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Some Remarks on Weak Amenability of Weighted Group Algebras

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Abstract. In [1] the authors consider the sufficient condition $\omega(n)\omega(-n) = o(n)$ for weak amenability of Beurling algebras on the integers. In this paper we show that this characterization does not generalize to non-abelian groups.

1. Introduction

The Banach algebra $\mathcal{A}$ is amenable if $\mathcal{H}^1(\mathcal{A}, \mathcal{X}') = 0$ for every Banach $\mathcal{A}$-bimodule $\mathcal{X}$, that is, every bounded derivation $D : \mathcal{A} \to \mathcal{X}'$ is inner. This definition was introduced by Johnson in (1972) [5]. The Banach algebra $\mathcal{A}$ is weakly amenable if $\mathcal{H}^1(\mathcal{A}, \mathcal{A}') = 0$. This definition generalizes the one which was introduced by Bade, Curtis and Dales in [1], where it was noted that a commutative Banach algebra $\mathcal{A}$ is weakly amenable if and only if $\mathcal{H}^1(\mathcal{A}, \mathcal{X}) = 0$ for every symmetric Banach $\mathcal{A}$-bimodule $\mathcal{X}$.

In [7] Johnson showed that $L^1(G)$ is weakly amenable for every locally compact group. In [9] Pourabbas proved that $L^1(G, \omega)$ is weakly amenable whenever $\sup\{\omega(g)\omega(g^{-1}) : g \in G\} < \infty$. Grønbæk [3] proved that the Beurling algebra $\ell^1(\mathbb{Z}, \omega)$ is weakly amenable if and only if

$$\sup\left\{\frac{|n|}{\omega(n)\omega(-n)} : n \in \mathbb{Z}\right\} = \infty.$$

In [3] he also characterized the weak amenability of $\ell^1(G, \omega)$ for abelian group G. He showed that

(∗) The Beurling algebra $\ell^1(G, \omega)$ is weakly amenable if and only if

$$\sup\left\{\frac{|f(g)|}{\omega(g)\omega(g^{-1})} : g \in G\right\} = \infty$$

for all $f \in \operatorname{Hom}_{\mathbb{Z}}(G, \mathbb{C})\backslash\{0\}$. The first author [8] generalizes the 'only if' part of $(*)$ for non-abelian groups. Borwick in [2] showed that Grønbæk's characterization does not generalize to non-abelian groups by exhibiting a group with non-zero additive functions but such that $\ell^1(G, \omega)$ is not weakly amenable.

For non-abelian groups, Borwick [2] gives a very interesting classification of weak amenability of Beurling algebras in term of functions defined on G.

Theorem 1.1. [2, Theorem 2.23] *Let $\ell^1(G, \omega)$ be a weighted non-abelian group algebra and let $\{\mathcal{C}_i\}_{i\in I}$ be the partition of G into conjugacy classes. For each $i \in I$, let F_i denote the set of nonzero functions $\psi : G \to \mathbb{C}$ which are supported on $\mathcal{C}_i$ and such that*

$$\sup\left\{\frac{\left|\psi(XY) - \psi(YX)\right|}{\omega(X)\omega(Y)} : X, Y \in G, XY \in \mathcal{C}_i\right\} < \infty.$$

Then $\ell^1(G, \omega)$ is weakly amenable if and only if for each $i \in I$ every element of F_i is contained in $\ell^\infty(G, \omega^{-1})$, that is, if and only if every $\psi \in F_i$ satisfies

$$\sup_{X\in G}\left\{\frac{\left|\psi(XYX^{-1})\right|}{\omega(XYX^{-1})}\right\} < \infty, \quad (Y \in \mathcal{C}_i).$$

In [1] the authors consider the sufficient condition $\omega(n)\omega(-n) = 0(n)$ for weak amenability of Beurling algebras on the integers. For abelian groups we have the following result:

Proposition 1.2. *Let G be a discrete abelian group and let ω be a weight on G such that $\lim_{n\to\infty} \frac{\omega(g^n)\omega(g^{-n})}{n} = 0$ for every $g \in G$. Then $\ell^1(G, \omega)$ is weakly amenable.*

