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The Weak Topology on the Space of Probability Capacities in $\mathbb{R}^d$ *

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Abstract. It is shown that the space of probability capacities in $\mathbb{R}^d$ equipped with the weak topology is separable and metrizable, and contains $\mathbb{R}^d$ topologically.

1. Introduction

Non-additive set functions plays an important role in several areas of applied sciences, including Artificial Intelligence, Mathematical Economics and Bayesian statistics. A special class of non-additive set functions, known as capacities, has been intensively studied during the last thirty years, see, e.g., [3, 5-7, 9]. Although some interesting results in the theory of capacities has been established for Polish spaces, the fundamental study of capacities has focused on $\mathbb{R}^d$, the $d-$dimensional Euclidean space (e.g., [7, 9]).

In this paper we investigate some topological properties of the space of probability capacities equipped with the weak topology. The main result of this paper shows that the space of probability capacities equipped with the weak topology is separable and metrizable, and contains $\mathbb{R}^d$ topologically.

2. Notation and Convention

We first recall some definitions and facts used in this paper. Let $\mathcal{K}(\mathbb{R}^d)$, $\mathcal{F}(\mathbb{R}^d)$,

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$\mathcal{G}(\mathbb{R}^d)$, $\mathcal{B}(\mathbb{R}^d)$ denote the family of all *compact sets, closed sets, open sets* and *Borel sets* in $\mathbb{R}^d$, respectively.

By a *capacity* in $\mathbb{R}^d$ we mean a set function $T: \mathbb{R}^d \to \mathbb{R}^+ = [0, +\infty)$ satisfying the following conditions:
(i) $T(\emptyset) = 0$;
(ii) T is alternating of infinite order: For any Borel sets $A_i,\ i = 1, 2, \dots, n;\ n \geq 2$, we have

$$T\big(\bigcap_{i=1}^{n} A_i\big) \leq \sum_{I \in \mathcal{I}(n)} (-1)^{\#I+1} T\big(\bigcup_{i \in I} A_i\big), \tag{2.1}$$

where $\mathcal{I}(n) = \{I \subset \{1, \dots, n\},\ I \neq \emptyset\}$ and $\#I$ denotes the cardinality of I;
(iii) $T(A) = \sup\{T(C):\ C \in \mathcal{K}(\mathbb{R}^d),\ C \subset A\}$ for any Borel set $A \in \mathcal{B}(\mathbb{R}^d)$;
(iv) $T(C) = \inf\{T(G):\ G \in \mathcal{G}(\mathbb{R}^d),\ C \subset G\}$, for any compact set $C \in \mathcal{K}(\mathbb{R}^d)$.

A capacity in $\mathbb{R}^d$ is, in fact, a generalization of a measure in $\mathbb{R}^d$. Clearly any capacity is a non-decreasing set function on Borel sets of $\mathbb{R}^d$.

By *support* of a capacity T we mean the smallest closed set $S \subset \mathbb{R}^d$ such that $T(\mathbb{R}^d \backslash S) = 0$. The support of a capacity T is denoted by supp T. We say that T is a *probability capacity* in $\mathbb{R}^d$ if T has a compact support and $T(\text{supp } T) = 1$. By $\tilde{\mathcal{C}}$ we denote the family of all probability capacities in $\mathbb{R}^d$.

Let T be a capacity in $\mathbb{R}^d$. Then for any measurable function $f: \mathbb{R}^d \to \mathbb{R}^+$ and $A \in \mathcal{B}(\mathbb{R}^d)$, the function $f_A: \mathbb{R} \to \mathbb{R}$ defined by

$$f_A(t) = T(\{x \in A:\ f(x) \geq t\}) \text{ for } t \in \mathbb{R} \tag{2.2}$$

is a non-increasing function in t. Therefore we can define the *Choquet integral* $\int_A f\, dT$ of f with respect to T by

$$\int_A f\, dT = \int_0^{\infty} f_A(t)dt = \int_0^{\infty} T(\{x \in A:\ f(x) \geq t\})dt. \tag{2.3}$$

If $\int_A f\, dT < \infty$, we say that f is *integrable*. In particular for $A = \mathbb{R}^d$ we write

$$\int_{\mathbb{R}^d} f\, dT = \int f\, dT.$$

Observe that if f is bounded, then

$$\int_A f\, dT = \int_0^{\alpha} T(\{x \in A:\ f(x) \geq t\}dt, \tag{2.4}$$

where $\alpha = \sup\{f(x):\ x \in A\}$.

