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Uniqueness Theorems for Harmonic and Separately Harmonic Entire Functions on $\mathbb{C}^N$

Bachir Djebbar

Department of Computer Sciences,
University of Sciences and Technology "M. B" of Oran,
B.P 1505, El M'naouer Oran 31000, Algeria

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Abstract. For harmonic and separately harmonic functions, we give results similar to the Carlson-Boas theorem. We give also harmonic analogous of the Polya and Guelfond theorems.

1. Introduction

The well known classical theorem of Carlson (see [2, p.153]) states that an entire holomorphic function of exponential type$< \pi$ (i.e. f satisfies an inequality of the form $|f(z)| \leq A\exp(\tau\,|z|)$ with $\tau < \pi$) must vanish identically if it vanishes on $\mathbb{N}$.

In [3] Boas extended Carlson's theorem to harmonic functions and proved the following theorem:

Theorem 1.1. (Boas theorem) *Let h be an entire harmonic function on $\mathbb{C}$ of exponential type $< \pi$.*
If
$$h(z) = 0 \quad for \quad z = 0, \pm 1, \pm 2, \dots, i, i \pm 1, i \pm 2, \dots \tag{1}$$
Then $h \equiv 0$.

Similarly, Ching in [5] *showed that the same conclusion holds under the conditions*

i) *h is of exponential type $< \pi$.*
ii) *$h(z) = 0$ for $z = 0, \pm 1, \pm 2, \dots, \pm i, \pm 2i, \dots$*
iii) *$h(z) = -h(-z)$ for all complex z.*

In [1] Armitage gives a similar result for harmonic entire function in $\mathbb{R}^N$. Let us recall the classical:

Theorem 1.2. (Polya Theorem [2]) *Let f be an entire function on $\mathbb{C}$, of exponential type $< \log 2$. If $\{f(n), n \in \mathbb{N}\} \subset \mathbb{Z}$, then f is a polynomial.*

Guelfond gives in [7] a similar result for an entire function that takes integers values on a sequence (β_n) under some growth condition near infinity.

Theorem 1.3. (Guelfond Theorem [7]) *Let g be an entire function on $\mathbb{C}$, β an integer greater than one. If $g(\beta^n)$ are integers for $n = 1, 2, \ldots$ and g satisfies the inequality:*

$$\log|g(z)| \leqslant \frac{\log^2|z|}{4\log\beta} - \frac{1}{2}\log|z| - \omega\big(|z|\big),$$

where $\omega : \mathbb{R}^+ \to \mathbb{R}$ satisfies $\lim\limits_{r\longrightarrow\infty} \omega(r) = \infty$ then g is a polynomial.

In this paper we give a result similar to Boas theorem but under different conditions. Our proof is based on the properties of a polynomial basis established in [6]. We extend this result to separately harmonic functions. We give also a Guelfond and Polya type theorem in the case of harmonic function.

2. Notations and Results

For all $z = re^{i\theta} \in \mathbb{C}$ and $n \in \mathbb{N}$ we put:

$$e_1(z) \equiv 1, \quad e_n(z) = \begin{cases} r^k \cos k\theta, & \text{if} \quad n = 2k \quad k \geq 1, \\ r^k \sin k\theta, & \text{if} \quad n = 2k+1 \quad k \geq 1. \end{cases} \tag{2}$$

The sequence $(e_j)_{j\geq 1}$ of harmonic polynomials with $\deg(e_j) = [j/2]$ ([ ] designates the entire part) is a basis for the space $H(\mathbb{C})$ of all entire harmonic functions. Moreover for all function $h \in H(\mathbb{C})$, we have the following relation between the growth of h and its coefficients in the basis (e_j).

Theorem 2.1. [6] *Let h be an entire harmonic function, and let $h(z) = \sum\limits_{j=1}^{\infty} a_j e_j(z)$ be an expansion according to the basis $(e_j)_{j\geq 1}$. Then the growth order ρ of h is given as follows*

$$\rho = \limsup_{j\to\infty} \frac{[j/2]\log[j/2]}{-\log|a_j|}. \tag{3}$$

When $\rho \in]0, +\infty[$, the growth type τ of h is given by

$$\tau = \limsup_{j\to\infty} \frac{[j/2]}{e\rho}(|a_j|)^{\dfrac{\rho}{[j/2]}}. \tag{4}$$

We will prove the following results.