Proof. If $\ell^1(G, \omega)$ is not weakly amenable, then by [3, Corollary 4.8] there exists a $\phi \in \operatorname{Hom}(G, \mathbb{C}) \setminus \{0\}$ such that $\sup_{g\in G} \frac{|\phi(g)|}{\omega(g)\omega(g^{-1})} = K < \infty$. Hence for every $g \in G$

$$\frac{|\phi(g^n)|}{\omega(g^n)\omega(g^{-n})} = \frac{n|\phi(g)|}{\omega(g^n)\omega(g^{-n})} \leq K,$$

or equivalently $\frac{\omega(g^n)\omega(g^{-n})}{n} \geq \frac{|\phi(g)|}{K}$. Therefore

$$\lim_{n\to\infty} \frac{\omega(g^n)\omega(g^{-n})}{n} = 0 \geq \frac{|\phi(g)|}{K},$$

which is a contradiction. ∎

Example 1.3. Let G be a subgroup of $GL(2, \mathbb{R})$ defined by

$$G = \left\{ \begin{bmatrix} e^{t_1} & t_2 \\ 0 & e^{t_1} \end{bmatrix} : t_1, t_2 \in \mathbb{R}\right\}$$

and let $\omega_\alpha : G \to \mathbb{R}^+$ be defined by

$$\omega_\alpha(T) = (e^{t_1} + |t_2|)^\alpha \qquad (\alpha > 0).$$

To show that ω_α is a weight, let us consider

$$T = \begin{bmatrix} e^{t_1} & t_2 \\ 0 & e^{t_1} \end{bmatrix} \quad S = \begin{bmatrix} e^{s_1} & s_2 \\ 0 & e^{s_1} \end{bmatrix}.$$

Then

$$\begin{aligned} \omega_\alpha(TS) &= (e^{t_1+s_1} + |t_2 e^{s_1} + s_2 e^{t_1}|)^\alpha \\ &\leq (e^{t_1+s_1} + |t_2| e^{s_1} + |s_2| e^{t_1} + |s_2||t_2|)^\alpha \\ &= (e^{t_1} + |t_2|)^\alpha (e^{s_1} + |s_2|)^\alpha = \omega_\alpha(T)\omega_\alpha(S), \end{aligned}$$

it is clear that $\omega_\alpha(I) = 1$. Also for $0 < \alpha < \frac{1}{2}$ we have

$$\begin{aligned} \frac{\omega_\alpha(T^n)\omega_\alpha(T^{-n})}{n} &= \frac{(e^{nt_1} + n|t_2|e^{(n-1)t_1})^\alpha (e^{-nt_1} + n|t_2|e^{-(n+1)t_1})^\alpha}{n} \\ &= \frac{(1 + n|t_2|e^{-t_1})^{2\alpha}}{n} \to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Therefore $\ell^1(G, \omega_\alpha)$ is weakly amenable for $0 < \alpha < \frac{1}{2}$. Note that in this example, we have

$$\begin{aligned} \sup_{T \in G}\{\omega_\alpha(T)\omega_\alpha(T^{-1})\} &= \sup_{t_1, t_2 \in \mathbb{R}} \left\{(e^{t_1} + |t_2|)^\alpha (e^{-t_1} + |t_2|e^{-2t_1})^\alpha\right\} \\ &= \sup_{t_1, t_2 \in \mathbb{R}} \left\{(1 + |t_2|e^{-t_1})^{2\alpha}\right\} = \infty, \quad (\alpha > 0). \end{aligned}$$

So by [4, Corollary 3.3] $\ell^1(G, \omega_\alpha)$ is not amenable.

Question 1.4. Is the condition

$$\lim_{n \to \infty} \frac{\omega(g^n)\omega(g^{-n})}{n} = 0 \tag{1.1}$$

sufficient for weak amenability of Beurling algebras on the not necessarily abelian group G?

It has been considered in [8] and [9].

Note that the condition $\sup\{\omega(g)\omega(g^{-1}) : g \in G\} < \infty$ implies the condition (1.1).

2. Main Results

Our aim in this section is to answer negatively the question 1.4 by producing an example of a group G which satisfies the condition (1.1), but it is not weakly amenable.