In the general case if $f:\ \mathbb{R}^d \to \mathbb{R}$ is a measurable function, we define

$$\int_A f\, dT = \int_A f^+ dT - \int_A f^- dT, \tag{2.5}$$

where $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$.

3. The Weak Topology on the Space of Probabilitiy Capacities

Let $\mathbf{B}$ be a family of sets of the form

$$\mathbf{B} = \{\mathcal{U}(T; f_1, \dots, f_k; \epsilon_1, \dots, \epsilon_k) : \ T \in \tilde{\mathcal{C}}, \ f_i \in C_0^+(\mathbb{R}^d), \ \epsilon_i > 0, \ i = 1, \dots, k\}, \tag{3.1}$$

where

$$\mathcal{U}(T; f_1, \dots, f_k; \epsilon_1, \dots, \epsilon_k) = \{S \in \tilde{\mathcal{C}} : \ |\int f_i dT - \int f_i \ dS| < \epsilon_i, \ i = 1, \dots, k\}$$
$$= \bigcap_{i=1}^{k} \mathcal{U}(T; f_i; \epsilon_i), \tag{3.2}$$

and $C_0^+(\mathbb{R}^d)$ denotes all continuous non-negative real valued functions with compact support in $\mathbb{R}^d$. Obviously the family $\mathbf{B}$ is a base of a topology on $\tilde{\mathcal{C}}$. This topology is called the *weak topology* on $\tilde{\mathcal{C}}$.

For any point $x \in \mathbb{R}^d$ let $T_x = \delta_x$ be the set function defined by

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases} \tag{3.3}$$

for $A \in \mathcal{B}(\mathbb{R}^d)$. Clearly that T_x is a probability capacity in $\mathbb{R}^d$. The following Lemma is proved in [9].

Lemma 3.1. *For* $x \in \mathbb{R}^d$ *we take* $T_x = \delta_x$ *with* δ_x *defined by (3.3). Then for any measurable function* $f : \ \mathbb{R}^d \to \mathbb{R}^+$ *we have*

$$\int f \ dT_x = f(x) \text{ for every } x \in \mathbb{R}^d. \tag{3.4}$$

Let $\tilde{\mathcal{C}}$ denotes the space of all probability capacities in $\mathbb{R}^d$ equipped with the weak topology. In this section we show that

Theorem 3.2. $\tilde{\mathcal{C}}$ *is separable and metrizable.*

The Theorem will be proved by Propositions 3.3 and 3.7 below.

Proposition 3.3. $\tilde{\mathcal{C}}$ *is a regular space.*

Proof. Assume that $\mathcal{A}$ is a closed set in $\tilde{\mathcal{C}}$, $T \in \tilde{\mathcal{C}}$ and $T \notin \mathcal{A}$. We will show that there are neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of T and $\mathcal{A}$ respectively, such that $\mathcal{U} \cap \mathcal{V} = \emptyset$. Since $\mathcal{A}$ is closed and $T \notin \mathcal{A}$, there exist $f_i \in C_0^+(\mathbb{R}^d)$, and $\epsilon_i > 0, \ i = 1, \dots, k$ such that

$$\mathcal{U}(T; f_1, \dots, f_k; \epsilon_1, \dots, \epsilon_k) \cap \mathcal{A} = \emptyset. \tag{3.5}$$

For each $i = 1, \dots, k$, we define

$$\mathcal{A}_i = \{S \in \mathcal{A} : S \notin \mathcal{U}(T; f_i; \epsilon_i)\}. \tag{3.6}$$

From (3.5) and (3.6) we get

$$\mathcal{A} = \bigcup_{i=1}^{k} \mathcal{A}_i. \tag{3.7}$$

We put

$$\mathcal{V}_i = \bigcup_{S \in \mathcal{A}_i} \mathcal{U}(S; f_i; \epsilon_i/3). \tag{3.8}$$

