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A Gagliardo-Nirenberg Inequality for Lorentz Spaces*

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Abstract. In this paper, essentially developing the method of [1] and [10], we give an extension of the Gagliardo-Nirenberg inequality to Lorentz spaces.

Let $\Phi : [0,\infty) \to [0,\infty)$ be a non-zero concave function, which is non-decreasing and $\Phi(0+) = \Phi(0) = 0$. We put $\Phi(\infty) = \lim_{t\to\infty} \Phi(t)$. For an arbitrary measurable function f we define

$$\|f\|_{N_\Phi} = \int_0^\infty \Phi\big(\lambda_f(y)\big)dy,$$

where $\lambda_f(y) = \text{mes}\{x \in \mathbb{R}^n : |f(x)| > y\}$, $(y \geq 0)$. The space $N_\Phi(\mathbb{R}^n)$ consisting of measurable functions f such that $\|f\|_{N_\Phi} < \infty$ is a Banach space. Denote by $M_\Phi(\mathbb{R}^n)$ the space of measurable functions g such that

$$\|g\|_{M_\Phi} = \sup\Big\{\frac{1}{\Phi(\text{mes }\Delta)}\int_\Delta |g(x)|dx : \ \Delta \subset \mathbb{R}^n, \ 0 < \ \text{mes }\Delta < \infty\Big\} < \infty.$$

Then $M_\Phi(\mathbb{R}^n)$ is a Banach space, see [7 - 9].

We have the following results [8 - 9]:

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Lemma 1. *If $f \in N_\Phi(\mathbb{R}^n)$, $g \in M_\Phi(\mathbb{R}^n)$ then $fg \in L_1(\mathbb{R}^n)$ and*

$$\int\limits_{\mathbb{R}^n} |f(x)g(x)|dx \le \|f\|_{N_\Phi}\|g\|_{M_\Phi}.$$

Lemma 2. *If $f \in N_\Phi(\mathbb{R}^n)$ then*

$$\|f\|_{N_\Phi} = \sup_{\|g\|_{M_\Phi}\le 1} \Big| \int\limits_{\mathbb{R}^n} f(x)g(x)dx \Big|.$$

Let $\ell \ge 2$. Denote by $W^{\ell,\infty}(\mathbb{R}^n)$ the set of all measurable functions f such that f and its generalized derivatives $D^\beta f$, $0 < |\beta| \le \ell$, belong to $L_\infty(\mathbb{R}^n)$. The following is the well-known Gagliardo-Nirenberg inequality:

Lemma 3. [6] *For fixed α, $0 < |\alpha| < \ell$, there is the best constant $C_{\alpha,\ell}$ such that*

$$\|D^\alpha f\|_\infty \le C_{\alpha,\ell}\|f\|_\infty^{1-\frac{|\alpha|}{\ell}} \big(\sum_{|\beta|=\ell} \|D^\beta f\|_\infty \big)^{\frac{|\alpha|}{\ell}},$$

for any $f \in W^{\ell,\infty}(\mathbb{R}^n)$.

The following result is an extension of the Gagliardo-Nirenberg inequality ([2 - 6]) to Lorentz spaces. Note that the Gagliardo-Nirenberg inequality has applications to partial differential equations and interpolation theory.

Theorem 1. *Let $\ell \ge 2$, f and its generalized derivatives $D^\beta f$, $|\beta| = \ell$ be in $N_\Phi(\mathbb{R}^n)$. Then $D^\alpha f \in N_\Phi(\mathbb{R}^n)$ for all α, $0 < |\alpha| < \ell$ and*

$$\|D^\alpha f\|_{N_\Phi} \le C_{\alpha,\ell}\|f\|_{N_\Phi}^{1-\frac{|\alpha|}{\ell}} \big(\sum_{|\beta|=\ell} \|D^\beta f\|_{N_\Phi} \big)^{\frac{|\alpha|}{\ell}}, \tag{1}$$

where the constant $C_{\alpha,\ell}$ is defined in Lemma 3.

