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Fibonacci Length of Direct Products of Groups

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Abstract. For a non-abelian finite group $G = \langle a_1, a_2, \dots, a_n \rangle$ the Fibonacci length of G with respect to the ordered generating set $A = \{a_1, a_2, \dots, a_n\}$ is the least integer l such that for the sequence of elements $x_i = a_i,\ 1 \le i \le n,\ x_{n+i} = \prod_{j=1}^{n} x_{i+j-1},\ i \ge 1$, of G, the equations $x_{l+i} = a_i,\ 1 \le i \le n$ hold. The question posed in 2003 by P. P. Campbell that "Is there any relationship between the lengths of finite groups G, H and $G \times H$?" In this paper we answer this question when at least one of the groups is a non-abelian 2-generated group.

1. Introduction

Let $G = \langle A \rangle$ be a finite non-abelian group where, $A = \{a_1, a_2, \dots, a_n\}$ is an ordered generating set. The sequence

$$x_i = a_i,\ 1 \le i \le n,\ x_{n+i} = \prod_{j=1}^{n} x_{i+j-1}, \quad i \ge 1$$

of the elements of G, denoted by $F_A(G)$, is called the Fibonacci orbit of G with respect to A, and the least integer l for which the equations $x_{l+i} = x_i,\ 1 \le i \le n$ hold, is called the Fibonacci length of G with respect to A and will be denoted by $LEN_A(G)$. The notions of basic Fibonacci orbit and basic Fibonacci length are almost similar. Indeed, the basic Fibonacci orbit of length m is also defined to be the same sequence of the elements of G such that m is the least integer where the equations $x_1\theta = x_{m+1},\ x_2\theta = x_{m+2},\ \dots,\ x_n\theta = x_{m+n}$ hold for some $\theta \in Aut(G)$. The integer m is called the basic Fibonacci length of G with respect to A and will be denoted by $BLEN_A(G)$. It is proved in [2] that $BLEN_A(G)$ divides $LEN_A(G)$, for 2-generated finite groups. Obviously $A \cup \{b_1, b_2, \dots, b_k\}, k \geq 1, b_1 = b_2 = \cdots = b_k = 1$ is also a generating set for G, however, the Fibonacci lengths with respect to A and with respect to this generating set are different. We will use this new generating set to make possible our calculations.

Since 1990 the Fibonacci length has been studied and calculated for certain classes of finite groups (one may see [1, 2, 4, 7], for examples), and certain original questions have been posed by Campbell in [5]. We answer one of these questions which is: how can one calculate the Fibonacci length of $G \times H$ (external direct product of groups G and H) in terms of the Fibonacci lengths of G and H?

For a finite number of the groups $G_1,\ G_2, \dots,\ G_k$ we use the notation $Dr^k G_i$ for the external direct product (or, simply the direct product) $G_1 \times G_2 \times \cdots \times G_k$.

Our first attempt is to calculate $LEN_{\{a,b,e\}}(G \times H)$ where $G = \langle a, b\rangle$ is a non-abelian finite group and $H = \langle e\rangle$ is a cyclic group of order m; then we will calculate $LEN_{\{a,b,c,d\}}(G \times H)$ where $G = \langle a, b\rangle$ and $H = \langle c, d\rangle$ are non-abelian finite groups.

For every positive integer m, we define the positive integer $k(m, 3)$ to be the minimal length of the period of series $(g_i \mod m)_{-\infty}^{+\infty}$, where

$$g_0 = g_1 = 0,\ g_2 = 1,\ g_i = g_{i-1} + g_{i-2} + g_{i-3}.$$

This number is similar to the Wall number $k(m)$ of Wall [10], where the Wall number is defined for the series $(f_i \mod m)_{-\infty}^{+\infty}$ such that

$$f_0 = f_1 = 1,\ f_i = f_{i-1} + f_{i-2}.$$

Following Wall [10] one may also prove the existence of $k(m, 3)$ for every positive integer m. We will use 1 for 1_G, the identity of the group G. Our main results are the following propositions. Propositions A, B and C give explicit formulas for computing the lengths of direct products of some groups, and Propositions D and E are the generalization of [2] for $LEN_{\{a,b\}}(D_{2n})$ and $LEN_{\{a,b\}}(Q_{2^n})$ (see [2]).