For any $S' \in \mathcal{V}_i$ and for any $T' \in \mathcal{U}(T; f_i; \epsilon_i/3)$ from (3.6) and (3.8) we have

$$\begin{aligned}|\int f_i\, dT' - \int f_i\, dS'| \geq |\int f_i\, dT - \int f_i\, dS| - |\int f_i\, dT - \int f_i\, dT'| \\ - |\int f_i\, dS - \int f_i\, dS'| \\ > \epsilon_i - \frac{\epsilon_i}{3} - \frac{\epsilon_i}{3} = \frac{\epsilon_i}{3} > 0\end{aligned}$$

for some $S \in \mathcal{A}_i$. That means

$$\mathcal{U}(T; f_i; \epsilon_i/3) \cap \mathcal{V}_i = \emptyset \text{ for } i = 1, \dots, k.$$

Hence

$$\mathcal{U}(T; f_1, \dots, f_k; \epsilon_1/3, \dots, \epsilon_k/3) \cap (\bigcup_{i=1}^{k} \mathcal{V}_i) = \emptyset.$$

Consequently, take $\mathcal{U} = \mathcal{U}(T; f_1, \dots, f_k; \epsilon_1/3, \dots, \epsilon_k/3)$ and $\mathcal{V} = \bigcup_{i=1}^{k} \mathcal{V}_i$ to complete the proof of the proposition. ■

Let T be a probability capacity in $\mathbb{R}^d$ and let $f: \mathbb{R}^d \to \mathbb{R}$ be a continuous function with compact support and let $\{x_i : i = 1, \dots, k\} \subset \operatorname{supp} f$ be a finite set in $\operatorname{supp} f$ for which we may assume that

$$0 < f(x_1) < f(x_2) < \cdots < f(x_k).$$

We take $x_0 \in \mathbb{R}^d$ with $f(x_0) = 0$, put $A = \{x_i : i = 0, 1, \dots, k\}$ and define

$$T_f^A = \sum_{i=1}^{k-1} (t_i - t_{i+1}) \delta_{x_i} + t_k \delta_{x_k} + (1 - t_1) \delta_{x_0},$$

where $t_i = T(\{x \in \mathbb{R}^d : f(x) > f(x_i)\})$ for $i = 1, \dots, k$ and δ_x is defined by (3.3).

Observe that $T_f^A \in \tilde{\mathcal{C}}$ and

$$\begin{aligned}\int f\, dT_f^A &= \sum_{i=1}^{k-1} (t_i - t_{i+1}) f(x_i) + t_k f(x_k) \\ &= \sum_{i=0}^{k-1} [f(x_{i+1}) - f(x_i)] t_{i+1}.\end{aligned} \tag{3.9}$$

To prove Proposition 3.7 we need Lemmas 3.4 and 3.6 below.

Lemma 3.4. *Let D be a countable dense set in $\mathbb{R}^d$. Then for any $T \in \tilde{\mathcal{C}}$, for any $f \in C_0^+(\mathbb{R}^d)$ and for any $\epsilon > 0$ there exists a finite set $A = \{x_i : i = 0, 1, \dots, k-1\} \subset D$ such that $T_f^A \in \mathcal{U}(T; f; \epsilon)$.*

Proof. For $T \in \tilde{\mathcal{C}}$ and $f \in C_0^+(\mathbb{R}^d)$ we have

$$\int f\, dT = \int_0^\alpha T(\{x \in \mathbb{R}^d :\ f(x) \geq t\})dt,$$

where $\alpha = \sup\{f(x) :\ x \in \mathbb{R}^d\} < \infty$. Note that by the compactness of supp$f$, we have

$$\alpha_0 = \inf\{f(x) :\ x \in \mathbb{R}^d\} = 0.$$

Since $f \in C_0^+(\mathbb{R}^d)$ and D is dense in $\mathbb{R}^d$, $D \cap (\text{supp } f)$ is dense in supp f. Therefore, for any $\epsilon > 0$ we can choose $x_i \in D \cap (\text{supp } f),\ i = 1, \dots, k-1$ with

$$\alpha_0 = 0 < \alpha_1 = f(x_1) < \cdots < \alpha_{k-1} = f(x_{k-1}) \leq \alpha_k = \alpha \tag{3.10}$$

such that

$$0 \leq \alpha_{i+1} - \alpha_i < \epsilon \text{ for every } i = 0, \dots, k-1. \tag{3.11}$$