Theorem 2.2. *Let h be an entire harmonic function on $\mathbb{C}$ of exponential type $< \pi$. If $h(z) = 0$ for $z = 0, 1, 2, \dots$ and $h(z) = h(\overline{z})$ then $h \equiv 0$ on $\mathbb{C}$.*

Theorem 2.3. *Let h be an entire separately harmonic function on $\mathbb{C}^N$ of exponential type $< \pi$ with respect to the norm $|z| = \sup_j |z_j|$ (i.e : $|h(z)| \leq A \exp(\tau |z|)$ with $\tau < \pi$).*

For $m \in \{0, 1, ..., N\} \subset \mathbb{N}$ let:

$E^m = \{(z_1, ..., z_N) \in \mathbb{C}^N : z_{m+1} = \cdots = z_N = 0\}$ and $L^m = \{(z_1, ...z_N) \in E^m : z_j \in \mathbb{N}.... \text{ for } j = 1, ...m\}$.

If $h \equiv 0$ on L^m and $h(z_1, ...z_{j-1}, z_j, z_{j+1,....} z_N) = h(z_1, ..., \overline{z_j}, ...z_N)$; $j = 1...., m$, then $h \equiv 0$ on E^m.

Corollary 2.4. *Let h be an entire separately harmonic function on $\mathbb{C}^N$ of exponential type $< \pi$. If $h(z_1, ..., z_N) = 0$ for z_j=0, 1...., and $h(z_1, ...z_{j-1}, z_j,_{j+1}, ..., z_N) = h(z_1, ..z_{j-1}, \overline{z_j}, z_{j+1,}...., z_N)$; $j = 1, .., N$ then $h \equiv 0$ on $\mathbb{C}^N$.*

Corollary 2.4 is a direct consequence of Theorem 2.3.

Theorem 2.5. [The harmonic analogous of Guelfond theorem] *Let h be an entire harmonic function on $\mathbb{R}^2 \approx \mathbb{C}$ and $q \in \mathbb{Z}$ such that $|q| > 1$. Suppose that*

i) $\left\{ h(q^n, 0) \dfrac{\partial h}{\partial y}(q^n, 0),\ \ n \in \mathbb{N} \right\} \subset \mathbb{Z}$,

ii) *There is a function $\omega : \mathbb{R}_+^* \longrightarrow \mathbb{R}_+$ such that: $\lim_{r \longrightarrow \infty} r^2 \omega(r) = 0$ and*

$$M(h, r) \leqslant \frac{\omega(r)}{\sqrt{r}} \exp\left(\frac{\log^2 r}{4 \log |q|}\right), \quad \forall r > 0,$$

where $M(h, r) = \sup_{|z|=r} |h(z)|$.

Then h is a polynomial.

Theorem 2.6. (The harmonic analogous of Polya Theorem) *Let h be an entire harmonic function on $\mathbb{R}^2$. If h satisfies:*

i) $\left\{ h(n, 0), \dfrac{\partial h}{\partial y}(n, 0),\ \ n \in \mathbb{N} \right\} \subset \mathbb{Z}$,

ii) $M(h, r) \leqslant A\ \exp(Cr), C < \log 2$,

then h is a polynomial.

3. Proofs

Proof of Theorem 2.2. Let h be an entire harmonic function on $\mathbb{C}$ of exponential type $\tau < \pi$, and let $h(z) = \sum_{j=1}^{\infty} a_j e_j(z)$ be its expansion in $(e_j)_{j \in \mathbb{N}}$. One can write

$$h(z) = \sum_{j=1}^{\infty} a_{2j} e_{2j}(z) + \sum_{j=1}^{\infty} a_{2j+1} e_{2j+1}(z).$$

The condition $h(z) = h(\overline{z}),\ \ \forall z \in \mathbb{C}$ implies that $a_{2j+1} = 0 \quad \forall j \geq 0$. However, for all $m \in \mathbb{N}$, we get $h(m) = \sum_{j=1}^{\infty} a_j e_j(m) = \sum_{j=1}^{\infty} a_{2j} m^j = 0$. Consider the function $f(z) = \sum_{j=1}^{\infty} a_{2j} z^j$ $(z \in \mathbb{C})$ which is entire on $\mathbb{C}$ and of exponential type $\beta < \tau < \pi$. We have: $f(m) = \sum_{j=1}^{\infty} a_{2j} m^j = h(m) = 0$ for all $m \in \mathbb{N}$, and hence $f \equiv 0$ by Carlson Theorem, so $a_{2j} = 0$ for $j = 1, 2, \ldots$, which finally implies that $h \equiv 0$. ■

