X \times Y : y \in Tx\}$.

3. Ky Fan Inequality and a Minimax Theorem

The two following results for hyperconvex metric spaces are similar to those of Ky Fan in [6].

Theorem 3.1. *Let X be a hyperconvex metric space, C a nonempty compact admissible set of X and $A \subset C \times C$ satisfying:*
1) $\{y \in C : (x, y) \notin A\} \in \mathcal{B}(X)$, *for all* $x \in C$,
2) $\{x \in C : (x, y) \in A\}$ *is closed, for all* $y \in C$,
3) $(x, x) \in A$, *for all* $x \in C$.

Then there exists $x_0 \in C$ such that $\{x_0\} \times C \subset A$.

Proof. For $y \in C$ we set $F(y) = \{x \in C : (x, y) \in A\}$ which is closed by condition 2. We shall prove by contradiction that F is a KKM-map. Suppose on the contrary that there exist a finite subset $\{y_1, \ldots, y_n\}$ of C and a point $z \in \text{ad}\{y_1, \ldots, y_n\}$ such that $z \notin \bigcup_{i=1}^{n} F(y_i)$. Then we have $(z, y_i) \notin A$, for all $i = 1, \ldots, n$. Putting

$$B = \{y \in C : (z, y) \notin A\},$$

we have $y_i \in B$ for each i. By condition 1, B is sub-admissible. Hence $z \in \text{ad}\{y_1, \ldots, y_n\} \subset B$. From this we have $(z, z) \notin A$, a contradiction to condition 3. So F is a KKM-map. Since C is compact, by Khamsi's theorem there exists $x_0 \in C$ such that $x_0 \in \bigcap_{y \in C} F(y)$. Hence $\{x_0\} \times C \subset A$ and the theorem is proved. ∎

From Theorem 3.1, we have the following result (Ky Fan inequality):

Theorem 3.2. *Let X, C be as in Theorem 3.1 and let $f : C \times C \longrightarrow \mathbb{R}$ be such that:*

i) *For each $x \in C$, the function $f(x,.) : C \longrightarrow \mathbb{R}$ is quasi-concave in y.*

ii) *For each $y \in C$, the function $f(.,y) : C \longrightarrow \mathbb{R}$ is lower semicontinuous in x,*

iii) *$f(x,x) \leq 0$, for all $x \in C$.*

Then there exists $x_0 \in C$ such that: $f(x_0, y) \leq 0$, for all $y \in C$.

Proof. Putting $A = \{(x,y) \in C \times C : f(x,y) \leq 0\}$, it is easy to verify that A satisfies all conditions of Theorem 3.1. Hence there is x_0 such that $\{x_0\} \times C \subset A$. This means that $f(x_0, y) \leq 0$, for all $y \in C$. ∎

Before proving the Browder-Fan fixed point theorem in hyperconvex metric spaces we introduce the following notation. Let C, D be two nonempty sub-admissible sets in two hyperconvex metric spaces X, Y respectively. We denote by $\mathcal{B}(C,D)$ the family of all mappings $T : C \longrightarrow \mathcal{B}(D)$ such that:

i) $Tx \neq \emptyset$, for all $x \in C$,

ii) $T^{-1}y$ is open in C for all $y \in D$,

where the mapping $T^{-1} : D \longrightarrow 2^C$ is defined by: $x \in T^{-1}y \quad \Leftrightarrow y \in Tx, \forall x \in C, \forall y \in D$.

The following is an analogue of a result due to Browder in [3] for hyperconvex metric spaces.

Theorem 3.3. *Let X be a hyperconvex metric space, C a nonempty compact admissible subset of X and $T \in \mathcal{B}(C,C)$. Then there exists $x_0 \in C$ such that $x_0 \in Tx_0$.*

Proof. For each $x \in C$, we set $F(x) = C \setminus T^{-1}x$. Since $T^{-1}x$ is open, we have that $F(x)$ is closed. Since $Tx \neq \emptyset$, for all $x \in C$, we get $C = \bigcup_{x \in C} T^{-1}x$.