Proof. We begin to prove (1) with the assumption that $D^\alpha f \in N_\Phi(\mathbb{R}^n), 0 \le |\alpha| \le \ell$.
Fix $0 < |\alpha| < \ell$. By Lemma 2 we have

$$\|D^\alpha f\|_{N_\Phi} = \sup_{\|v\|_{M_\Phi}\le 1} \big| \int\limits_{\mathbb{R}^n} D^\alpha f(x)v(x)dx \big|.$$

Let $\epsilon > 0$. We choose a function $v_\epsilon \in M_\Phi(\mathbb{R}^n)$ such that $\|v_\epsilon\|_{M_\Phi} = 1$ and

$$\big| \int\limits_{\mathbb{R}^n} f(x)v_\epsilon(x)dx \big| \ge \|f\|_{N_\Phi} - \epsilon/2.$$

By Lemma 1, there is $\mathcal{H} := [-H, H]^n$ such that

$$\big| \int\limits_{\mathbb{R}^n} f(x)v(x)dx \big| \ge \|f\|_{N_\Phi} - \epsilon, \tag{2}$$

where $v = v(\mathcal{H}, \epsilon) := \chi_{\mathcal{H}} v_\epsilon$ and $\chi_{\mathcal{H}}$ is the characteristic function of $\mathcal{H}$. Put

$$F_\epsilon(x) = \int_{\mathbb{R}^n} f(x+y)v(y)dy.$$

Then $F_\epsilon \in L_\infty(\mathbb{R}^n)$ by virtue of Lemma 1, and it is easy to check that

$$D^\beta F_\varepsilon(x) = \int_{\mathbb{R}^n} D^\beta f(x+y)v(y)dy, \ 0 \le |\beta| \le \ell \tag{3}$$

in the distribution sense.

For all $x \in \mathbb{R}^n$, clearly,

$$|D^\beta F_\varepsilon(x)| \le \|D^\beta f(x+\cdot)\|_{N_\Phi} \|v\|_{M_\Phi} \le \|D^\beta f\|_{N_\Phi}. \tag{4}$$

Now we prove the continuity of $D^\beta F_\varepsilon$ on $\mathbb{R}^n$ ($0 \le |\beta| \le \ell$). We show this for $\beta = 0$. Clearly, it suffices to prove that for any $x \in \mathbb{R}^n$,

$$\lim_{t\to 0} \|\chi_{\mathcal{H}}(\cdot)\big(f(x+t+\cdot) - f(x+\cdot)\big)\|_{N_\Phi} = 0.$$

Assume the contrary that for some $\delta > 0$, point x^0 and sequence $t_m \to 0$,

$$\|\chi_{\mathcal{H}}(\cdot)\big(f(x^0+t_m+\cdot) - f(x^0+\cdot)\big)\|_{N_\Phi} \ge \delta, \quad m \ge 1. \tag{5}$$

For simplicity of notation we suppose $x^0 = 0$. Since $f \in N_\Phi(\mathbb{R}^n)$, $f \in L_{1,\ell oc}(\mathbb{R}^n)$. So, it is known that

$$\int_{\mathcal{H}} |f(x+t_m) - f(x)|dx \to 0 \quad \text{as} \quad m \to \infty.$$

Therefore, there exists a subsequence $\{t_{m_j}\}$, which we still denote by $\{t_m\}$, such that $f(\cdot + t_m) \to f$ a.e. on $\mathcal{H}$. Define

$$g_n(x) = \inf_{m \ge n} |f(x+t_m)|, \ x \in \mathcal{H};$$

then $\{g_n\}$ is a non-decreasing sequence and $g_n \to |f|$ a.e. on $\mathcal{H}$. It is easy to see that

$$\lambda_{\chi_{\mathcal{H}} g_n}(t) \to \lambda_{\chi_{\mathcal{H}}|f|}(t) \ \text{ as } n \to \infty, \ \text{for every } t > 0.$$

We have

$$\Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) = \lim_{m\to\infty} \Phi(\lambda_{\chi_{\mathcal{H}}|g_m|}(t)) \le \varliminf_{m\to\infty} \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)|}(t)), \ t > 0. \tag{6}$$

It follows from the definition of Φ that $\Phi(a+b) \le \Phi(a) + \Phi(b)$ for $a, b \ge 0$. Observing that, for any $f, g \in N_\Phi(\mathbb{R}^n)$ and $t > 0$ we have $\lambda_{\chi_{\mathcal{H}}(f+g)}(2t) \le \lambda_{\chi_{\mathcal{H}} f}(t) + \lambda_{\chi_{\mathcal{H}} g}(t)$, then

$$\Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(2t)) \le \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)|}(t)) + \Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t)).$$