Proposition A. *For every 2-generated non-abelian finite group $G = \langle a, b\rangle$ and every cyclic group $H = \langle e\rangle$ of order m,*

$$LEN_{\{a,b,e\}}(G \times H) = l.c.m.(k(m, 3), LEN_{\{a,b,1\}}(G)).$$

Proposition B. *For every 2-generated non-abelian finite groups $G = \langle a, b\rangle$ and $H = \langle c, d\rangle$,*

$$LEN_{\{a,b,c,d\}}(G \times H) = l.c.m.(LEN_{\{a,b,1,1\}}(G), LEN_{\{1,1,c,d\}}(H)).$$

Proposition C. *For a positive integer n, let $G_i = \langle a_i, b_i\rangle$, $1 \le i \le n$ be non-abelian 2-generated finite groups. For every i ($1 \le i \le n$) define $A_i = \{a_{i,1}, a_{i,2}, \dots, a_{i,2n}\}$, as follows,*

$$a_{i,j} = \begin{cases} b_i, & \text{if } j = 2i, \\ a_i, & \text{if } j = 2i-1, \\ 1_{G_i}, & \text{if } j \neq 2i, 2i-1, \end{cases}$$

($j = 1, 2, \dots, 2n$). Then,

$$LEN_{\{a_1,\ b_1,\ a_2,\ b_2,\dots,\ a_n,\ b_n\}}(Dr^n G_i) = l.c.m_{1\le i\le n} LEN_{A_i}(G_i).$$

Consider the dihedral groups $D_{2n} = \langle a, b | a^2 = b^n = (ab)^2 = 1\rangle$ and the generalized quaternion groups $Q_{2^n} = \langle a, b | a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, b^{-1}ab = a^{-1}\rangle$, where $n \ge 3$. The followings are two numerical results on these groups.

Proposition D.

(i) *For every $n \ge 3$ and $k \ge 1$, $LEN_{\{a_1,a_2,\dots,a_k,a,b\}}(D_{2n}) = 2k + 6$ where, $a_1 = a_2 = \cdots = a_k = 1$.*

(ii) *For every $n \ge 3$, $LEN_{\{a,b,1\}}(D_{2n}) = \frac{8n}{g.c.d(4,n)}$, $LEN_{\{a,b,1,1\}}(D_{2n}) = \frac{10n}{g.c.d(4,n)}$, and in general, for an integer $k \ge 3$ and for every i ($1 \le i \le k$) we define $A_i = \{a_1, a_2, \dots, a_{k+1}, a_{k+2}\}$, where*

$$a_m = \begin{cases} a, & \text{if } m = i, \\ b, & \text{if } m = i+1, \\ 1, & \text{otherwise.} \end{cases}$$

Then, $LEN_{A_i}(D_{2n}) = \frac{(2k+6)n}{g.c.d(n,4)} q_{i,k,n}$, where, $q_{i,k,n}$ is the least positive integer such that for all values of r ($3 \le r \le k - i + 1$),

$$r! \mid 2^r \Big(\frac{2q_{i,k,n}}{g.c.d(4,n)}\Big)\Big(\frac{2nq_{i,k,n}}{g.c.d(4,n)} + 1\Big) \dots \Big(\frac{2nq_{i,k,n}}{g.c.d(4,n)} + r - 1\Big).$$

Proposition E. *For an integer $k \ge 3$ and for every i ($1 \le i \le k$) let A_i be as in the Proposition D. Then, $LEN_{A_i}(Q_{2^n}) = 2k + 6$.*

2. Proofs

Proof of Proposition A. Let $H = \langle e\rangle$ and $G = \langle a, b\rangle$, so $G = \langle a, b, 1\rangle$. Consider the Fibonacci sequence of the elements of $G = \langle a, b, 1\rangle$ and $G \times H = \langle a, b, e\rangle$ as

$$x_1 = a,\ x_2 = b,\ x_3 = 1,\ x_i = x_{i-3}x_{i-2}x_{i-1},\ i \ge 4,$$

and

$$y_1 = a,\ y_2 = b,\ y_3 = e,\ y_i = y_{i-3}y_{i-2}y_{i-1},\ i \ge 4,$$

respectively. For every $n \ge 1$, $y_n = x_n.e^{g_n}$, for,

$$y_1 = a = x_1 = x_1.e^0 = x_1.e^{g_1},$$
$$y_2 = b = x_2 = x_2.e^0 = x_2.e^{g_2},$$
$$y_3 = e = 1.e = x_3.e^1 = x_3.e^{g_3}.$$

