A Formula About Tree

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Abstract. Let $G$ be a tree. It is proved that for any vertex $v$ of $G$

$$|V| + \sum_{q \in V} [d(q) - 2]l(v, q) = 1$$

in which $d(q)$ is the degree of the vertex $q$, and $l(v, q)$ is the distance between $v$ and $q$ in $G$. This result enable us to derive a formula concering the average distance for some particular trees.

1. Introduction

We begin by recalling some definitions and notations. A tree $G$ is a connected graph that contains no simple cycles [1]. The degree of a vertex $v$ of $G$, denoted by $d(v)$, is the number of edges incident with $v$. The distance between two vertices $v$ and $q$, denoted $l(v, q)$, is the number of edges from $v$ to $q$ on the unique $(v, q)$-path in the tree $G$. The transmission of a vertex $v$ of $G$, is the value $\sigma(v) = \sum_{u \in V} l(v, u)$. The transmission of $G$ is the value $\sigma(G) = \sum_{v \in V} \sigma(v)$ [3].

The Wiener index of a connected graph $G$, denoted by $W(G)$, is defined by $W(G) = \sum_{u, v \in V} l(v, u)$, in which the summation is taken over all unordered pairs $\{u, v\}$ of distinct vertices of $G$. It is evident that $W(G) = \frac{1}{2} \sigma(G)$. Finally,
the average distance of $G$, denoted by $\mu(G) = W(G)/|G|$, where $p$ is order of $G$ [2]. The purpose of this paper is to establish the following formula

$$|V| + \sum_{q \in V} [d(q) - 2]l(v, q) = 1$$

for any $v \in V$. Consequently, this enables us to evaluate the average distance of some particular trees.

2. The Formula

**Theorem 2.1.** Let $G$ be a tree with finite vertex set $V$. For every vertex $v$ of $G$, we have

$$|V| + \sum_{q \in V} (d(q) - 2)l(v, q) = 1.$$

**Proof.** We use induction on $D(v) = \max_{q \in V} l(v, q)$. If $D(v) = 1$ then $G$ is a star and we have

$$|V| + \sum_{q \in V} (d(q) - 2)l(v, q) = 1.$$

Now suppose that $|V| + \sum_{q \in V} (d(q) - 2)l(v, q) = 1$ for all trees $G$ and for vertex $v \in V(G)$ such that $D(v) = k \geq 1$. We consider a tree $G$ with a vertex $v \in V(G)$ such that $D(v) = k + 1$. Let $X$ be the set of the vertices $q$ with $l(v, q) = k + 1$ and $Y$ be the set of neighbors of $X$. It is easy to see that $l(v, q) = k$ for $q \in Y$ and that $\sum_{q \in Y} d_x = |X|$, where $d_x(q)$ is the number of the neighbors of $q$ in $X$. For the tree $G' = G - X$ we have by induction hypothesis

$$1 = |V - X| + \sum_{q \in V - X} (d_{G'}(q) - 2)l(q, v)$$

$$= |V| - |X| + \sum_{q \in V - X - Y} (d_{G'}(q) - 2)l(q, v) + \sum_{q \in Y} (d_{G'}(q) - 2)l(q, v)$$

$$= |V| + \sum_{q \in V - X - Y} (d_{G'}(q) - 2)l(q, v) + \sum_{q \in Y} (d_{G'}(q) - 2)l(q, v) - \sum_{q \in Y} d_x(q)l(q, v) - |X|$$

$$= |V| + \sum_{q \in V - X} (d_G(q) - 2)l(q, v) - |X|l(q, v) - |X|$$

$$= |V| + \sum_{q \in V} (d_G(q) - 2)l(q, v),$$

since $d_G(q) = 1$ for all $q \in X$. Thus, Theorem 2.1 holds for any finite tree $G$. □

3. The Average Distance

The previous formula may be used to find the average distance in some special trees. Let us start by some preliminary results.
Proposition 3.1. If $G$ is a tree with order $p > 1$, then

