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On the Representation Categories of Matrix Quantum Groups of Type A*

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Dedicated to Professor Yu. I. Manin

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Abstract. A quantum groups of type A is defined in terms of a Hecke symmetry. We show in this paper that the representation category of such a quantum group is uniquely determined as an abelian braided monoidal category by the bi-rank of the Hecke symmetry.

1. Introduction

A matrix quantum group of type A is defined as the "spectrum" of the Hopf algebra associated to a closed solution of the (quantized) Yang-Baxter equation and the Hecke equation (called a Hecke symmetry). Explicitly, let V be a vector space (over a field) of finite dimension d. An invertible operator $R : V \otimes V \longrightarrow V \otimes V$ is called a Hecke symmetry if it satisfies the equations

$$R_1 R_2 R_1 = R_2 R_1 R_2, \tag{1}$$

where $R_1 := R \otimes \mathrm{id}_V, R_2 := \mathrm{id}_V \otimes R$ (the Yang-Baxter equation),

$$(R+1)(R-q) = 0, \quad q \neq 0; -1, \tag{2}$$

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(the Hecke equation) and is closed in the sense that the half dual operator

$$R^\sharp : V^* \otimes V \longrightarrow V \otimes V^*, \quad \langle R^\sharp(\xi \otimes v), w\rangle = \langle \xi, R(v \otimes w)\rangle,$$

is invertible.

Given such a Hecke symmetry one constructs a Hopf algebra H as follows. Fix a basis $\{x_i; 1 \leqslant i \leqslant d\}$ of V and let R^{ij}_{kl} be the matrix of R with respect to this basis. As an algebra H is generated by two sets of generators $\{z^i_j, t^i_j; 1 \leqslant i \leqslant d\}$, subject to the following relations (we will always adopt the convention of summing over the indices that appear in both upper and lower places):

$$\begin{aligned} R^{ij}_{pq} z^p_k z^q_l &= z^i_m z^j_n R^{mn}_{kl}, \\ z^i_k t^k_j &= t^i_k z^k_j = \delta^i_j. \end{aligned}$$

In case R is the usual symmetry operator: $R(v \otimes w) = w \otimes v$ (thus $q = 1$), H is isomorphic to the function algebra on the algebraic group $GL(V)$.

The most well-known Hecke symmetry is the Drinfeld–Jimbo solutions of series A to the Yang–Baxter equation (fix a square root $\sqrt{q}$ of q)

$$R^d_q(x_i \otimes x_j) = \begin{bmatrix} q x_i \otimes x_i & \text{if } i = j \\ \sqrt{q} x_j \otimes x_i & \text{if } i > j \\ \sqrt{q} x_j \otimes x_i - (q-1) x_i \otimes x_j & \text{if } i < j. \end{bmatrix} \tag{3}$$

In the "classical" limit $q \to 1$, R^d_q reduces to the usual symmetry operator. There is also a super version of these solutions due to Manin [12]. Let V be a vector superspace of super-dimension $(r|s)$, $r + s = d$, and let $\{x_i\}$ be a homogeneous basis of V, the parity of x_i is denoted by $\hat{\imath}$. The Hecke symmetry $R^{r|s}_q$ is given by

$$R^{r|s}_q(x_i \otimes x_j)b = \begin{bmatrix} (-1)^{\hat{\imath}} q x_i \otimes x_i & \text{if } i = j \\ (-1)^{\hat{\imath}\hat{\jmath}} \sqrt{q} x_j \otimes x_i & \text{if } i > j \\ (-1)^{\hat{\imath}\hat{\jmath}} \sqrt{q} x_j \otimes x_i - (q-1) x_i \otimes x_j & \text{if } i < j. \end{bmatrix} \tag{4}$$

In the "classical" limit $q \to 1$, $R^{r|s}_q$ reduces to the super-symmetry operator $R(x_i \otimes x_j) = (-1)^{\hat{\imath}\hat{\jmath}} x_j \otimes x_i$.

The quantum group associated to the Drinfeld–Jimbo solution (3) is called the standard quantum deformation of the general linear group or simply standard quantum general linear group. Similarly, the quantum general linear super-group is determined in terms of the solution (4) (actually, some signs must be inserted in the definition, see [12] for details).