Hence $\bigcap_{x \in C} Fx = C \setminus \bigcup_{x \in C} T^{-1}x = \emptyset$. From Khamsi's theorem, F cannot be a KKM map. Then there exist $x_1, x_2, \ldots, x_n \in C$ and $x_0 \in \text{ad}\{x_1, \ldots, x_n\} \subset C$ such that $x_0 \notin Fx_i$ for each i. This is equivalent to $x_0 \in T^{-1}x_i$ for each i. Hence $x_i \in Tx_0$ for each i. Since Tx_0 is sub-admissible, we have $x_0 \in \text{ad}\{x_1, \ldots, x_n\} \subset Tx_0$. The theorem is proved. ∎

Before proving a minimax theorem we establish a coincidence theorem which is analogous to a modified version of a theorem of Ben-El-Mekhaiekh et al. in [2].

Theorem 3.4. *Let C, D be two nonempty sub-admissible sets in two hyperconvex metric spaces X, Y respectively. Suppose that C or D is compact and A, $B : C \longrightarrow 2^D$ are two mappings such that $B \in \mathcal{B}(C,D)$ and $A^{-1} \in \mathcal{B}(D,C)$.*

Then there exists $x_0 \in C$ such that

$$Ax_0 \cap Bx_0 \neq \emptyset.$$

Proof. Suppose that D is compact. Because $A^{-1}y \neq \emptyset$ for all $y \in D$, we have $D = \bigcup_{x \in C} Ax$.

Since D is compact, there exists a finite subset $\{x_1, \dots, x_n\}$ of C such that $D = \bigcup_{i=1}^{n} Ax_i$.

Denote by $\{\beta_1, \dots, \beta_n\}$ a partition of unity subordinate to the above covering. We indentify the imbedding $i : X \hookrightarrow X_\infty = \mathrm{ad}(X) \in \mathcal{A}(l^\infty(X))$ (see [8]). Define a continuous map: $p : D \to C$ by putting

$$p(y) = \sum_{j=1}^{n} \beta_j(y) x_j.$$

Set $L = \mathrm{ad}\{x_1, \dots, x_n\} \subset C$ and $M = \mathrm{conv}\{x_1, \dots, x_n\}$ in X_∞. Letting r be the nonexpansive retract $r : X_\infty \to X$, we have $r(M) \subset L$ (see [8]).

For each j, $\beta_j(y) \neq 0$ implies $y \in Ax_j$. Hence $x_j \in A^{-1}y$. Since $A^{-1}y$ is sub-admissible, we have

$$rp(y) \in L_y \subset A^{-1}y,$$

where $L_y = \mathrm{ad}\{x_i : \beta_i(y) \neq 0\}$. So we have

$$y \in Arp(y), \quad \text{for all } y \in D. \tag{3.1}$$

We define a map $T : D \to 2^D$ by setting $Ty = Brp(y)$.

Since r, p are continuous, from the property of B, we get $T \in \mathcal{B}(D, D)$.

By Theorem 3.3, there exists $y_0 \in D$ such that

$$y_0 \in Ty_0 = Brp(y_0). \tag{3.2}$$

Putting $x_0 = rp(y_0)$, from (3.1), (3.2) we have $y_0 \in Ax_0 \cap Bx_0$.

The case when C is compact can be proved similarly. The proof is complete. ∎

Now we are in a position to proof a version of the well known minimax theorem due to Neumann-Sion [11, 15] in the case of hyperconvex metric spaces.

