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A Formula About Tree

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Abstract. Let G be a tree. It is proved that for any vertex v of G

$$|V| + \sum_{q \in V} [d(q) - 2] l(v,q) = 1$$

in which $d(q)$ is the degree of the vertex q, and $l(v,q)$ is the distance between v and q in G. This result enable us to derive a formula concering the average distance for some particular trees.

1. Introduction

We begin by recalling some definitions and notations. A tree G is a connected graph that contains no simple cycles [1]. The *degree* of a vertex v of G, denoted by $d(v)$, is the number of edges incident with v. The *distance* between two vertices v anh q, denoted $l(v,q)$, is the number of edges from v to q on the unique (v,q)-path in the tree G.

The transmission of a vertex v of G, is the value $\sigma(v) = \sum_{u \in V} l(v,u)$. The *transmission* of G is the value $\sigma(G) = \sum_{v \in V} \sigma(v)$ [3].

The Wiener index of a connected graph G, denoted by $W(G)$, is defined by $W(G) = \sum_{u,v \in V} l(v,u)$, in which the summation is taken over all unordered pairs $\{u,v\}$ of distinct vertices of G. It is evident that $W(G) = \frac{1}{2}\sigma(G)$. Finally, the *average distance* of G, denoted by $\mu(G) = W(G)/\binom{p}{2}$, where p is order of G [2]. The purpose of this paper is to establish the following formula

$$|V| + \sum_{q \in V} [d(q) - 2] l(v,q) = 1$$

for any $v \in V$. Consequently, this enables us to evaluate the average distance of some particular trees.

2. The Formula

Theorem 2.1. *Let G be a tree with finite vertex set V. For every vertex v of G, we have*

$$|V| + \sum_{q \in V} (d(q) - 2).l(v,q) = 1.$$

Proof. We use induction on $D(v) = max_{q \in V} l(v,q)$. If $D(v) = 1$ then G is a star and we have

$$|V| + \sum_{q \in V} (d(q) - 2) l(v,q) = |V| + \sum_{q \in V} (1 - 2) l(v,q) = 1.$$

Now suppose that $|V| + \sum_{q \in V} (d(q) - 2) l(v,q) = 1$ for all trees G and for vertex $v \in V(G)$ such that $D(v) = k \geq 1$. We consider a tree G with a vertex $v \in V(G)$ such that $D(v) = k + 1$. Let X be the set of the vertices q with $l(v,q) = k + 1$ and Y be the set of neighbors of X. It is easy to see that $l(v,q) = k$ for $q \in Y$ and that $\sum_{q \in Y} d_x = |X|$, where $d_x(q)$ is the number of the neighbors of q in X. For the tree $G' = G - X$ we have by induction hypothesis

$$\begin{aligned}
1 &= |V - X| + \sum_{q \in V - X} (d_{G'}(q) - 2) l(q,v) \\
&= |V| - |X| + \sum_{q \in V - X - Y} (d_{G'}(q) - 2) l(q,v) + \sum_{q \in Y} (d_{G'}(q) - 2) l(q,v) \\
&= |V| + \sum_{q \in V - X - Y} (d_G(q) - 2) l(q,v) + \sum_{q \in Y} (d_G(q) - 2) l(q,v) - \sum_{q \in Y} d_X(q) l(q,v) - |X| \\
&= |V| + \sum_{q \in V - X} (d_G(q) - 2) l(q,v) - |X| l(q,v) - |X| \\
&= |V| + \sum_{q \in V} (d_G(q) - 2) l(q,v),
\end{aligned}$$

since $d_G(q) = 1$ for all $q \in X$. Thus, Theorem 2.1 holds for any finite tree G. ■

3. The Average Distance

The previous formula may be used to find the average distance in some special trees. Let us start by some preliminary results.

Proposition 3.1. *If G is a tree with order $p > 1$, then*

$$\mu(G) = \frac{1}{2} + \frac{1}{2p(p-1)} \sum_{q \in V} d(q)\sigma(q).$$

Proof. According to Theorem 2.1, we have

$$|V| + \sum_{q \in V} (d(q) - 2).l(v, q) = 1.$$

Summing for all $v \in V$, we obtain

$$p^2 + \sum_{v \in V} \sum_{q \in V} (d(q) - 2) l(v, q) = p.$$

Since $\sigma(q) = \sum_{v \in V} l(v, q)$, then

$$\sum_{q \in V} (2 - d(q))\sigma(q) = p^2 - p.$$

Therefore

$$2 \sum_{q \in V} \sigma(q) - \sum_{q \in V} d(q)\sigma(q) = p(p-1).$$

It follows that

$$W(G) = \frac{p^2 - p}{4} + \frac{1}{4} \sum_{q \in V} d(q)\sigma(q).$$

Thus

$$\mu(G) = \frac{1}{2} + \frac{1}{2p(p-1)} \sum_{q \in V} \sigma(q).$$

■

Proposition 3.2. *If G is a tree with order $p > 0$ and of maximum degree $d > 2$. If every vertex of G has degree 1 or d, then*