Using an induction method, the hypothesis gives us

$$y_{n+1} = y_{n-2}y_{n-1}y_n = x_{n-2}.e^{g_{n-2}}.x_{n-1}.e^{g_{n-1}}.x_n.e^{g_n}$$

and since e is a central element of $G \times H$, then

$$y_{n+1} = x_{n-2}.x_{n-1}.x_n.e^{g_n+g_{n-1}+g_{n-2}} = x_{n+1}e^{g_{n+1}}.$$

If $l = LEN_{\{a,b,e\}}(G \times H)$ then l is the least integer such that $y_{l+1} = a$, $y_{l+2} = b$ and $y_{l+3} = e$. This leads to the equations $x_{l+1}.e^{g_{l+1}} = a$, $x_{l+2}.e^{g_{l+2}} = b$ and $x_{l+3}.e^{g_{l+3}} = e$. Equivalently, we get two classes of the equations as follows:

$$x_{l+1} = a,\ x_{l+2} = b,\ x_{l+3} = 1$$

and

$$g_{l+1} \equiv 0 \mod m,\ g_{l+2} \equiv 0 \mod m,\ g_{l+3} \equiv 1 \mod m.$$

The first and second classes of the equations prove the divisablity of l by $LEN_{\{a,b,1\}}(G)$ and $k(m,3)$, respectively. Since l is the least integer satisfying these properties, so the result follows. ∎

Proofs of Propositions B and C. Let $G = \langle a,b \rangle$, $H = \langle c,d \rangle$ and $G \times H = \langle a,b,c,d \rangle$ where, $[a,c] = [a,d] = [b,c] = [b,d] = 1$. Consider G and H as $G = \langle a,b,1,1 \rangle$ and, $H = \langle 1,1,c,d \rangle$. Then, the sequences of elements of these groups with respect to these ordered generating sets are

$$x_1 = a,\ x_2 = b,\ x_3 = x_4 = 1,\ x_{i+1} = x_{i-3}x_{i-2}x_{i-1}x_i,\ i \geq 4,$$
$$y_1 = y_2 = 1,\ y_3 = c,\ y_4 = d,\ y_{i+1} = y_{i-3}y_{i-2}y_{i-1}y_i,\ i \geq 4,$$
$$z_1 = a,\ z_2 = b,\ z_3 = c,\ z_4 = d,\ z_{i+1} = z_{i-3}z_{i-2}z_{i-1}z_i,\ i \geq 4,$$

respectively.

We can then prove $z_n = x_ny_n$ by an induction method on n. Now let l be the least positive integer such that all of the equations

$$z_{l+1} = a,\ z_{l+2} = b,\ z_{l+3} = c,\ z_{l+4} = b.$$

hold (i.e. $l = LEN_{\{a,b,c,d\}}(G)$), then by substituting for z_{l+1}, z_{l+2}, z_{l+3} and z_{l+4} the result follows by a similar way to that of Theorem A. Theorem C may be proved in a similar way. ∎

Proof of Proposition D.

(i) For $k \geq 1$ we consider the Fibonacci sequence of the elements. For instance if $D_{2n} = \langle 1,1,a,b \rangle$ i.e., $k = 2$ then we have $x_1 = x_2 = 1,\ x_3 = a,\ x_4 = b,\ x_5 = ab,\ x_6 = (ab)^2 = 1,\ x_7 = (ab)^4 = 1,\ x_8 = bab = a,\ x_9 = aba = b^{-1},\ x_{10} = ab^{-1},\ x_{11} = x_{12} = 1,\ x_{13} = a,\ x_{14} = b$. So, $LEN_{\{1,1,a,b\}}(D_{2n}) = 10$. In general every element of the sequence $x_1 = x_2 = \cdots = x_k = 1_G,\ x_{k+1} = a,\ x_{k+2} = b,\ x_{k+3} = x_1x_2\ldots x_{k+2}, \ldots$ of the group $D_{2n} =< a_1, \ldots, a_k, a, b >$ may be represented as