Hence for all $m \ge 1$,

$$0 \le \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)|}(t)) + \Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) - \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(2t)), \ \forall t > 0.$$

It is easy to check that

$$\|\chi_{\mathcal{H}} f(\cdot + t_m)\|_{N_\Phi} = \|\chi_{\mathcal{H}} f\|_{N_\Phi},\ \forall m \geq 1.$$

Applying Fatou's lemma to the sequence

$$\{\Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)|}(t)) + \Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) - \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(2t))\},$$

we obtain

$$\begin{aligned}
&\int_0^\infty \varliminf_{m\to\infty} \big[\Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)|}(t)) + \Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) - \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(2t))\big]dt \\
&\leq \varliminf_{m\to\infty} \int_0^\infty \big[\Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)|}(t)) + \Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) - \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(2t)\big]dt \\
&= 2\int_0^\infty \Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t))dt - \frac{1}{2}\varlimsup_{m\to\infty} \int_0^\infty \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(t))dt. \qquad (7)
\end{aligned}$$

On the other hand,

$$\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(t) = \mathrm{mes}\{x \in \mathcal{H} : |f(x+t_m) - f(x)| > t\}.$$

Therefore, since $f(\cdot + t_m) \to f$ a.e. on $\mathcal{H}$, we have

$$\lim_{m\to\infty} \lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(t) = 0$$

and then

$$\lim_{m\to\infty} \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(t)) = 0.$$

So, by (6) we get for any $t > 0$

$$\begin{aligned}
2\Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) &= \lim_{m\to\infty} \Phi(\lambda_{\chi_{\mathcal{H}}|g_m|}(t)) + \Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) - \lim_{m\to\infty} \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(2t)) \\
&\leq \varliminf_{m\to\infty} \big[\Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)|}(t)) + \Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t)) - \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(2t))\big]. \qquad (8)
\end{aligned}$$

From (7) and (8), we have

$$2\int_0^\infty \Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t))dt \leq 2\int_0^\infty \Phi(\lambda_{\chi_{\mathcal{H}}|f|}(t))dt - \frac{1}{2}\varlimsup_{m\to\infty} \int_0^\infty \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(t))dt.$$

Hence

$$\int_0^\infty \Phi(\lambda_{\chi_{\mathcal{H}}|f(\cdot+t_m)-f|}(t))dt \to 0 \quad \text{as } m \to \infty,$$

i.e.,

$$\lim_{m\to\infty} \|\chi_{\mathcal{H}}\big(f(\cdot + t_m) - f\big)\|_{N_\Phi} = 0,$$

which contradicts (5).

The cases $1 \le |\beta| \le \ell$ are proved similarly. The continuity of $D^\alpha F_\varepsilon,\ 0 \le |\beta| \le \ell$ has been proved.

The functions $D^\beta F_\varepsilon,\ 0 \le |\beta| \le \ell$ are continuous and bounded on $\mathbb{R}^n$. Therefore, it follows from Lemma 3 and (2) - (3) that

$$\big(\|D^\alpha f||_{N_\Phi} - \epsilon\big) \le |D^\alpha F_\varepsilon(0)| \le \|D^\alpha F_\varepsilon\|_\infty \le \\ \le C_{\alpha,\ell}\|F_\varepsilon\|_\infty^{1-\frac{|\alpha|}{\ell}}\big(\sum_{|\beta|=\ell}\|D^\beta F_\varepsilon\|_\infty\big)^{\frac{|\alpha|}{\ell}},$$

which together with (4) implies

$$||D^\alpha f||_{N_\Phi} - \epsilon \le C_{\alpha,\ell}\|f\|_{N_\Phi}^{1-\frac{|\alpha|}{\ell}}\big(\sum_{|\beta|=\ell}\|D^\beta f\|_{N_\Phi}\big)^{\frac{|\alpha|}{\ell}}.$$

By letting $\epsilon \to 0$ we have (1).