For every $i = 0, \dots, k$ let

$$t_i = T(\{x \in \mathbb{R}^d :\ f(x) > \alpha_i\}). \tag{3.12}$$

Then for $t \in (\alpha_i, \alpha_{i+1}]$ we have

$$t_{i+1} \leq\ T(\{x \in \mathbb{R}^d :\ f(x) \geq t\}) \leq\ t_i,\ i = 0, \dots, k-1.$$

Hence, by virtue of (3.11) and with noting that $t_0 \leq 1$ and $t_k = 0$ (see (3.12)) we have

$$\begin{aligned} 0 \leq \int f\, dT - \sum_{i=0}^{k-1} (\alpha_{i+1} - \alpha_i)t_{i+1} &\leq \sum_{i=0}^{k-1} [(\alpha_{i+1} - \alpha_i)t_i - (\alpha_{i+1} - \alpha_i)t_{i+1}] \\ &= \sum_{i=0}^{k-1} (\alpha_{i+1} - \alpha_i)(t_i - t_{i+1}) \\ &< \epsilon \sum_{i=0}^{k-1} (t_i - t_{i+1}) \leq \epsilon. \end{aligned}$$

That means

$$|\int f\, dT - \sum_{i=0}^{k-1} (\alpha_{i+1} - \alpha_i)t_{i+1}| < \epsilon. \tag{3.13}$$

We take $x_0 \in D$ such that $f(x_0) = 0$. Note that for $A = \{x_i :\ i = 0, \dots, k-1\}$ from (3.9) we have

$$\int f\ dT_f^A = \sum_{i=0}^{k-2} (\alpha_{i+1} - \alpha_i) t_{i+1}. \tag{3.14}$$

Thus, from (3.13) and (3.14) we get

$$|\int f\ dT - \int f\ dT_f^A| < \epsilon + (\alpha_k - \alpha_{k-1}) t_k = \epsilon.$$

Therefore $T_f^A \in \mathcal{U}(T; f; \epsilon)$. The lemma is proved. ∎

Note that the set function T_f^A defined in the proof of Lemma 3.4 is a probability capacity with finite support. Hence, Lemma 3.4 immediately implies the following corollary.

Corollary 3.5. *The probability capacities with finite support are weakly dense in the space* $\tilde{\mathcal{C}}$.

Lemma 3.6. *Let* $f, g \in C_0^+(\mathbb{R}^d)$ *and* ϵ *be a positive real such that*

$$|f(x) - g(x)| < \epsilon \ for\ all\ x \in \mathbb{R}^d. \tag{3.15}$$

Then we have

$$|\int f\ dT - \int g\ dT| \leq \epsilon \ for\ every\ \ T \in \tilde{\mathcal{C}}.$$

Proof. Note that if $f(x) \ \leq g(x)$ for all $x \in \mathbb{R}^d$, then $\int f\ dT \leq \int g\ dT$ for every $T \in \tilde{\mathcal{C}}$. Let

$$\beta = \sup\{g(x) : x \in \mathbb{R}^d\}.$$

From (3.15) we have

$$\begin{aligned}
\int f\ dT &\leq \int (g + \epsilon) dT \\
&= \int_0^{\beta+\epsilon} T(\{x \in \mathbb{R}^d : g(x) + \epsilon \geq t\}) dt \\
&= \int_0^{\epsilon} T(\{x \in \mathbb{R}^d : g(x) + \epsilon \geq t\}) dt + \int_{\epsilon}^{\beta+\epsilon} T(\{x \in \mathbb{R}^d : g(x) \geq t - \epsilon\}) dt \\
&= \epsilon + \int_0^{\beta} T(\{x \in \mathbb{R}^d : g(x) \geq t\}) dt \\
&= \epsilon + \int g\ dT \text{ for every } \ T \in \mathcal{C}.
\end{aligned}$$

Similarly

$$\int g\, dT \le \epsilon + \int f\, dT.$$

Thus, the lemma is proved. ∎

Let C and Q denote a countable dense set of $C_0^+(\mathbb{R}^d)$ and $(0,1)$, respectively. Denote

$$\mathbf{G} = \{\bigcap_{i=1}^{k} \mathcal{U}(T_{g_i}^{A_i}; g_i; \delta_i) : \ A_i \in \mathcal{F}(D),\ g_i \in C,\ \delta_i \in Q,\ i = 1, \dots, k\},$$

where $\mathcal{F}(D)$ is the family of all finite sets of D. Using Lemmas 3.4 and 3.6 we will show that

Proposition 3.7. $\mathbf{G}$ *is a countable base of the weak topology in* $\tilde{\mathcal{C}}$.