$$x_i = \begin{cases} a, & \text{if } i \equiv -2 \text{ or } k+1 \mod 2k+6, \\ b, & \text{if } i \equiv k+2 \mod 2k+6, \\ b^{-1}, & \text{if } i \equiv -1 \mod 2k+6, \\ ab^{-1}, & \text{if } i \equiv 0 \mod 2k+6, \\ 1, & \text{otherwise.} \end{cases}$$

where, $i \geq k+3$. This may be proved by induction on i. So, considering the definition of the Fibonacci length gives the result at once.
(ii) Consider the Fibonacci sequence of $D_{2n} = \langle a, b, 1\rangle$ as

$$x_1 = a,\ x_2 = b, x_3 = 1,\ x_i = x_{i-3}x_{i-2}x_{i-1},\ i \geq 4.$$

For every $k \geq 4$, every element of this sequence can be represented by

$$x_k = \begin{cases} a, & \text{if } k \equiv 1 \mod 4, \\ b^{(-1)^{\frac{k-2}{4}}}, & \text{if } k \equiv 2 \mod 4, \\ b^{(-1)^{\frac{k-3}{4}} \cdot \frac{k-3}{2}}, & \text{if } k \equiv 3 \mod 4, \\ ab^{(-1)^{\frac{k+4}{4}} \cdot \frac{k-2}{2}}, & \text{if } k \equiv 0 \mod 4. \end{cases}$$

If $l = LEN_{\{a,b,1\}}(D_{2n})$ then $x_{l+1} = a$, $x_{l+2} = b$, and $x_{l+3} = 1$. So $l \equiv 0 \mod 4$, and then $b^{(-1)^{\frac{l}{4}}} = b$ yields $l \equiv 0 \mod 8$. Moreover, $b^{(-1)^{\frac{l}{4}} \cdot \frac{l}{2}} = 1$ holds if and only if $\frac{l}{2}$ is divisible by n. Consequently, considering three cases for n as $n \equiv 0 \mod 4$, $n \equiv 1 \mod 4$ or $n \equiv 2 \mod 4$, we get $l = 2n$, $l = 8n$ or $l = 4n$, respectively. i.e., $l = \frac{8n}{g.c.d.(n,4)}$, as desired.

The Fibonacci sequence of the group $G = \langle a, b, 1, 1\rangle$ is the sequence

$$x_1 = a,\ x_2 = b, x_3 = x_4 = 1,\ x_i = x_{i-4}x_{i-3}x_{i-2}x_{i-1},\ i \geq 5.$$

It is easy to prove that for every $k \geq 5$,

$$x_k = \begin{cases} ab^{2(\frac{k}{5})^2-1}, & \text{if } k \equiv 0 \mod 5, \\ a, & \text{if } k \equiv 1 \mod 5, \\ b^{(-1)^{\frac{k-2}{5}}}, & \text{if } k \equiv 2 \mod 5, \\ b^{2(-1)^{\frac{k-3}{5}} \cdot \frac{k-3}{5}}, & \text{if } k \equiv 3 \mod 5, \\ b^{2(-1)^{\frac{k-4}{5}} \cdot \frac{k-4}{5} \cdot \frac{k+1}{5}}, & \text{if } k \equiv 4 \mod 5. \end{cases}$$

If $l = LEN_{\{a,b,1,1\}}(D_{2n})$ then in a similar way as for (ii) we deduce that $l \equiv 0 \mod 5$, and almost a simple computation gives us $l = \frac{10n}{g.c.d.(n,4)}$.