$$\mu(G) = \frac{1}{2} + \frac{1}{2p(p-1)} \sum_{q \in V} d(q) \sigma(q).$$

Proof. According to Theorem 2.1, we have

$$|V| + \sum_{q \in V} (d(q) - 2)l(v, q) = 1.$$ 

Summing for all $v \in V$, we obtain

$$p^2 + \sum_{v \in V} \sum_{q \in V} (d(q) - 2)l(v, q) = p.$$ 

Since $\sigma(q) = \sum_{v \in V} l(v, q)$, then

$$\sum_{q \in V} (2 - d(q))\sigma(q) = p^2 - p.$$ 

Therefore

$$2 \sum_{q \in V} \sigma(q) - \sum_{q \in V} d(q)\sigma(q) = p(p - 1).$$ 

It follows that

$$W(G) = \frac{p^2 - p}{4} + \frac{1}{4} \sum_{q \in V} d(q)\sigma(q).$$ 

Thus

$$\mu(G) = \frac{1}{2} + \frac{1}{2p(p-1)} \sum_{q \in V} \sigma(q).$$

Proposition 3.2. If $G$ is a tree with order $p > 0$ and of maximum degree $d > 2$. If every vertex of $G$ has degree 1 or $d$, then

$$\mu(G) = \frac{d - 1}{p(p-1)(d-2)} \sum_{q, d(q) = 1} \sigma(q) - \frac{1}{d-2}.$$ 

Proof. In view of the above proof, we have $\sum_{q \in V} (2 - d(q))\sigma(q) = p^2 - p$. Since every vertex of $G$ has degree 1 or $d$, then the last relation becomes

$$\sum_{q, d(q) = 1} \sigma(q) + (2 - d) \sum_{q, d(q) = d} \sigma(q) = p(p - 1).$$ 

Thus

$$\sum_{q, d(q) = d} \sigma(q) = \frac{1}{d-2} \left\{ \sum_{q, d(q) = 1} \sigma(q) - p(p - 1) \right\}.$$ 

But

$$\mu(G) = \frac{1}{2} + \frac{1}{2p(p-1)} \left\{ \sum_{d(q) = 1} \sigma(q) + d \sum_{d(q) = d} \sigma(q) \right\},$$ 

so it follows that
\[ \mu(G) = \frac{1}{2} + \frac{1}{2p(p-1)(d-2)} \left\{ 2(d-1) \sum_{d(q)=1} \sigma(q) - dp(p-1) \right\}. \]

That is
\[ \mu(G) = \frac{d-1}{p(p-1)(d-2)} \sum_{q,d(q)=1} \sigma(q) - \frac{1}{d-2}. \]

We can notice for such graph that the average distance depends only on vertices of degree 1.

**Example 3.3.** Let \( G(h, d) \) be a rooted tree of height \( h(h > 2) \) such that
(i) The degree of every internal vertex of \( G \) is \( d \) (\( d > 2 \));
(ii) At level 0, there is only one vertex (the root \( o \));
(iii) At level \( i \) (\( 1 \leq i \leq h \)), there are \( d(d-1)^{i-1} \) vertices.

Our goal is to evaluate the average distance of \( G(h, d) \). By applying Proposition 3.2, it is sufficient to calculate the transmission of each vertex of degree 1. In fact, we will compute \( \sigma(v) \) for one fixed vertex of \( G(h, d) \). Any other vertex of degree 1 has the same transmission. To this end, consider two sub-graphs \( L(h, d) \) and \( T(h, d) \) with the following descriptions:
1) \( L(h, d) \) is a rooted tree of height \( h \) such that
   (i) The degree of \( o \) is \( d-1 \) while the degree of every other vertex of \( G(h, d) \) is \( d \).
   (ii) At level 0, there is only one vertex (the root \( o \)).
   (iii) At level \( i \) (\( 1 \leq i \leq h \)), there are \( (d-1)^i \) vertices.
2) \( T(h, d) \) is a rooted tree of height \( h \) such that
   (i) The degree of \( o \) is 1 while the degree of every other vertex of \( G(h, d) \) is \( d \).
   (ii) At level 0, there only one vertex (the root \( o \)).
   (iii) At level \( i \) (\( 1 \leq i \leq h \)), there are \( (d-1)^{i-1} \) vertices.
3) \( i \) \( v \) is an end vertex of \( T(h, d) \).
   \( ii \) \( o \) is the unique common vertex of \( T(h, d) \) and \( L(h, d) \).
   \( iii \) \( G(h, d) = L(h, d) \cup T(h, d) \).