Proof. Clearly $\mathbf{G}$ is countable. We prove that $\mathbf{G}$ is a base of the weak topology in $\tilde{\mathcal{C}}$.

Given $\mathcal{U}(T; f_1, \dots, f_k; \epsilon_1, \dots, \epsilon_k) \in \mathbf{B}$. Since C is dense in $C_0^+(\mathbb{R}^d)$, for each $i = 1, \dots, k$ there exists $g_i \in C$ such that

$$|f_i(x) - g_i(x)| \ < \delta_i \text{ for all } \ x \in \mathbb{R}^d,$$

where $\delta_i \in Q, \delta_i \ < \epsilon/4$ and $\epsilon = \min\,\{\epsilon_i : i = 1, \dots, k\}$. By Lemma 3.6,

$$|\int f_i\, dT - \int g_i\, dT| \ \le \delta_i \text{ for every } \ T \in \tilde{\mathcal{C}}, i = 1, \dots, k. \tag{3.16}$$

On the other hand, by Lemma 3.4 for each $i = 1, \dots, k$ we can choose $A_i = \{x_j^i : \ j = 1, \dots, n_i\} \ \in \mathcal{F}(D)$ such that

$$|\int g_i\, dT - \int g_i\, dT_{g_i}^{A_i}| < \delta_i. \tag{3.17}$$

Therefore for every $S \in \mathcal{U}(T_{g_i}^{A}; g_i; \delta_i)$, from (3.16) and (3.17) we have

$$\begin{aligned} |\int f_i\, dT - \int f_i\, dS| \le |\int f_i\, dT - \int g_i\, dT| + |\int g_i\, dT - \int g_i\, dT_{g_i}^{A_i}| \\ + |\int g_i\, dT_{g_i}^{A_i} - \int g_i\, dS| + |\int g_i\, dS - \int f_i\, dS| \\ < \delta_i + \delta_i + \delta_i + \delta_i = 4\delta_i < \epsilon \end{aligned}$$

for every $i = 1, \dots, k$. Thus

$$S \in \mathcal{U}(T; f_i; \epsilon_i) \text{ for every } i = 1, \dots, k.$$

Consequently

$$\bigcap_{i=1}^{k} \mathcal{U}(T_{g_i}^{A_i}; g_i; \delta_i) \subset \mathcal{U}(T; f_1, \dots, f_k; \epsilon_1, \dots, \epsilon_k),$$

and the proposition is proved. ∎

The proof of the theorem is finished. Thus, since $\tilde{\mathcal{C}}$ equipped with the weak topology is a metric space, we can define the notion of weak convergence of a sequence in $\tilde{\mathcal{C}}$ as follows.

Definition 3.8. *A sequence of capacities* $\{T_n\}_{n=1}^{\infty} \subset \tilde{\mathcal{C}}$ *is said to be weakly convergent to the capacity* $T \in \tilde{\mathcal{C}}$ *if and only if* $\int f\ dT_n \to \int f\ dT$ *for every* $f \in C_0^+(\mathbb{R}^d)$.

In comparison with the notion of the weak topology, we have the following proposition.

Proposition 3.9. *The convergence in the weak topology and the weak convergence are equivalent.*

Proof. Let T_n be a sequence of probability capacities in $\mathbb{R}^d$ and $T \in \tilde{\mathcal{C}}$. Assume that T_n is weakly convergent to T. Let $\mathcal{U}(T; f_1, \ldots, f_k; \epsilon_1, \ldots, \epsilon_k), f_i \in C_0^+(\mathbb{R}^d), \epsilon_i > 0, i = 1, \ldots, k$ be a neighborhood of T in the weak topology. For every $i = 1, \ldots, k$ there exists $n_i \in \mathbb{N}$ such that

$$|\int f_i\ dT_n - \int f_i\ dT| < \epsilon_i \text{ for all } n \geq n_i.$$

Let $n_0 = \max\{n_i, i = 1, \ldots, k\}$. Then we have

$$|\int f_i\ dT_n - \int f_i\ dT| < \epsilon_i \text{ for all } n \geq n_0 \text{ and for all } i = 1, \ldots, k.$$

Hence

$$T_n \in \mathcal{U}(T; f_1, \ldots, f_k; \epsilon_1, \ldots, \epsilon_k) \text{ for all } n \geq n_0.$$

That means $T_n \to T$ in the weak topology.