In general case every element of the Fibonacci sequence of D_{2n} with respect to A_i may be represented by

$$x_j = \begin{cases} a, & \text{if } j \equiv i \mod k+3, \\ b^{(-1)^s}, & \text{if } j \equiv i+1 \mod k+3, \\ b^{(-1)^s 2s}, & \text{if } j \equiv i+2 \mod k+3, \\ b^{(-1)^s 2s(s+1)}, & \text{if } j \equiv i+3 \mod k+3, \\ b^{(-1)^s \frac{2^3}{3!} s(s+1)(s+2)}, & \text{if } j \equiv i+4 \mod k+3, \\ \vdots & \\ b^{(-1)^s \frac{2^{k+1-i}}{(k+1-i)!} s(s+1)(s+2)\dots(s+k-i)}, & \text{if } j \equiv k+2 \mod k+3, \\ ab^{(-1)^s \alpha_j}, & \text{if } j \equiv 0 \mod k+3, \\ 1, & \text{otherwise,} \end{cases}$$

where $s = \left[\frac{j}{k+3}\right]$ and

$$\alpha_l = 1 + \sum_{r=1}^{k-i+1} \frac{2^r}{r!} t(t+1)\dots(t+r-1)$$

that $t = \frac{j}{k+3} - 1$. ($[x]$ is used for the integer part of the real x.) This may be proved by induction method on j. Let $l = LEN_{A_i}(D_{2n})$, by a similar method as above we get $l \equiv 0 \mod k+3$ and l must satisfy all of the relations

(i) $2 \mid \frac{l}{k+3}$,

(ii) $n \mid \frac{2l}{k+3}$,

(iii) $n \mid \frac{2^r}{r!}\left(\frac{l}{k+3}\right)\left(\frac{l}{k+3}+1\right)\dots\left(\frac{l}{k+3}+r-1\right)$, for all $3 \le r \le k-i+1$.

The relations (i) and (ii) yield $l = \frac{(2k+6)n}{g.c.d(4,n)} q_{i,k,n}$, where, $q_{i,k,n}$ is a positive integer, and by (iii) we get

$$n \mid \frac{2^r}{r!}\Big(\frac{2nq_{i,k,n}}{g.c.d(4,n)}\Big)\Big(\frac{2nq_{i,k,n}}{g.c.d(4,n)}+1\Big)\dots\Big(\frac{2nq_{i,k,n}}{g.c.d(4,n)}+r-1\Big),$$

for all r where, $3 \le r \le k-i+1$, and this holds if and only if

$$r! \mid 2^r\Big(\frac{2q_{i,k,n}}{g.c.d(4,n)}\Big)\Big(\frac{2nq_{i,k,n}}{g.c.d(4,n)}+1\Big)\dots\Big(\frac{2nq_{i,k,n}}{g.c.d(4,n)}+r-1\Big),$$

for all r where, $3 \le r \le k-i+1$. This completes the proof. ■

Note. Certain values of the sequence $\{q_{i,k,n}\}$ is given as follows, for instance,

$q_{i,k,n} = 1$ if $i = k$ or $i = k-1$, and $n \ge 3$,

$$q_{i,4,3} = \begin{cases} 3, & \text{if } i = 1 \textit{ or } 2, \\ 1, & \text{if } i = 3, \\ 4, & \text{if } i = 4, \end{cases}$$

$q_{i,4,4} = 1$, where, $i = 1,2,3,4$,

$$q_{i,4,6} = \begin{cases} 3, & \text{if } i = 1 \textit{ or } 2, \\ 1, & \text{if } i = 3 \textit{ or } 4. \end{cases}$$

Proof of Proposition E. Let $\{x_j\}_{j=1}^{\infty}$ be the Fibonacci sequence of Q_{2^n} with respect to A_i. We consider three cases

Case I. $i = 1$. In this case we have $x_1 = a,\ x_2 = b,\ x_3 = 1,\ \ldots,\ x_{k+2} = 1$ and for every j where, $k+3 \le j \le 2k+6$, we get

$$x_j = \begin{cases} ba^{-1}, & \text{if} \quad j = k+3, \\ b^2a^{-1}, & \text{if} \quad j = k+4, \\ b^3a^{-2}, & \text{if} \quad j = k+5, \\ b^2, & \text{if} \quad j = k+6, \\ b^3a^{-1}, & \text{if} \quad j = 2k+6, \\ 1, & \text{otherwise.} \end{cases}$$

Since then $x_{2k+7} = a,\ x_{2k+8} = b,\ x_{2k+9} = x_{2k+10} = \ldots = x_{3k+8} = 1$, so, $LEN_{A_1}(Q_{2^n}) = 2k+6$.