\( G(3, 3) \) is shown in Fig. 1. \( T(3, 3) \) is drawn in bold lines while \( L(3, 3) \) is drawn in normal lines.
Step 1. Set $L_k$ be the transmission of the vertex $o$ in the graph $L(k,d)$. We have

$$
L_1 = d - 1,
L_2 = L_1 + 2(d - 1)^2,
L_3 = L_2 + 3(d - 1)^3,
\ldots
L_k = L_{k-1} + k(d - 1)^k \quad \text{for every} \quad k \geq 2.
$$

Therefore we can deduce that

$$
L_k = L_1 + \sum_{j=1}^{k-1} (L_{j+1} - L_j) = \sum_{j=1}^{k} j(d - 1)^j.
$$

It follows that

$$
L_k = \frac{d - 1}{(d - 2)^2}[(k(d - 2) - 1)(d - 1)^k + 1].
$$

Step 2. Let $T_k$ be the transmission of the root $v$ in the graph $T(k,d)$. We have

$$
T_1 = 1,
T_2 = T_1 + 2(d - 1),
T_3 = T_2 + (d - 2)L_1 + 3(d - 2)[1 + (d - 1)] + 3,
T_4 = T_3 + (d - 2)L_2 + 4(d - 2)[1 + (d - 1) + (d - 1)^2]) + 4,
\ldots
T_k = T_{k-1} + (d - 2)L_{k-2} + k(d - 2)[1 + (d - 1) + \ldots + (d - 1)^{k-2}] + k,
$$

that is $T_k = T_{k-1} + (d - 1)L_{k-2} + k(d - 1)^{k-1}$ for every $k \geq 3$.

Using the expression of $L_{k-2}$, we can write

$$
T_k - T_{k-1} = \frac{d - 1}{d - 2} + 2(k - 1)(d - 1)^{k-1} - \frac{1}{d - 2}(d - 1)^{k-1}.
$$

It results that

$$
T_k = T_2 + \sum_{j=2}^{k-1} (T_{j+1} - T_j)
$$

$$
= 1 + 2(d - 1) + \frac{d - 1}{d - 2}(k - 2) + 2 \sum_{j=2}^{k-1} j(d - 1)^j - \frac{1}{d - 2} \sum_{j=2}^{k-1} (d - 1)^j
$$

$$
= 1 + \frac{d - 1}{d - 2}(k - 2) + 2 \sum_{j=1}^{k-1} j(d - 1)^j - \frac{1}{d - 2} \sum_{j=2}^{k-1} (d - 1)^j
$$

$$
= \frac{d - 1}{d - 2} k + 2 \sum_{j=1}^{k-1} j(d - 1)^j - \frac{1}{d - 2} \sum_{j=0}^{k-1} (d - 1)^j
$$

$$
= 2L_{k-1} + \frac{d - 1}{d - 2} k - \frac{1}{(d - 2)^2}[(d - 1)^k - 1].
$$
**Step 3.** We are ready to produce the transmission of the vertex \( v \) in \( G(h, d) \).

\[
\sigma(v) = L_h + T_h + h(d - 1)[1 + (d - 1) + (d - 1)^2 + \ldots + (d - 1)^{h-1}]
= L_h + T_h + \frac{h(d - 1)}{d - 2}[(d - 1)^h - 1].
\]

Replacing the value of \( T_h \) from step 2, we get

\[
\sigma(v) = L_h + \frac{2h(d - 1)(d - 1)^h}{d - 2} - \frac{1}{(d - 2)^2}[(d - 1)^h - 1]
= 3L_{h-1} + h(d - 1)^h + \frac{h(d - 1)(d - 1)^h}{d - 2} - \frac{1}{(d - 2)^2}[(d - 1)^h - 1].
\]

Now, replacing the value of \( L_{h-1} \) from step 1, we obtain

\[
\sigma(v) = 2hd(d - 1)^h - \frac{(3d - 2)}{(d - 2)^2}[(d - 1)^h - 1].
\]

One can verify easily that this formula is still true when the height \( h \) is 1 or 2.

**Step 4.** According to Proposition 3.2

\[
\mu(G) = \frac{d - 1}{p(p - 1)(d - 2)} \sum_{q, d(q) = 1} \sigma(q) - \frac{1}{d - 2}.
\]

Note that the number of vertices of degree 1 in \( G \) is \( d(d - 1)^h \), and that each of them has the same transmission as \( v \). Then

\[
\mu(G) = \frac{d(d - 1)^{h+1}}{p(p - 1)(d - 2)} \sigma(v) - \frac{1}{d - 2}.
\]

Finally, we find

\[
\mu(G) = \frac{d(d - 1)^{h+1}}{p(p - 1)(d - 2)^2} \{2hd(d - 2)(d - 1)^h - (3d - 2)[(d - 1)^h - 1]\} - \frac{1}{d - 2}
\]

**References**