There are many other non-standard Hecke symmetries and there is so far no classification of these solutions except for the case the dimension of V is 2. On the other hand, many properties of the associated quantum groups to these solutions are obtained in an abstract way. The aim of this work is to study representation category of the matrix quantum group associated to a Hecke symmetry, by this we understand the comodule category over the corresponding Hopf algebra. The pair (r, s), where r is the number of roots and s is the number of poles of $P_\wedge(t)$ (see 2.1.4), is called the *bi-rank* of the Hecke symmetry. The main result of this paper is that the category of comodules over the Hopf algebra associated to a Hecke symmetry, as a braided monoidal abelian category, depends only on the parameter q and the bi-rank.

The proof of the main result is inspired by the work [1] of Bichon, whose idea was to use a result of Schauenburg on the relationship between equivalences of comodule categories a pair of Hopf algebras and bi-Galois extensions.

The main result implies that the study of representations of a matrix quantum group of type A can be reduced to the study of that of a standard quantum general linear group. The latter has been studied by Zhang [14]. In particular we show that the homological determinant is always one-dimensional.

2. Matrix Quantum Group of Type A

Let V be a vector space of finite dimension d over a field k of characteristic zero. Let $R : V \otimes V \longrightarrow V \otimes V$ be a *Hecke symmetry*. Throughout this work we will assume that q *is not a root of unity other then the unity itself*. The entries of the matrix $R^\sharp$ are given by ${R^\sharp}^{kl}_{ij} = R^{ik}_{jl}$. Therefore, the invertibility of $R^\sharp$ can be expressed as follows: there exists a matrix P such that $P^{im}_{jn} R^{nk}_{ml} = \delta^i_l \delta^k_j$. Define the following algebras:

$$
\begin{aligned}
S &:= k\langle x_1, x_2, \dots, x_d\rangle / (x_k x_l R^{kl}_{ij} = q x_i x_j),\\
\wedge &:= k\langle x_1, x_2, \dots, x_d\rangle / (x_k x_l R^{kl}_{ij} = -x_i x_j),\\
E &:= k\langle z^1_1, z^1_2, \dots, z^d_d\rangle / (z^i_m z^j_n R^{mn}_{kl} = R^{ij}_{pq} z^p_k z^q_l),\\
H &:= k\langle z^1_1, z^1_2, \dots, z^d_d, t^1_1, t^1_2, \dots, t^d_d\rangle \Big/ \begin{pmatrix} z^i_m z^j_n R^{mn}_{kl} = R^{ij}_{pq} z^p_k z^q_l \\ z^i_k t^k_j = t^i_k z^k_j = \delta^i_j \end{pmatrix},
\end{aligned}
$$

where $\{x_i\}$, $\{z^i_j\}$ and $\{t^i_j\}$ are sets of generators.

The algebras $\wedge$ and S are called the *quantum anti-symmetric* and *quantum symmetric algebras* associated to R. Together they "define" a quantum vector space.

The algebra E is in fact a bialgebra with coproduct and counit given by

$$\Delta(z^i_j) = z^i_k \otimes z^k_j, \quad \varepsilon(z^i_j) = \delta^i_j.$$

The algebra H is a Hopf algebra with $\Delta(z^i_j) = z^i_k \otimes z^k_j$, $\Delta(t^i_j) = z^k_j \otimes z^i_k$, $\varepsilon(z^i_j) = \varepsilon(t^i_j) = \delta^i_j$. For the antipode, let $C^i_j := P^{im}_{jm}$. Then $S(z^i_j) = t^i_j$ and

$$S(t^i_j) = C^i_k z^k_l {C^{-1}}^l_j \tag{5}$$

[7, Thm. 2.1.1]. The matrix C plays an important role in our study, its trace is called the quantum rank of the Hecke symmetry, $\mathrm{Rank}_q R := \mathrm{tr}\ (C)$, see 2.2.1.

The bialgebra E is considered as *the function algebra on a quantum semi-group of type A* and the Hopf algebra H is considered as *the function algebra on a matrix quantum groups of A*. Representations of this (semi-)group are thus comodules over H (resp. E).