Step 2. To complete the proof, it remains to show that $D^\alpha f \in N_\Phi(\mathbb{R}^n),\ \forall\alpha :$ $0 < |\alpha| < \ell$ if $f, D^\alpha f \in N_\Phi(\mathbb{R}^n),\ |\alpha| = \ell$. Since $f \in L_{1,\ell oc}(\mathbb{R}^n)$ and $D^\alpha f \in L_{1,\ell oc}(\mathbb{R}^n),\ |\alpha| = \ell$, we get $D^\alpha f \in L_{1,\ell oc}(\mathbb{R}^n),\ 0 < |\alpha| < \ell$ (see [5], p. 7). Let $\psi(x) \in C_0^\infty(\mathbb{R}^n), \psi(x) \ge 0, \psi(x) = 0$ for $|x| \ge 1$ and $\int\limits_{\mathbb{R}^n}\psi(x)dx = 1$. We put $\psi_\lambda(x) = \frac{1}{\lambda^n}\psi(\frac{x}{\lambda}),\ \lambda > 0$ and $f_\lambda = f * \psi_\lambda$. Then $D^\alpha f_\lambda = f * D^\alpha\psi_\lambda,\ |\alpha| \ge 0$ and $D^\alpha f_\lambda = D^\alpha f * \psi_\lambda, 0 \le |\alpha| \le \ell$ in the $\mathcal{D}'(\mathbb{R}^n)$ sense. Actually, for every $\varphi \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned}\langle D^\alpha f_\lambda(x), \varphi(x)\rangle &= (-1)^{|\alpha|}\langle f_\lambda(x), D^\alpha\varphi(x)\rangle \\ &= (-1)^{|\alpha|}\int\limits_{\mathbb{R}^n}\big(\int\limits_{\mathbb{R}^n} f(x-y)\psi_\lambda(y)dy\big)D^\alpha\varphi(x)dx \\ &= (-1)^{|\alpha|}\int\limits_{\mathbb{R}^n}\psi_\lambda(y)\big(\int\limits_{\mathbb{R}^n} f(x-y)D^\alpha\varphi(x)dx\big)dy \\ &= \int\limits_{\mathbb{R}^n}\psi_\lambda(y)\big(\int\limits_{\mathbb{R}^n} D^\alpha f(x-y)\varphi(x)dx\big)dy \\ &= \int\limits_{\mathbb{R}^n}\big(\int\limits_{\mathbb{R}^n} D^\alpha f(x-y)\psi_\lambda(y)dy\big)\varphi(x)dx \\ &= \langle\int\limits_{\mathbb{R}^n} D^\alpha f(x-y)\psi_\lambda(y)dy\ ,\ \varphi(x)\rangle\quad .\end{aligned}$$

Taking Lemma 1 into account, we get easily $D^\beta f_\lambda = f * D^\beta\psi_\lambda \in N_\Phi(\mathbb{R}^n),\ 0 \le |\beta| \le \ell$ and

$$\|f_\lambda\|_{N_\Phi} = \|f * \psi_\lambda\|_{N_\Phi} \le \|f(x-\cdot)\|_{N_\Phi}\|\psi_\lambda\|_1 = \|f\|_{N_\Phi}, \tag{9}$$

$$\|D^\alpha f_\lambda\|_{N_\Phi} = \|D^\alpha f * \psi_\lambda\|_{N_\Phi} \le \|D^\alpha f(x-\cdot)\|_{N_\Phi}\|\psi_\lambda\|_1 = \|D^\alpha f\|_{N_\Phi}. \tag{10}$$

Applying (1) for f_λ, we have from (9) - (10) for every α $(0 < |\alpha| < \ell)$

$$\begin{aligned}||D^\alpha f_\lambda||_{N_\Phi} &\le C_{\alpha,\ell}\|f_\lambda\|_{N_\Phi}^{1-\frac{|\alpha|}{\ell}}\Big(\sum_{|\beta|=\ell}\|D^\beta f_\lambda\|_{N_\Phi}\Big)^{\frac{|\alpha|}{\ell}}\\ &\le C_{\alpha,\ell}\|f\|_{N_\Phi}^{1-\frac{|\alpha|}{\ell}}\Big(\sum_{|\beta|=\ell}\|D^\beta f\|_{N_\Phi}\Big)^{\frac{|\alpha|}{\ell}}. \qquad (11)\end{aligned}$$