$$\mu(G) = \frac{d-1}{p(p-1)(d-2)} \sum_{q, d(q)=1} \sigma(q) - \frac{1}{d-2}.$$

Proof. In view of the above proof, we have $\sum_{q \in V} (2 - d(q))\sigma(q) = p^2 - p$. Since every vertex of G has degree 1 or d, then the last relation becomes

$$\sum_{q, d(q)=1} \sigma(q) + (2 - d) \sum_{q, d(q)=d} \sigma(q) = p(p-1).$$

Thus

$$\sum_{q, d(q)=d} \sigma(q) = \frac{1}{d-2} \Big\{ \sum_{q, d(q)=1} \sigma(q) - p(p-1) \Big\}.$$

But

$$\mu(G) = \frac{1}{2} + \frac{1}{2p(p-1)} \Big\{ \sum_{d(q)=1} \sigma(q) + d \sum_{d(q)=d} \sigma(q) \Big\},$$

so it follows that

$$\mu(G) = \frac{1}{2} + \frac{1}{2p(p-1)(d-2)}\Big\{2(d-1)\sum_{d(q)=1}\sigma(q) - dp(p-1)\Big\}.$$

That is

$$\mu(G) = \frac{d-1}{p(p-1)(d-2)}\sum_{q,d(q)=1}\sigma(q) - \frac{1}{d-2}. \qquad \blacksquare$$

We can notice for such graph that the average distance depends only of vertices of degree 1.

Example 3.3. Let $G(h,d)$ be a rooted tree of height $h (h > 2)$ such that
(i) The degree of every internal vertex of G is d $(d > 2)$;
(ii) At level 0, there is only one vertex (the root o);
(iii) At level i $(1 \leqslant i \leqslant h)$, there are $d(d-1)^{i-1}$ vertices.

Our goal is to evaluate the average distance of $G(h,d)$. By applying Proposition 3.2, it is sufficient to calculate the transmission of each vertex of degree 1. In fact, we will compute $\sigma(v)$ for one fixed vertex of $G(h,d)$. Any other vertex of degree 1 has the same transmission. To this end, consider two sub-graphs $L(h,d)$ and $T(h,d)$ with the following descriptions:

1) $L(h,d)$ is a rooted tree of height h such that
   (i) The degree of o is $d-1$ while the degree of every other vertex of $G(h,d)$ is d.
   (ii) At level 0, there is only one vertex (the root o).
   (iii) At level $i(1 \leqslant i \leqslant h)$, there are $(d-1)^i$ vertices.
2) $T(h,d)$ is a rooted tree of height h such that
   (i) The degree of o is 1 while the degree of every other vertex of $G(h,d)$ is d.
   (ii) At level 0, there only one vertex (the root o).
   (iii) At level $i(1 \leqslant i \leqslant h)$, there are $(d-1)^{i-1}$ vertices.
3) i) v is an end vertex of $T(h,d)$.
   ii) o is the unique common vertex of $T(h,d)$ and $L(h,d)$.
   iii) $G(h,d) = L(h,d) \cup T(h,d)$.

$G(3,3)$ is shown in Fig. 1. $T(3,3)$ is drawn in bold lines while $L(3,3)$ is drawn in normal lines.

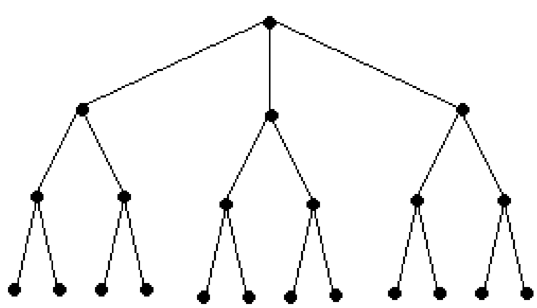

Fig. 1

Step 1. Set L_k be the transmission of the vertex o in the graph $L(k,d)$. We have

$$\begin{aligned}
&L_1 = d-1,\\
&L_2 = L_1 + 2(d-1)^2,\\
&L_3 = L_2 + 3(d-1)^3,\\
&\quad\cdots\\
&L_k = L_{k-1} + k(d-1)^k \quad \text{for every} \quad k \geq 2.
\end{aligned}$$