2.1. Comodules Over E

The space V is a comodule over E by the map $\delta : V \longrightarrow V \otimes E; x_i \longmapsto x_j \otimes z_i^j$. Since E is a bialgebra, any tensor power of V is also a comodule over E. The map $R : V \otimes V \longrightarrow V \otimes V$ is a comodule map. The classification of E-comodules is done with the help of the action of the Hecke algebra.

2.1.1. The Hecke Algebra

The Hecke algebra $\mathcal{H}_n = \mathcal{H}_{q,n}$ has generators $t_i, 1 \leqslant i \leqslant n-1$, subject to the relations:

$$\begin{aligned} t_i t_j &= t_j t_i, |i-j| \geq 2; \\ t_i t_{i+1} t_i &= t_{i+1} t_i t_{i+1}, 1 \leqslant i \leqslant n-2; \\ t_i^2 &= (q-1)t_i + q. \end{aligned}$$

There is a k-basis in $\mathcal{H}_n$ indexed by permutations of n elements: $t_w, w \in \mathfrak{S}_n$ ($\mathfrak{S}_n$ is the permutation group), in such a way that $t_{(i,i+1)} = t_i$ and $t_w t_v = t_{wv}$ if the length of wv is equal to the sum of the length of w and the length of v.

If q is not a root of unity of degree greater than 1, $\mathcal{H}_n$ is a semisimple algebra. It is isomorphic to the direct product of its minimal two-sided ideals, which are themselves simple algebras. The minimal two-sided ideals can be indexed by partitions of n. Thus

$$\mathcal{H}_n \cong \prod_{\lambda \vdash n} A_\lambda,$$

where A_λ denotes the minimal two-sided ideal corresponding to λ. Each A_λ is a matrix ring over the ground field and one can choose a basis $\{e_\lambda^{ij}; 1 \leqslant i, j \leqslant d_\lambda\}$ such that

$$e_\lambda^{ij} e_\lambda^{kl} = \delta_k^j e_\lambda^{il},$$

where d_λ is the dimension of the simple $\mathcal{H}_n$-comodule corresponding to λ and can be computed by the combinatorics of λ-tableaux. In particular, $\{e_\lambda^{ii}, 1 \leqslant i \leqslant d_\lambda\}$ are mutually orthogonal conjugate primitive idempotents of $\mathcal{H}_n$. For more details, the reader is referred to [2, 3].

2.1.2. An Action of $\mathcal{H}_n$

R induces an action of the Hecke algebra $\mathcal{H}_n = \mathcal{H}_{q,n}$ on $V^{\otimes n}$, $t_i \longmapsto R_i = \mathrm{id}^{i-1} \otimes R \otimes \mathrm{id}^{n-i-1}$ which commutes with the coaction of E. The action of t_w will be denoted by R_w.

Thus, each element of $\mathcal{H}_n$ determines an endomorphism of $V^{\otimes n}$ as an E-comodule. For q not a root of unity of degree greater 1, the converse is also true: each endomorphism of $V^{\otimes n}$ represents the action of an element of $\mathcal{H}_n$, moreover $V^{\otimes n}$ is semi-simple and its simple subcomodules can be given as the images of the endomorphisms determined by primitive idempotents of $\mathcal{H}_n$, conjugate idempotents (i.e. belonging to the same minimal two-sided ideal) determine isomorphic comodules [7].

Since conjugate classes of primitive idempotents of $\mathcal{H}_n$ are indexed by partitions of n, simple subcomodules of $V^{\otimes n}$ are indexed by a subset of partitions of n. Thus E is cosemisimple and its simple comodules are indexed by a subset of partitions.

2.1.3. Quantum Symmetrizers

Denote

$$[n]_q := \frac{q^n - 1}{q - 1}.$$

The primitive idempotent corresponding to partition (n) of n,

$$X_n := \frac{1}{[n]_q} \sum_{w \in \mathfrak{S}_n} R_w,$$

determines a simple comodule isomorphic to the n-th homogeneous component S_n of the quantum symmetric algebra S (the n-th quantum symmetric power) and the primitive idempotent corresponding to partition (1^n) of n,

$$Y_n := \frac{1}{[n]_{1/q}} \sum_{w \in \mathfrak{S}_n} (-q)^{-l(w)} R_w,$$

determines a simple comodule isomorphic to the n-th homogeneous component $\wedge_n$ of the quantum exterior algebra $\wedge$ (the n-th quantum anti-symmetric power).