Case II. $i = k+1$. Then we get $x_1 = x_2 = x_3 = \ldots = x_k = 1,\ x_{k+1} = a,\ x_{k+2} = b$ and for every j where, $k+3 \le j \le 2k+6$,

$$x_j = \begin{cases} ba^{-1}, & \text{if} \quad j = k+3, \\ b^2, & \text{if} \quad j = k+4, \\ a^{-1}, & \text{if} \quad j = 2k+4, \\ b^3a^{-2}, & \text{if} \quad j = 2k+5, \\ ba^{-1}, & \text{if} \quad j = 2k+6, \\ 1, & \text{otherwise.} \end{cases}$$

So, $x_{2k+7} = x_{2k+8} = x_{2k+9} = \cdots = x_{3k+6} = 1,\ x_{3k+7} = a$ and $x_{3k+8} = b$. Then $LEN_{A_{k+1}}(Q_{2^n}) = 2k+6$.

Case III. Let $i \ne 1$ and $i \ne k+1$. Then $x_1 = a_1,\ x_2 = a_2, \ldots,\ x_k = a_k,\ x_{k+1} = a_{k+1},\ x_{k+2} = a_{k+2}$ and for every j where, $k+3 \le j \le 2k+6$,

$$x_j = \begin{cases} ba^{-1}, & \text{if} \quad j = k+3, \\ b^2 & \text{if} \quad j = k+4, \\ a^{-1} & \text{if} \quad j = k+i+3, \\ b^3a^{-2} & \text{if} \quad j = k+i+4, \\ b^2 & \text{if} \quad j = k+i+5, \\ b^3a^{-1} & \text{if} \quad j = 2k+6, \\ 1 & \text{otherwise.} \end{cases}$$

So $x_{2k+7} = a_1,\ x_{2k+8} = a_2, \ldots,\ x_{3k+8} = a_{k+2}$. Consequently $LEN_{A_i}(Q_{2^n}) = 2k+6$. This completes the proof. ■

3. Conclusions

We give here certain numerical results on LEN and $BLEN$ of the groups D_{2n}^k $(= Dr^k D_{2n}$, the direct product of k copies of $D_{2n})$ and $Q_{2^n}^k$ $(= Dr^k Q_{2^n})$, by applying the propositions of Sec. 2.

Remark 1. $LEN(Q_{2^n}^k) = 4k+2$.

Proof. By the Propositions C and E. ■

Remark 2. $LEN(D_{2n}^2) = \frac{10n}{g.c.d(4,n)}$ and $LEN(D_{2n}^k) = \frac{(4k+2)n}{g.c.d(4,n)} q_{1,2k-2,n}$, $k \geq 3$.

Proof. To the first part we use Proposition D and get $LEN_{\{a,b,1,1\}}(D_{2n}) = \frac{10n}{g.c.d(4,n)}$, and $LEN_{\{1,1,a,b\}}(D_{2n}) = 10$. Then Proposition B yields

$$LEN_{\{a,b,c,d\}}(D_{2n} \times D_{2n}) = l.c.m.(10, \frac{10n}{g.c.d(4,n)}) = \frac{10n}{g.c.d(4,n)}.$$

To prove the second part, consider the definition of $q_{i,k,n}$ and get

$$q_{j,2k-2,n} \mid q_{1,2k-2,n}, \;\; 2 \leq j \leq 2k-2.$$

Then $LEN_{A_j}(D_{2n}) \mid LEN_{A_1}(D_{2n})$, for every j $(2 \leq j \leq 2k-2)$ and Proposition C yields the result $LEN(D_{2n}^k) = \frac{(4k+2)n}{g.c.d(4,n)} q_{1,2k-2,n}$ as desired. ■

Remark 3. Let A_i be defined as in Proposition D. Then, for every $n \geq 3$
(i) $2 \times BLEN_{A_i}(D_{2n}) = LEN_{A_i}(D_{2n})$,
(ii) $BLEN_{A_i}(Q_{2^n}) = LEN_{A_i}(Q_{2^n})$.

Proof. Considering the Fibonacci sequences of elements and using the similar method as in Propositions D and E we get the results immediately. ■

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