Example 2.1. Let H be a Heisenberg group of matrices of the form

$$a = \begin{bmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 0 & 1 \end{bmatrix},$$

where $a_1, a_2, a_3 \in \mathbb{R}$. Let

$$a = \begin{bmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 0 & 1 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 & b_1 & b_2 \\ 0 & 1 & b_3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then we see that

$$ab = \begin{bmatrix} 1 & a_1 + b_1 & a_2 + b_2 + a_1 b_3 \\ 0 & 1 & a_3 + b_3 \\ 0 & 0 & 1 \end{bmatrix}, \quad a^{-1} = \begin{bmatrix} 1 & -a_1 & a_1 a_3 - a_2 \\ 0 & 1 & -a_3 \\ 0 & 0 & 1 \end{bmatrix},$$

and for every $n \geq 2$

$$a^n = \begin{bmatrix} 1 & na_1 & \sum_{i=1}^n i a_1 a_3 + n a_2 \\ 0 & 1 & n a_3 \\ 0 & 0 & 1 \end{bmatrix}, \quad a^{-n} = \begin{bmatrix} 1 & -na_1 & \sum_{i=1}^n i a_1 a_3 - n a_2 \\ 0 & 1 & -n a_3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let define $\omega_\alpha : H \to \mathbb{R}^+$ by

$$\omega_\alpha(a) = (1 + |a_3|)^\alpha, \quad (\alpha > 0).$$

Since

$$\begin{aligned} \omega_\alpha(ab) &= (1 + |a_3 + b_3|)^\alpha \\ &\leq \big(1 + |a_3| + |b_3| + |a_3||b_3|\big)^\alpha \\ &= (1 + |a_3|)^\alpha (1 + |b_3|)^\alpha = \omega_\alpha(a)\omega_\alpha(b), \end{aligned}$$

then ω_α is a weight on H, which satisfies the condition (1.1), because for every $0 < \alpha < \frac{1}{2}$, we have

$$\begin{aligned} \lim_{n\to\infty} \frac{\omega_\alpha(a^n)\omega_\alpha(a^{-n})}{n} &= \lim_{n\to\infty} \frac{\big(1 + |na_3|\big)^\alpha (1 + |-na_3|)^\alpha}{n} \\ &= \lim_{n\to\infty} \frac{\big(1 + n|a_3|\big)^{2\alpha}}{n} = 0. \end{aligned}$$

Lemma 2.2. *Suppose that $0 < \alpha < \frac{1}{2}$. Then $\ell^1(H, \omega_\alpha)$ is not weakly amenable.*

Proof. Let $e = \begin{bmatrix} 1 & e_1 & e_2 \\ 0 & 1 & e_3 \\ 0 & 0 & 1 \end{bmatrix}$. The conjugacy class of e is denoted by $\tilde{e}$ and has the following form

$$\tilde{e} = \big\{aea^{-1} : a \in H\big\} = \left\{ \begin{bmatrix} 1 & e_1 & -a_3 e_1 + e_2 + a_1 e_3 \\ 0 & 1 & e_3 \\ 0 & 0 & 1 \end{bmatrix} : a_1, a_2, a_3 \in \mathbb{R} \right\}.$$

In particular if $E = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, then $\widetilde{E} = \left\{ \begin{bmatrix} 1 & 1 & 1-a_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : a_3 \in \mathbb{R} \right\}$

If $a, b \in H$, then $ab \in \widetilde{E}$ if and only if $a_1 + b_1 = 1$ and $a_3 + b_3 = 0$. Note also that if $ab \in \widetilde{E}$, then $ba = a^{-1}(ab)a \in \widetilde{E}$.