2.1.4. The Bi-Rank

There is a determinantal formula in the Grothendieck ring of finite dimensional E-comodules which computes simple comodules in terms of quantum symmetric tensor powers [7]:

$$I_\lambda = \det |S_{\lambda_i - i + j}|_{1 \leqslant i,j \leqslant k}; \quad k \text{ is the length of } \lambda. \tag{6}$$

Consequently, we have a similar form for the dimensions of simple comodules. It follows from this and a theorem of Edrei on Pólya frequency sequences that the Poincaré series of $\wedge$ is rational function with negative zeros and positive poles [6]. The pair (r, s) where r is the number of zeros and s is the number of poles is called the *bi-rank* of the Hecke symmetry. It then follows from (6) that the E-comodule I_λ is non-zero, and hence simple, if and only if $\lambda_r \leqslant s$. The set of partitions of n satisfying this property is denoted by $\Gamma_n^{r,s}$. Simple E-comodules are thus completely classified in terms of the bi-rank.

2.2. The Hopf Algebra H and Its Comodules

2.2.1. The Koszul Complex

Through the natural map $E \longrightarrow H$ E-comodules are comodule over H. Since H is a Hopf algebra, for the comodules S_n and $\wedge_n$, their dual spaces ${S_n}^*, {\wedge_n}^*$ are also comodules over H. One can define H-comodule maps

$$d^{k,l} : K^{k,l} := \wedge_k \otimes {S_l}^* \longrightarrow K^{k+1,l+1} := \wedge_{k+1} \otimes {S_{l+1}}^*, \quad k, l \in \mathbb{Z},$$

in such a way that the sequence

$$K^a : \cdots \longrightarrow \wedge_{k-1} \otimes {S_{l-1}}^* \longrightarrow \wedge_k \otimes {S_l}^* \longrightarrow \wedge_{k+1} \otimes {S_{l+1}}^* \cdots$$

($a = k - l$) is a complex. This complexes were introduced by Manin for the case of standard Hecke symmetry [12] and studied by Gurevich, Lyubashenko, Sudbery [5, 10].

It is expected that the homology of this complexes is concentrated at a certain term where it has dimension one, in this case it induces a group-like element in H, called *the homological determinant* as suggested by Manin. Gurevich showed that all the complexes K^a, $a \in \mathbb{Z}$, except might be for the complex K^b with $[-b]_q = -\mathrm{rank}_q R$, are exact. For the case of even Hecke symmetries he showed that the homology is one-dimensional [5]. The homology of the complex K^b was shown to be one-dimensional by Manin for the case of standard Hecke symmetry [12]. This fact has also been shown by Lyubashenko-Sudbery for Hecke sums of odd and even Hecke symmetries [10]. A combinatorial proof for Hecke symmetries of birank (2.1) was given in [4].

In [9] the author showed that the homology should be non-vanishing at the term $\wedge_r \otimes S_s{}^*$ and consequently the quantum rank $\mathrm{rank}_q R := \mathrm{tr}\ (C)$ is equal to $-[s-r]_q$.

2.2.2. The Integral

In the study of the category H-comod, the integral over H plays an important role as shown in [4]. By definition, a right integral over H is a (right) comodule map $H \longrightarrow k$ where H coacts on itself by the coproduct and on the base field k by the unit map. The existence of the integral on H was proven in [8, Thm.3.2], under the assumption that $\mathrm{rank}_q R = -[s-r]_q$, which was later shown in [9] for an arbitrary Hecke symmetry. In fact, an explicit form for the integral was given. Since we will need it later on, let us recall it here.