Proof of Theorem 2.3. We prove Theorem 2.3 by induction on m. The case $m = 1$ is an immediate consequence of Theorem 2.2 applied to the function $v(z) = h(z, 0 \ldots, 0),\ \ z \in \mathbb{C}$. Suppose the theorem is true for m such that $1 \leq m < N$. Assume that h satisfies the hypotheses of the theorem for $m+1$. Hence h is an entire separately harmonic function of exponential type $\sigma < \pi$ and satisfies the condition:

$$\text{if } h \equiv 0 \text{ on } L^{m+1} \text{ and } h(z_1, \ldots, z_j, \ldots, z_N) = h(z_1, \ldots, \overline{z_j}, \ldots z_N),\quad j = 1, \ldots, m+1,$$
$$\text{then}\quad h(z_1, \ldots, z_m, 0, \ldots 0) = 0, \quad \forall (z_1, \ldots, z_m \in \mathbb{N}),$$

since $h \equiv 0$ on L^m then $h \equiv 0$ on E^m. So $h(z_1, \ldots, z_m, 0, \ldots, 0) = 0,\ \ \forall (z_1, \ldots, z_m) \in \mathbb{C}^m$. Let $k \in \mathbb{N}$ and consider the translation:

$$T_k : \mathbb{C}^N \to \mathbb{C}^N$$
$$(z_1, \ldots, z_N) \to (z_1, \ldots, z_m,\ z_{m+1} + k, z_{m+2}, ..., z_N)$$

$h \circ T_k(z_1, \ldots, z_m, 0, \ldots, 0) = h(z_1, z_2, \ldots z_m, k, 0, ...0)$ then $h \circ T_k \equiv 0$ on L^m.$h \circ T_k$ is a entire separately harmonic function of exponential type $< \pi$ which satisfies:

$$\begin{aligned} h \circ T_k(z_1, ..., \overline{z_j}, ..., z_N) &= h(z_1, ..., \overline{z_j}, ...z_m\ ,\ z_{m+1} + k, ..., z_N) \\ &= h(z_1, ..., z_j, ..., z_m, z_{m+1} + k, ...z_N), \quad j = 1, .., m \end{aligned}$$

then $h \circ T_k \equiv 0$ on E^m,

$$\text{i.e. } h \circ T_k(z_1, \ldots, z_m, 0, \ldots, 0) = h(z_1, \ldots, z_m, k, 0, \ldots, 0) = 0,\ \forall z_j \in \mathbb{C}; j = 1, \ldots, m$$

and, $k \in \mathbb{N}$. For $z_1, \ldots, z_m$ fixed in $\mathbb{C}$, we consider the function: $g(z) = h(z_1, \ldots, z_m, z, 0, \ldots, 0)\ \ z \in \mathbb{C}$. g is an entire separately harmonic function of exponential type $\leq \sigma < \pi$, and satisfies:

$$\begin{cases} g(\overline{z}) = h(z_1, ...z_m, \overline{z}, 0, \ldots, 0) = h(z_1, \ldots, z_m, z, 0, \ldots, 0) = g(z)\ \forall z \in \mathbb{C} \\ g(k) = h(z_1, \ldots, z_m, k, 0, \ldots, 0) = h \circ T_k(z_1, ...z_m, 0.....0) = 0, \ \ \forall k \in N. \end{cases}$$

By Theorem 2.2 we deduce that $g(z) = 0,\ \ \forall z \in \mathbb{C}$. Since $(z_1, \ldots, z_m)$ is arbitrarily fixed in $\mathbb{C}^m$ then

$$h(z_1, \dots, z_{m,} z, 0, \dots, 0) = 0, \quad \forall (z_1, \dots, z_m) \in \mathbb{C}^m \quad \text{and} \quad \forall z \in \mathbb{C}.$$

Consequently $h(z_1, \dots, z_m, z_{m+1}, 0, \dots, 0) = 0 \quad \forall (z_1, \dots, z_{m+1}) \in \mathbb{C}^{m+1}$. So $h \equiv 0$ on E^{m+1}. The induction is complete. ∎

Proof of Theorem 2.5. Let h be an entire harmonic function and let $f(z) = \sum_{k=0}^{\infty} (a_k + ib_k)\, z^k$ be its Taylor series expansion.