Fix α $(0 < |\alpha| < \ell)$. Since $D^\alpha f \in L_{1,\ell oc}(\mathbb{R}^n)$, for each $j = 1, 2, \ldots$, we have $D^\alpha f * \psi_\lambda \to D^\alpha f$ in $L_1([-j, j]^n)$ as $\lambda \to 0$. Therefore, there is a sequence $\{\lambda_k^j\}_{k=1}^\infty$, $\lambda_k^j \searrow 0$ such that $D^\alpha f_{\lambda_k^j} \to D^\alpha f$ a.e. in $[-j, j]^n$ as $k \to \infty$. So, by the diagonal process, there exists a subsequence denoted by $\{\lambda_m\}_1^\infty : \lambda_m \to 0$ such that

$$\lim_{m\to\infty} D^\alpha f_{\lambda_m}(x) = D^\alpha f(x) \qquad (12)$$

a.e. in $\mathbb{R}^n$. For each function $v \in M_\Phi(\mathbb{R}^n)$, $\|v\|_{M_\Phi} \le 1$ and $m \ge 1$, by (11) - (12) and the definition of the Lorentz norm we get

$$\int_{\mathbb{R}^n}\big|(D^\alpha f_{\lambda_m})(x)v(x)\big|dx \le C_{\alpha,\ell}\|f\|_{N_\Phi}^{1-\frac{|\alpha|}{\ell}}\Big(\sum_{|\beta|=\ell}\|D^\beta f\|_{N_\Phi}\Big)^{\frac{|\alpha|}{\ell}}. \qquad (13)$$

Therefore, using Fatou's lemma, (12) and (13), we obtain

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} D^\alpha f(x)v(x)dx\Big| &\le \int_{\mathbb{R}^n}\liminf_{m\to\infty}\big|\, D^\alpha f_{\lambda_m}(x)v(x)\big|dx\\ &\le \liminf_{m\to\infty}\int_{\mathbb{R}^n}\big|(D^\alpha f_{\lambda_m})(x)v(x)\big|dx\\ &\le C_{\alpha,\ell}\|f\|_{N_\Phi}^{1-\frac{|\alpha|}{\ell}}\Big(\sum_{|\beta|=\ell}\|D^\beta f\|_{N_\Phi}\Big)^{\frac{|\alpha|}{\ell}}. \qquad (14)\end{aligned}$$

Because (14) is true for all $v \in M_\Phi(\mathbb{R}^n)$, $\|v\|_{M_\Phi} \le 1$, by the definition of $\|\cdot\|_{N_\Phi}$ we have

$$||D^\alpha f||_{N_\Phi} \le C_{\alpha,\ell}\|f\|_{N_\Phi}^{1-\frac{|\alpha|}{\ell}}\Big(\sum_{|\beta|=\ell}\|D^\beta f\|_{N_\Phi}\Big)^{\frac{|\alpha|}{\ell}} < \infty,\ 0 < |\alpha| < \ell.$$

The proof is complete. ∎

By Theorem 1, we have

Theorem 2. *Let $\ell \ge 2$, f and its generalized derivatives $D^\beta f$, $|\beta| = \ell$ be in $N_\Phi(\mathbb{R}^n)$. Then $D^\alpha f \in N_\Phi(\mathbb{R}^n)$ for all α, $0 < |\alpha| = r < \ell$ and*

$$\sum_{|\alpha|=r}\|D^\alpha f\|_{N_\Phi} \le C_{r,\ell}\|f\|_{N_\Phi}^{1-\frac{r}{\ell}}\Big(\sum_{|\beta|=\ell}\|D^\beta f\|_{N_\Phi}\Big)^{\frac{r}{\ell}}.$$

Corollary 1. *Let $\ell \geq 2$, f and its generalized derivatives $D^\beta f$, $|\beta| = \ell$ be in $N_\Phi(\mathbb{R}^n)$. Then $D^\alpha f \in N_\Phi(\mathbb{R}^n)$ for all α, $0 < |\alpha| = r < \ell$ and*

$$\sum_{|\alpha|=r} \|D^\alpha f\|_{N_\Phi} \leq Ch^{-\frac{r}{\ell-r}} \|f\|_{N_\Phi} + Ch \sum_{|\beta|=\ell} \|D^\beta f\|_{N_\Phi},$$

for all $h > 0$ and C does not depend on f.

Remark 1. Note that the Gagliardo-Nirenberg inequality for Orlicz spaces $L_\Phi(\mathbb{R}^n)$ was proved in [1] but the techniques used there cannot be applied to Lorentz spaces $N_\Phi(\mathbb{R}^n)$.

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