For a partition λ of n, let $[\lambda]$ be the corresponding tableau and for any node $x \in [\lambda]$, $c(x)$ be its content, $h(x)$ its hook-length, $n(\lambda) := \sum_{x\in[\lambda]} c(x)$ (see [11] for details). Let

$$p_\lambda := \prod_{x\in[\lambda]\setminus[(s^r)]} q^{r-s}[c(x)+r-s]_q{}^{-1}, \quad k_\lambda := q^{n(\lambda)} \prod_{x\in[\lambda]} [h(x)]_q^{-1},$$

where (s^r) is the sub-tableau of λ consisting of nodes in the i-th row and j-th column with $i \leqslant s, j \leqslant r$. In particular, $p_\lambda = 0$ if $\lambda_r < s$. Let $\Omega_n^{r,s}$ denote the set of partitions from $\Gamma_n^{r,s}$ such that $p_\lambda \neq 0$. Thus $\Omega_n^{r,s} = \{\lambda \vdash n; \lambda_r = s\}$.

Denote for each set of indices $I = (i_1, i_2, \ldots, i_n)$, $J = (j_1, j_2, \ldots, j_n)$

$$Z_J^I := z_{j_1}^{i_1} z_{j_2}^{i_2} \ldots z_{j_n}^{i_n}; \quad T_{J'}^{I'} := t_{j_n}^{i_n} \ldots t_{j_2}^{i_2} t_{j_1}^{i_1}.$$

Then the value of the integral on $Z_J^I T_{L'}^{K'}$ can be given as follows

$$\int (Z_I^J T_{K'}^{L'}) = \sum_{\substack{\lambda\in\Omega_n^{r,s}\\ 1\leqslant i,j\leqslant d_\lambda}} \frac{p_\lambda}{k_\lambda} (C^{\otimes n} E_\lambda^{ij})_I^L (E_\lambda^{ji})_K^J, \tag{7}$$

where E_λ^{ij} is the matrix of the basis element e_λ^{ij} in the representation ρ_n, the matrix C is given in (5). In particular, the left hand-side is zero if $n < rs$.

3. Bi-Galois Extensions

Let A be a Hopf algebra over a field k. A right A-comodule algebra is a right A-comodule with the structure of an algebra on it such that the structure maps (the multiplication and the unit map) are A-comodule maps. A right A-Galois extension M/k is a right A-comodule algebra M such that the Galois map

$$\kappa_r : M \otimes M \longrightarrow M \otimes A; \quad \kappa_r(m \otimes n) = \sum_{(n)} mn_{(0)} \otimes n_{(1)}, \tag{8}$$

is bijective. Similarly one has the notion of left A-Galois extension, in which M is a left A-comodule algebra and the Galois map is $\kappa_l : M \otimes M \longrightarrow A \otimes M$; $m \otimes n \longmapsto \sum_{(m)} m_{(-1)} \otimes m_{(0)}n$.

Lemm 3.1. *Let M be a right A-comodule algebra. Assume that there exists an algebra map $\gamma : A \longrightarrow M^{\mathrm{op}} \otimes M$, $a \longmapsto \sum_{(a)} a^- \otimes a^+$ such that the following equations in $M \otimes M$ hold true*

$$\begin{aligned} &\sum_{(m)} m_{(0)}{m_{(1)}}^- \otimes {m_{(1)}}^+ = 1 \otimes m, \quad m \in M, \\ &\sum_{(a)} a^-{a^+}_{(0)} \otimes {a^+}_{(1)} = 1 \otimes a; \quad a \in A. \end{aligned} \tag{9}$$

Then M is a right A-Galois extension of k. For left Galois extension the conditions read: $\gamma : A \longrightarrow M \otimes M^{\mathrm{op}}$, $a \longmapsto \sum_{(a)} a^+ \otimes a^-$,

$$\begin{aligned} &\sum_{(m)} {m_{(-1)}}^+ \otimes {m_{(-1)}}^- m_{(0)} = m \otimes 1; \quad m \in M, \\ &\sum_{(a)} {a^+}_{(-1)} \otimes {a^+}_{(0)}a^- = a \otimes 1; \quad a \in A. \end{aligned} \tag{10}$$

Proof. The inverse to κ_r is given in terms of γ as follows: $m \otimes a \longmapsto \sum_{(a)} ma^- \otimes a^+$. For κ_l, the inverse is given by $a \otimes m \longmapsto \sum_{(a)} a^+ \otimes a^- m$. ■

Remark. We see from the proof that the map γ can be obtained from κ_r as follows: $\gamma(a) = {\kappa_r}^{-1}(1 \otimes a)$. Then, one can show that γ is an algebra homomorphism. In fact, in the above proof, we do not use the fact that γ is an algebra homomorphism. We assume it however, since the equations in (9) and (10) respect the multiplications in A and M, that is, if an equation holds true for a and a' in A or m and m' in M then it holds true for the products aa' or mm' respectively. Therefore it is sufficient to check this conditions on a set of generators of A and M.