We consider the function $F(z) = \frac{1}{2}\big[f(z) + \overline{f(\overline{z})}\big]$. Then F is an holomorphic entire function and $F(z) = \sum_{k=0}^{\infty} a_k\ z^k$, $F(q^n) = \text{Re}\ f(q^n) = h(q^n, 0) \in \mathbb{Z}$. By the following Carathéodory's inequality [2]

$$M(f, r) \leq \big|f(0)\big| + \frac{2r}{R - r}\big[M(\text{Re}\, f, R) - \text{Re}\, f(0)\big], \quad 0 < r < R,$$

we deduce that F satisfies conditions of the theorem of Gurelfond in the holomorphic case, so F is a polynomial.

There is an integer N such that $a_k = 0, \quad \forall k > N$. Consider now the holomorphic entire function H defined by:

$$H(z) = \frac{1}{2}\big[if^{'}(z) + \overline{if'(\overline{z})}\big] = \sum_{k=1}^{\infty} -2kb_k z^{k-1}.$$

Then

$$H(q^n) = \frac{1}{2}\big[if^{'}(q^n) + \overline{if'(q^n)}\big] = -2\frac{\partial h}{\partial y}(q^n, 0) \in \mathbb{Z}, \quad \forall\ n \in \mathbb{N}.$$

The classical result

$$\begin{cases} \text{if } g \text{ is holomorphic in } |z| < R + \varepsilon \text{ then we have :} \\ |g^{'}(z)| \leq \dfrac{R}{(R - r)^2} M(g, R) \ \text{ for } \ |z| \leqslant r < R, \end{cases}$$

gives

$$M(f^{'}, r) \leqslant (r + 1) M(f, r + 1), \quad \forall r > 0.$$

H satisfies the Gurelfond's Theorem conditions in the holomorphic case, so H is a polynomial; there exist N' such $b_k = 0, \ \forall k > N'$. Then f is a polynomial, and consequently h is also a polynomial. ∎

Proof of Theorem 2.6. Very similar to the proof of Theorem 2.5.

Remark. It would certainly be interesting to give Gelfond and Polya type theorems in the general case of harmonic entire functions in $\mathbb{R}^N$.

References

1. D. H. Armitage, Uniqueness theorems for harmonic functions which vanish at lattice points, *J. Approximation Theory* **26** (1979) 259–268.
2. R. Boas, Entire functions, Academic Press., New York, 1954.
3. R. Boas, A uniqueness theorems for harmonic functions, *J. Approximation Theory* **5** (1972) 425–427.
4. M. Brelot, *Eléments de la théorie du Potentiel*, Centre de Documentation Universitaire, Paris, 1969
5. C. H.Ching, An interpolation formula for harmonic functions, *J. Approximation Theory* **15** (1975) 50–53.
6. B. Djebbar, *Approximation Polynomiale et Croissance des Fonctions N-harmoniques - thèse de Doctorat de 3^{eme} Cycle*, Univ Paul-Sabatier, Toulouse, 1987.
7. A. O. Guelfond, *Calcul des Différences finies,* Dunod Paris, 1963.
8. Ü. Kuran, On Brelot, Choquet axial polynomials, *J. Lond. Math. Soc* **4** (1971) 15–26.
9. P. Lelong and L. Gruman, *Entire Functions of Several Complex Variables,* Springer Verlag, Berlin–Heidelberg, 1986.
10. Th. V. Nguyen, Bases communes pour certains espaces de fonctions harmoniques, *Bull. Sci. Math.* **97** (1973) 33–49.
11. Th. V. Nguyen, Bases polynomiales et approximation des fonctions séparément harmoniques dans $\mathbb{C}^\nu$, *Bull. Sci. Math.* 2^e Série **113** (1989) 349–361.
12. Th. V. Nguyen and B. Djebbar, Propriétés Assymptotiques d'une suite orthonormale de polynômes harmoniques, *Bull. Sci. Math.* 2^e Série **113** (1989) 239–251.