Now define $\psi : H \to \mathbb{C}$ by $\psi(a) = |a_2|^\alpha$, where $a = \begin{bmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_3 \\ 0 & 0 & 1 \end{bmatrix}$. Then since $a_1 + b_1 = 1$ and $a_3 + b_3 = 0$, by replacing a_3 by $-b_3$ and a_1 by $1 - b_1$ respectively, we get

$$\begin{aligned} \sup_{a,b \in H} \left\{ \frac{|\psi(ab) - \psi(ba)|}{\omega_\alpha(a)\omega_\alpha(b)} : ab \in \tilde{E} \right\} &= \sup \left\{ \frac{\big| |a_2+b_2+a_1b_3|^\alpha - |a_2+b_2+b_1a_3|^\alpha \big|}{(1+|a_3|)^\alpha (1+|b_3|)^\alpha} \right\} \\ &= \sup \left\{ \frac{\big| |a_2+b_2+b_3-b_1b_3|^\alpha - |a_2+b_2-b_1b_3|^\alpha \big|}{(1+|b_3|)^{2\alpha}} \right\} \\ &\le \sup \left\{ \frac{|b_3|^\alpha}{(1+|b_3|)^{2\alpha}} : b_3 \in \mathbb{R} \right\} < \infty. \qquad (2.1) \end{aligned}$$

But for every $a \in H$ and $b \in \tilde{E}$ we have

$$aba^{-1} = \begin{bmatrix} 1 & 1 & b_2 - a_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

so

$$\sup \left\{ \frac{|\psi(aba^{-1})|}{\omega_\alpha(aba^{-1})} : a \in H \right\} = \sup \left\{ |b_2 - a_3|^\alpha : a_3 \in \mathbb{R} \right\} = \infty.$$

Thus by Theorem 1.1 if $0 < \alpha < \frac{1}{2}$, then $\ell^1(H, \omega_\alpha)$ is not weakly amenable. ∎

Borwick in [2] showed that Grønbæk's characterization $(*)$ does not generalize to non-abelian groups. Here we will give a simple example of a non-abelian group that satisfies condition of $(*)$, but $\ell^1(G, \omega)$ is not weakly amenable.

Example 2.3. Let H be a Heisenberg group on the integers. Consider the weight function ω_α that was defined in the previous Example. Suppose $\phi \in \mathrm{Hom}\,(H, \mathbb{C}) \setminus \{0\}$, and let $a = \begin{bmatrix} 1 & r & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$. Then $a = E_1^r E_2^t E_3^{s-rt}$, where

$$E_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \; E_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore

$$\sup_{a \in H} \frac{|\phi(a)|}{\omega_\alpha(a)\omega_\alpha(a^{-1})} = \sup_{r,s,t \in \mathbb{Z}} \frac{|r\phi(E_1) + t\phi(E_2) + (s - rt)\phi(E_3)|}{(1+|t|)^{2\alpha}}. \qquad (2.2)$$

Since $\phi \neq 0$ without loss of generality we can assume that $\phi(E_2) \neq 0$, then for $r = s = 0$ the equation (2.2) reduces to

$$\sup_{t\in\mathbb{Z}} \frac{|t\phi(E_2)|}{(1+|t|)^{2\alpha}} = \infty, \quad \Big(0 < \alpha < \frac{1}{2}\Big).$$

Thus $\sup\left\{\frac{|\phi(a)|}{\omega_\alpha(a)\omega_\alpha(a^{-1})} : a \in H\right\} = \infty$. But by Lemma 2.2, $\ell^1(H, \omega_\alpha)$ is not weakly amenable for $0 < \alpha < \frac{1}{2}$.

In the following theorem we will determine the connection between derivations and a family of additive maps for every discrete weighted group algebra.

Theorem 2.4. *Let G be a not necessarily abelian discrete group. Then every bounded derivation $D : \ell^1(G, \omega) \to \ell^\infty(G, \omega^{-1})$ is described uniquely by a family $\{\phi_t\}_{t\in Z(G)} \subset Hom_{\mathbb{Z}}(G, \mathbb{C})$ such that*

$$\sup\Big\{\frac{|\phi_t(g)|}{\omega(g)\omega(g^{-1}t)} : g \in G, t \in Z(G)\Big\} < \infty.$$

Proof. Suppose that $D : \ell^1(G, \omega) \to \ell^\infty(G, \omega^{-1})$ is a bounded derivation. Then D corresponds via the equation $\tilde{D}(g, h) = D(\delta_g)(\delta_h)$ to an element $\tilde{D}$ of $\ell^\infty(G \times G, \omega^{-1} \times \omega^{-1})$ which satisfies

$$\tilde{D}(gh, k) = \tilde{D}(g, hk) + \tilde{D}(h, kg), \quad (g, h, k \in G). \tag{2.3}$$