Theorem 3.5. *Let C, D be as in Theorem* 3.4 *and $f, g : C \times D \to \mathbb{R}$ be two functions such that:*

i) *$f(x, y) \leq g(x, y)$, for all $(x, y) \in C \times D$.*

ii) *For each $x \in C$, the function $g(x, .)$ is quasi-convex and $f(x, .)$ is lower semicontinuous in y.*

iii) *For each $y \in D$, the function $f(., y)$ is quasi-concave and $g(., y)$ is upper semicontinuous in x.*

Then

$$\inf_{y \in D} \sup_{x \in C} f(x, y) \leq \sup_{x \in C} \inf_{y \in D} g(x, y).$$

Proof. Suppose that the conclusion is false. Then there exists a real number λ such that

$$\inf_{y \in D} \sup_{x \in C} f(x, y) > \lambda > \sup_{x \in C} \inf_{y \in D} g(x, y). \tag{3.3}$$

For each $x \in C$ we set

$$F(x) = \{y \in D : f(x,y) > \lambda\}$$

and

$$G(x) = \{y \in D : g(x,y) < \lambda\}.$$

Hence

$$F^{-1}y = \{x \in C : f(x,y) > \lambda\}$$

and

$$G^{-1}(y) = \{x \in C : g(x,y) < \lambda\}.$$

From (3.3), we have

$$F^{-1}y \neq \emptyset, \quad G(x) \neq \emptyset \quad \text{for all } x \in C,\ y \in D.$$

Furthermore, from conditions (ii) and (iii) we get that $F(x)$ and $G^{-1}(y)$ are open, and $G(x)$ and $F^{-1}y$ are sub-admissible.

From Theorem 3.4, there exists $(x_0, y_0) \in C \times D$ such that: $y_0 \in Fx_0 \cap Gx_0$. This is equivalent to $f(x_0, y_0) > \lambda$ and $g(x_0, y_0) < \lambda$, a contradiction to i). The theorem is proved. ∎

Corollary 3.6. *Let C, D be as in Theorem* 3.4 *and let $f : C \times D \to \mathbb{R}$ satisfy*

i) *For each $x \in C$, the function $f(x,.)$ is quasi-convex and lower semi continuous.*

ii) *For each $y \in D$, the function $f(.,y)$ is quasi-concave and upper semicontinuous.*

Then

$$\inf_{y\in D}\sup_{x\in C} f(x,y) = \sup_{x\in C}\inf_{y\in D} f(x,y).$$

4. Fixed Point Theorems for Multivalued Maps

In [12] the authors have established a generalization of the well-known Ky Fan fixed point theorem for multivalued usc mappings, using the KKM-maps principle for mappings with open values due to Shih in [14]. In this section, following the same idea of [12], we establish an analogous result in hyperconvex metric spaces.

Lemma 4.1. (KKM-maps principle for open sets) *Let C be a nonempty compact admissible subset in a hyperconvex metric space X and A be a finite subset of C. Suppose that $G : A \to 2^C$ is a KKM-map with open values. Then $\bigcap_{x\in A} G(x) \neq \emptyset$.*

Proof. To prove this lemma, it suffices to show that there exists a KKM-map $F : A \to 2^C$ with closed values such that $F(x) \subset G(x)$, for all $x \in A$.

For each $y \in G(A) = \bigcup_{x\in A} G(x)$, there exists $r_y > 0$ such that the open ball $\overset{\circ}{B}(y, r_y) \subset G(x)$ for such $x \in A$ that $y \in G(x)$. Taking any subset α of A, we have

$$G(\alpha) = \bigcup\{G(x) : x \in \alpha\} \subset \bigcup\{\overset{\circ}{B}(y, r_y) : y \in G(\alpha)\}.$$

Since G is a KKM-map, we get

$$\mathrm{ad}(\alpha) \subset G(\alpha) \subset \bigcup\{\overset{\circ}{B}(y, r_y) : y \in G(\alpha)\}.$$

Because $\mathrm{ad}(\alpha)$ is closed in the compact set C, hence $\mathrm{ad}(\alpha)$ is also compact. So, there exists a finite subset $B(\alpha)$ of $G(\alpha)$ such that

$$\mathrm{ad}(\alpha) \subset \bigcup\{\overset{\circ}{B}(y, r_y) : y \in B(\alpha)\}.$$

Set $B = \bigcup_{\alpha} B_\alpha$. Clearly B is a finite set.