Now let A and B be Hopf algebras and M an $A - B$-bi-comodule, i.e. M is left A-comodule and right B-comodule and the two coactions are compatible. M is said to be an $A - B$-bi-Galois extension of k if it is both a left A-Galois extension and a right B-Galois extension of k. We will make use of the following fact [13, Cor. 5.7]:

There exists a 1-1 correspondence between the set of isomorphic classes of (non-zero) $A-B$-bi-Galois extension of k and k-linear monoidal equivalences between the categories of comodules over A and B.

The equivalence functor is given in terms of the co-tensor product with the bi-comodule. Recall that each $A-B$-bi-comodule M defines an additive functor from the category A-comod of right A-comodules to the category B-comod $X \longmapsto X\Box_A M$, where the co-tensor product $X\Box_A M$ is defined as the equalizer of the two maps induced from the coactions on A:

$$X\Box_A M \longrightarrow X\otimes_k M \underset{\delta_X\otimes\mathrm{id}}{\overset{\mathrm{id}\otimes\delta_M}{\rightrightarrows}} X\otimes_k A\otimes_k M,$$

or, explicitly,

$$X\Box_A M = \{x\otimes m\in X\otimes_k M|\sum_{(m)} x\otimes m_{(-1)}\otimes m_{(0)} = \sum_{(x)} x_{(0)}\otimes x_{(1)}\otimes m\}.$$

The coaction of B on $X\Box_A M$ is induced from that on M.

4. A Bi-Galois Extension for Matrix Quantum Groups

Let R and $\bar{R}$ be Hecke symmetries and H, $\bar{H}$ be the associated Hopf algebras. We construct in this subsection an $H-\bar{H}$-bi-Galois extension.

Assume that R is defined on a vector space of dimension d and $\bar{R}$ is defined over a vector space of dimension $\bar{d}$. Consider the algebra $M = M_{R,\bar{R}}$ generated by elements $a^i_\lambda, b^\lambda_i; 1\leqslant i\leqslant d, 1\leqslant\lambda\leqslant\bar{d}$, subject to the following relations

$$\begin{aligned} &R^{ij}_{pq}a^p_\lambda a^q_\mu = a^i_\nu a^j_\gamma \bar{R}^{\nu\gamma}_{\lambda\mu},\\ &a^i_\lambda b^\lambda_j = \delta^i_j;\quad b^\lambda_k a^k_\mu = \delta^\lambda_\mu. \end{aligned}$$

The following equations can also be deduced from the equations above

$$\begin{aligned} &R^{mn}_{kl}b^\lambda_n b^\mu_m = b^\gamma_k b^\nu_l \bar{R}^{\lambda\mu}_{\nu\gamma},\\ &P^{qp}_{lk}a^l_\nu b^\gamma_q = b^\mu_k a^p_\lambda \bar{P}^{\gamma\lambda}_{\nu\mu},\\ &a^l_\gamma C^q_l b^\nu_q = \bar{C}^\nu_\gamma. \end{aligned}$$

The proof is completely similar to that of [7, Thm. 2.1.1].

Lemma 4.1. *Assume that the algebra M constructed above is non-zero. Then it is an $H-\bar{H}$-bi-Galois extension of k.*

Proof. The coactions of H and $\bar{H}$ on M are given by

$$\begin{aligned} &\delta: M\longrightarrow H\otimes M;\quad a^i_j\longmapsto z^i_k\otimes a^k_j,\quad b^j_i\longmapsto t^k_i\otimes b^j_k,\\ &\bar{\delta}: M\longrightarrow M\otimes\bar{H};\quad a^i_j\longmapsto a^i_k\otimes\bar{z}^k_j,\quad b^j_i\longmapsto b^k_i\otimes\bar{t}^j_k. \end{aligned}$$

The verification that this maps induce a structure of left H-comodule (resp. right $\bar{H}$-comodule) algebra over H and an $H-\bar{H}$ bi-comodule structure is straightforward.