Therefore we can deduce that

$$L_k = L_1 + \sum_{j=1}^{k-1}(L_{j+1} - L_j) = \sum_{j=1}^{k} j(d-1)^j.$$

It follows that

$$L_k = \frac{d-1}{(d-2)^2}[(k(d-2)-1)(d-1)^k + 1].$$

Step 2. Let T_k be the transmission of the root v in the graph $T(k,d)$. We have

$$\begin{aligned}
&T_1 = 1,\\
&T_2 = T_1 + 2(d-1),\\
&T_3 = T_2 + (d-2)L_1 + 3(d-2)[1+(d-1)] + 3,\\
&T_4 = T_3 + (d-2)L_2 + 4(d-2)[1+(d-1)+(d-1)^2]) + 4,\\
&\quad\cdots\\
&T_k = T_{k-1} + (d-2)L_{k-2} + k(d-2)[1+(d-1)+...+(d-1)^{k-2}] + k,
\end{aligned}$$

that is $T_k = T_{k-1} + (d-1)L_{k-2} + k(d-1)^{k-1}$ for every $k \geq 3$.

Using the expression of L_{k-2}, we can write

$$T_k - T_{k-1} = \frac{d-1}{d-2} + 2(k-1)(d-1)^{k-1} - \frac{1}{d-2}(d-1)^{k-1}.$$

It results that

$$\begin{aligned}
T_k &= T_2 + \sum_{j=2}^{k-1}(T_{j+1} - T_j)\\
&= 1 + 2(d-1) + \frac{d-1}{d-2}(k-2) + 2\sum_{j=2}^{k-1} j(d-1)^j - \frac{1}{d-2}\sum_{j=2}^{k-1}(d-1)^j\\
&= 1 + \frac{d-1}{d-2}(k-2) + 2\sum_{j=1}^{k-1} j(d-1)^j - \frac{1}{d-2}\sum_{j=2}^{k-1}(d-1)^j\\
&= \frac{d-1}{d-2}k + 2\sum_{j=1}^{k-1} j(d-1)^j - \frac{1}{d-2}\sum_{j=0}^{k-1}(d-1)^j\\
&= 2L_{k-1} + \frac{d-1}{d-2}k - \frac{1}{(d-2)^2}[(d-1)^k - 1].
\end{aligned}$$

Step 3. We are ready to produce the transmission of the vertex v in $G(h,d)$.

$$\begin{aligned}\sigma(v) &= L_h + T_h + h(d-1)[1+(d-1)+(d-1)^2+...+(d-1)^{h-1}]\\ &= L_h + T_h + \frac{h(d-1)}{d-2}[(d-1)^h - 1].\end{aligned}$$

Replacing the value of T_h from step 2, we get

$$\begin{aligned}\sigma(v) &= L_h + 2L_{h-1} + \frac{h(d-1)(d-1)^h}{d-2} - \frac{1}{(d-2)^2}[(d-1)^h - 1]\\ &= 3L_{h-1} + h(d-1)^h + \frac{h(d-1)(d-1)^h}{d-2} - \frac{1}{(d-2)^2}[(d-1)^h - 1].\end{aligned}$$

Now, replacing the value of L_{h-1} from step 1, we obtain

$$\sigma(v) = \frac{2hd(d-1)^h}{d-2} - \frac{(3d-2)}{(d-2)^2}[(d-1)^h - 1].$$

One can verify easily that this formula is still true when the height h is 1 or 2.

Step 4. According to Proposition 3.2

$$\mu(G) = \frac{d-1}{p(p-1)(d-2)} \sum_{q, d(q)=1} \sigma(q) - \frac{1}{d-2}.$$

Note that the number of vertices of degree 1 in G is $d(d-1)^h$, and that each of them has the same transmission as v. Then

$$\mu(G) = \frac{d(d-1)^{h+1}}{p(p-1)(d-2)}\sigma(v) - \frac{1}{d-2}.$$

Finally, we find

$$\mu(G) = \frac{d(d-1)^{h+1}}{p(p-1)(d-2)^3}\{2hd(d-2)(d-1)^h - (3d-2)[(d-1)^h - 1]\} - \frac{1}{d-2}$$

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