Now for every t in $Z(G)$ (the center of G) we define

$$\phi_t(g) = \tilde{D}(g, g^{-1}t), \quad (g \in G).$$

For every g and h in G we have

$$\begin{aligned}\phi_t(gh) &= \tilde{D}(gh, h^{-1}g^{-1}t)\\ &= \tilde{D}(g, hh^{-1}g^{-1}t) + \tilde{D}(h, h^{-1}g^{-1}tg)\\ &= \phi_t(g) + \phi_t(h)\end{aligned}$$

and

$$\begin{aligned}\sup\Big\{\frac{|\phi_t(g)|}{\omega(g)\omega(g^{-1}t)} : g \in G, t \in Z(G)\Big\} &= \sup\Big\{\frac{|\tilde{D}(g, g^{-1}t)|}{\omega(g)\omega(g^{-1}t)} : g \in G, t \in Z(G)\Big\}\\ &\leq \big\|\tilde{D}\big\|_\omega^\infty.\end{aligned}$$

So D corresponds to the family $\{\phi_t\}_{t\in Z(G)} \subset \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C})$.

Conversely, we consider a family $\{\phi_t\}_{t\in Z(G)} \subset \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C})$ such that

$$\sup\Big\{\frac{|\phi_t(g)|}{\omega(g)\omega(g^{-1}t)} : g \in G, t \in Z(G)\Big\} < \infty.$$

We define a function $\tilde{D}$ by

$$\tilde{D}(g, h) = \sum_{t\in Z(G)} \phi_t(g)\chi_t(gh), \quad (g, h \in G),$$

where χ_t is the characteristic function. We show that $\tilde{D} \in \ell^\infty(G\times G, \omega^{-1}\times\omega^{-1})$:

$$\sup\Big\{\frac{|\tilde{D}(g,h)|}{\omega(g)\omega(h)} : g,h\in G\Big\} = \sup\Big\{\frac{|\sum_{t\in Z(G)}\phi_t(g)\chi_t(gh)|}{\omega(g)\omega(h)} : g,h\in G\Big\}$$
$$= \sup\Big\{\frac{|\phi_{t'}(g)|}{\omega(g)\omega(g^{-1}t')} : g\in G, t'\in Z(G)\Big\} < \infty.$$

Also $\tilde{D}$ corresponds to the derivation $D : \ell^1(G,\omega) \to \ell^\infty(G,\omega^{-1})$ which satisfies equation (2.3). Since $gh = t$ if and only if $hg = t$ for every $t \in Z(G)$, then

$$\begin{aligned}\tilde{D}(gh,k) &= \sum_{t\in Z(G)} \phi_t(gh)\chi_t(ghk)\\ &= \sum_{t\in Z(G)} \phi_t(g)\chi_t(ghk) + \sum_{t\in Z(G)} \phi_t(h)\chi_t(hkg)\\ &= \tilde{D}(g,hk) + \tilde{D}(h,kg).\end{aligned}$$

Finally let $\{\phi_t\}_{t\in Z(G)}$ correspond to $\tilde{D}'$ and let $\tilde{D}'$ correspond to $\{\phi'_t\}_{t\in Z(G)}$. Then

$$\phi'_{t'}(g) = \tilde{D}'(g, g^{-1}t') = \sum_{t\in Z(G)} \phi_t(g)\chi_t(gg^{-1}t') = \phi_{t'}(g).$$

On the other hand if $\tilde{D}$ corresponds to $\{\phi'_t\}_{t\in Z(G)}$ and if $\{\phi'_t\}_{t\in Z(G)}$ corresponds to $\tilde{D}'$, then

$$\tilde{D}(g,h) = \sum_{t\in Z(G)} \phi'_t(g)\chi_t(gh) = \sum_{t\in Z(G)} \tilde{D}'(g,g^{-1}t)\chi_t(gh) = \tilde{D}'(g,h).$$

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