According to Lemma 3.1 and the remark following it, to show that M is a left H-Galois extension of k it suffices to construct the map γ satisfying the condition of the lemma. Define

$$\begin{aligned}\gamma(z^i_j) &= \ a^i_\mu \otimes b^\mu_j,\\ \gamma(t^i_j) &= \ b^\mu_j \otimes \bar{C}^{-1}{}^\nu_\mu a^l_\nu C^j_l,\end{aligned}$$

and extend them to algebra maps. Using the relations on M one can check easily that this map gives rise to an algebra homomorphism $H \longrightarrow M \otimes M^{\mathrm{op}}$. Since M is now an H-comodule algebra and since γ is algebra homomorphism, the equations in Lemma 3 respect the multiplications in M and in H, that is, it suffices to check them for the generators z^i_j and t^i_j which follows immediately from the relations mentioned above on the a^i_λ and b^μ_j. ∎

Notice that in the proof of this lemma the Hecke equation is not used.

Lemma 4.2. *Let R and $\bar{R}$ be Hecke symmetries defined over V and $\bar{V}$ respectively. Assume that they are defined for the same value q and have the same bi-rank. Then the associated algebra $M = M_{R,\bar{R}}$ is non-zero.*

Proof. To show that M is non-zero we construct a linear functional on M and show that this linear functional attains a non-zero value at some element of M. The construction of the linear functional resembles the integral on the Hopf algebra H given in the previous section. In fact, using the same method as in the proof of Theorem 3.2 and Equation 3.6 of [8] we can show that there is a linear functional on M given by

$$\int (A^J_\Lambda B^{\Gamma'}_{K'}) = \sum_{\substack{\lambda\in\Omega^{r,s}_n\\ 1\leqslant i,j\leqslant d_\lambda}} \frac{p_\lambda}{k_\lambda} (\bar{C}^{\otimes n}\bar{E}^{ij}_\lambda)^\Gamma_\Lambda (E^{ji}_\lambda)^J_K,$$

where Λ, Γ, I, J are multi-indices of length n and (r,s) is the bi-rank of R and $\bar{R}$.

According to Subsecs. 1.2.4 and 1.3.1 for $n \geq rs$ and $\lambda \in \Omega^{r,s}_n$ the matrices E^{ji}_λ and $\bar{E}^{ij}_\lambda$ are all non-zero, therefore the linear functional $\int$ does not vanish identically on M, for example

$$\int ((E^{ii} A \bar{E}^{ii})^J_\Lambda (\bar{E}^{ii}_\lambda B E^{ii}_\lambda)^{\Gamma'}_{K'}) = \frac{p_\lambda}{k_\lambda} (\bar{C}^{\otimes n}\bar{E}^{ii})^\Gamma_\Lambda E^{ii}_\lambda{}^J_K$$

is non-zero for a suitable choice of indices K, J, Γ, Λ. ∎

Theorem 4.3. *Let R and $\bar{R}$ be Hecke symmetries defined respectively on V and $\bar{V}$. Then there is a monoidal equivalence between H-comod and $\bar{H}$-comod sending V to $\bar{V}$ and presvering the braiding if and only if R and $\bar{R}$ are defined with the same parameter q and have the same bi-rank.*

Proof. Assume that R and $\bar{R}$ satisfies the condition of the theorem. According to the lemma above it remains to prove that the monoidal functor given by co-tensoring with M sends V to $\bar{V}$ and R to $\bar{R}$. Indeed, by the definition of $V\Box_H M$, the map $\bar{V} \longrightarrow V\Box_H M$ given by $\bar{x}_\lambda \longmapsto x_j \otimes a^j_\lambda$ is an injective $\bar{H}$-comodule homomorphism. According to Lemma 4.2 and Schauenburg's result, $V\Box_H M$ is a simple $\bar{H}$-comodule, therefore $\bar{V}$ is isomorphic to $V\Box_H M$. It is then easy to see that R is mapped to $\bar{R}$.