For each $x \in A$, we set

$$F(x) = \bigcup\{B(y, r_y) : y \in B \cap G(x)\}.$$

Obviously Fx is a closed set.

For each $y \in G(x)$, from $B(y, r_y) \subset G(x)$ we get $F(x) \subset G(x)$.

Now we shall prove that F is a KKM-map.

For $\alpha \subset A$ and $z \in \mathrm{ad}(\alpha)$, there exists $y \in B_\alpha \subset B$ such that $z \in B(y, r_y)$. Since $B_\alpha \subset G(\alpha)$, there exists $x \in \alpha$ such that $y \in G(x)$. This implies $y \in B \cap G(x)$. Hence $z \in F(x)$. So $\mathrm{ad}(\alpha) \subset \bigcup_{x \in \alpha} F(x)$. The lemma is proved. ■

Theorem 4.2. *Let X be a hyperconvex metric space and C a nonempty compact admissible subset of X. Suppose that $T : C \to \mathcal{A}(C)$ is an upper semicontinuous map. Then T has a fixed point.*

Proof. Since C is compact, for each $r > 0$ there exists a finite subset $\{x_1, x_2, \ldots, x_n\}$ of C such that

$$C \subset \bigcup_{i=1}^{n} \overset{\circ}{B}(x_i, r). \tag{4.1}$$

We set

$$F(x_i) = \{x \in C : Tx \cap B(x_i, r) = \emptyset\}.$$

Since T is upper semicontinuous, we have that $F(x_i)$ is open. Since (4.1) implies $\bigcap_{i=1}^{n} F(x_i) = \emptyset$, so F cannot be a KKM-map. Hence, there exist x_r^*, $x_{i_1}, \ldots, x_{i_k}$ such that

$$x_r^* \in \mathrm{ad}\{x_{i_1}, \ldots, x_{i_k}\} \quad \text{and} \quad x_r^* \notin \bigcup_{j=1}^{k} F(x_{ij}), \tag{4.2}$$

From (4.2), we have

$$Tx_r^* \cap B(x_{ij}, r) \neq \emptyset, \quad \text{for all} \quad j = 1, 2, \ldots, k.$$

Set $L = \mathrm{ad}\{x_1, \ldots, x_n\} \subset C$ and $M = \{x \in L : Tx_r^* \cap B(x, r) \neq \emptyset\}$. Because $Tx_r^* \in \mathcal{A}(C)$ we get $N_r(Tx_r^*) \in \mathcal{A}(C)$. On the other hand, $M = N_r(Tx_r^*) \cap$

$L \in \mathcal{A}(C)$ and $x_{ij} \in M$ for each $j = 1, \ldots, k$. This implies $x_r^* \in M$. So $d(x_r^*, Tx_r^*) \leq r$, and there is $y_r^* \in Tx_r^*$, such that $d(x_r^*, y_r^*) \leq r$. Let $r = \dfrac{1}{n}$, we get two sequences $\{x_n^*\}$, $\{y_n^*\}$ such that

$$d(x_n^*, y_n^*) \leq \frac{1}{n} \quad \text{and} \quad y_n^* \in Tx_n^*.$$

Since C is compact, we may suppose that there exists $x_0^* \in C$ such that $x_n^* \to x_0^*$. Hence $y_n^* \to x_0^*$. Since T is upper semicontinuous and C is compact, we have that the graph of T is closed. From the above observation we get $x_0^* \in Tx_0^*$ and the theorem is proved. ■

Remark 4.3. By a quite different method, this result was obtained by Yuan in [17] (see also [16]).

Remark 4.4. The notion of condensing mappings was introduced by Sadovski for Banach spaces in [13]. Kirk and Shin in [9] have obtained analogous results for single-valued mappings in hyperconvex spaces. Combining Theorem 4.2 and the method used in [9] one easily gets the following result for multivalued mappings.

Theorem 4.5. *Let C be an admissible set in a hyperconvex metric space, T a multivalued usc condensing in C with admissible values. Then T has a fixed point.*

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