The converse statement is obvious. First, since R is mapped to $\bar{R}$ they should be defined for the same value of q. Further, according to Subsec. 2.1.4, let (r,s) and $(\bar{r},\bar{s})$ be the bi-ranks of R and $\bar{R}$, respectively. Then $\Gamma^{r,s}_n = \Gamma^{\bar{r},\bar{s}}_n$ for all n, whence $(r,s) = (\bar{r},\bar{s})$. ■

Notice that if $(r,s) \neq (\bar{r},\bar{s})$ and $r - s = \bar{r} - \bar{s}$ then $\Omega^{r,s}_n \cap \Omega^{\bar{r},\bar{s}}_n = \emptyset$. This implies also that the linear functional in Lemma 4.2 is zero.

Theorem 4.3 states that the study of comodules over a Hopf algebra associated to a Hecke symmetry of bi-rank (r,s) can be reduced to the study of the Hopf algebra associated to the standard solution $R^{r,s}_q$. For the latter Hopf algebra simple comodules were classified by Zhang [14]. As an immediate consequence of Theorem 4.3, we have:

Corollary 4.4. *Let R be a Hecke symmetry of bi-rank (r,s). Then the homology of the associated Koszul complex (cf. Subsection 2.2.1) is concentrated at the term $K^{r,s}$ and has dimension one. Thus one has a homological determinant.*

Proof. In fact, the statement for $\bar{R} = R^{r,s}_q$ was proved by Manin [12]. Now, according to Theorem 4.3, for $M = M_{R,\bar{R}}$ the functor $-\Box_R M$ is fully faithful and exact hence the homology of the Koszul complex associated to R is concentrated at the term r,s as the one associated to $\bar{R}$ is. Since the homology group of the complex associated to $\bar{R}$ is one dimensional and being an $\bar{H}$-comodule, it is an invertible comodule. Therefore the homology group of the complex associated to R is also invertible as an H-comodule, hence is one-dimensional. ■

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References

1. J. Bichon, The representation category of the quantum group of a non-degenerate bilinear form, *Communications in Algebra* **31** (2003) 4831–4851.
2. R. Dipper and G. James, Representations of Hecke algebras of general linear groups, *Proc. London Math. Soc.* **52** (1986) 20–52.
3. R. Dipper and G. James, Block and idempotents of Hecke algebras of general linear groups. *Proc. London Math. Soc.* **54** (1987) 57–82.

4. N. P. Dung and P. H. Hai, On the Poincaré series of quadratic algebras associated to Hecke symmetries, *Intern. Res. Math. Notices* **40** (2003) 2193–2203.
5. D. I. Gurevich, Algebraic aspects of the quantum Yang-Baxter equation, *Leningrad Math. J.* **2** (1991) 801–828.
6. P. H. Hai, Poincaré series of quantum qpaces associated to Hecke operators, *Acta Math. Vietnam.* **24** (1999) 236–246.
7. P. H. Hai, On matrix quantum groups of type A_n, *Intern. J. Math.* **11** (2000) 1115–1146.
8. P. H. Hai, The integral on quantum super groups of type $A_{r|s}$, *Asian J. Math.* **5** (2001) 751–770.
9. P. H. Hai, The homological determinant of a quantum general linear supergroup, *Proc. of the AMS* **133** (2005) 1897–1905.
10. V. V. Lyubashenko and A. Sudbery, Quantum supergroups of $\mathrm{GL}(n|m)$ type: Differential forms, Koszul complexes and Berezinians, *Duke Math. J.* **90** (1997) 1–62.
11. I. G. Macdonald, *Symmetric Functions and the Hall Polynomials*, Oxford University Press, New York, 1979 (2^{nd} Edition 1995).
12. Yu. I. Manin, Multiparametric quantum deformation of the general linear supergroups, *Comm. Math. Phys.* **123** (1989) 163–175.
13. P. Schauenburg, Hopf bi-Galois extensions, *Comm. Algebra* **24** (1996) 3797–3825.
14. R. B. Zhang, Structure and representations of the quantum general linear supergroup, *Comm. Math. Phys.* **195** (1998) 525–547.