Conversely, let $T_n \to T$ in the weak topology. For given $f \in C_0^+(\mathbb{R}^d)$ and $\epsilon > 0$, let $\mathcal{U}(T; f; \epsilon)$ be a neighborhood of T in the weak topology. Then there exists $n_0 \in \mathbb{N}$ such that

$$\left|\int f\ dT_n - \int f\ dT\right| < \epsilon \text{ for all } n \geq n_0,$$

i.e., T_n converges weakly to T.

The following proposition shows that the convergence on compact sets implies the weak convergence.

Proposition 3.10. *Let* $\{T_n\}_{n=1}^{\infty}$ *be a sequence of probability capacities in* $\mathbb{R}^d$ *and* $T \in \tilde{\mathcal{C}}$. *If* $T_n(C) \to T(C)$ *for every* $C \in \mathcal{K}(\mathbb{R}^d)$, *then* T_n *is weakly convergent to* T.

Proof. Assume that $T_n(C) \to T(C)$ for every $C \in \mathcal{K}(\mathbb{R}^d)$. For $f \in C_0^+(\mathbb{R}^d)$ we put

$$\alpha = \sup\{f(x) : x \in \mathbb{R}^d\} < \infty.$$

Since $\{x \in \mathbb{R}^d : \ f(x) \geq t\}$ is compact for any $t > 0$, we have

$$T_n(\{x \in \mathbb{R}^d : f(x) \geq t\}) \to T(\{x \in \mathbb{R}^d : \ f(x) \geq t\}) \text{ for every } t \in [0, 1].$$

Since

$$g_n(t) = T_n(\{x \in \mathbb{R}^d : \ f(x) \geq t\}) \leq 1 \text{ for every } t \in \mathbb{R} \text{ and } n \in \mathbb{N},$$

by the Lebesgue's bounded convergence Theorem [4] we get

$$\int_0^\alpha T_n(\{x \in \mathbb{R}^d : \ f(x) \geq t\})dt \to \int_0^\alpha T(\{x \in \mathbb{R}^d : \ f(x) \geq t\})dt.$$

Therefore

$$\int f \, dT_n \to \int f \, dT \text{ for every } f \in C_0^+(\mathbb{R}^d).$$

That means T_n is weakly convergent to T. ∎

4. Topological Embedding $\mathbb{R}^d$ into $\tilde{\mathcal{C}}$

Note that the corresponding $x \mapsto T_x = \delta_x$, with δ_x defined by (3.3), is one-to-one between $\mathbb{R}^d$ and the set of the probability measures $\{T_x : \ x \in \mathbb{R}^d\} \subset \tilde{\mathcal{C}}$. Therefore, in some sense the class of capacities in $\mathbb{R}^d$ also contains $\mathbb{R}^d$. In a such way, let $V : \ \mathbb{R}^d \to \tilde{\mathcal{C}}$ be a transform defined by

$$V(x) = T_x \text{ for every } x \in \mathbb{R}^d. \tag{4.1}$$

We now show that

Theorem 4.1. *The map* $V : \ \mathbb{R}^d \to \tilde{\mathcal{C}}$ *is a topological embedding, i.e,* $\mathbb{R}^d$ *is homeomorphic to* $V(\mathbb{R}^d)$*, which is the closed subset of* $\tilde{\mathcal{C}}$*.*

Proof. Clearly $V(x) \neq V(y)$ for $x \neq y$. Moreover, if $x_n \to x$ then for any $f \in C_0^+(\mathbb{R}^d)$, from (3.4) we have

$$\int f \, dT_{x_n} = f(x_n) \to f(x) = \int f \, dT_x.$$

Therefore $V(x_n) \to V(x)$, and so V is continuous in the weak topology.

Conversely, assume that $T_{x_n} \to T \in \tilde{\mathcal{C}}$ in the weak topology. We claim that $x_n \to x$ and $T = T_x$ for some $x \in \mathbb{R}^d$. In fact, $T_{x_n} \to T$ implies $\int f \, dT_{x_n} = f(x_n) \to \int f \, dT$ for every $f \in C_0^+(\mathbb{R}^d)$.

We will now show that there exists $f \in C_0^+(\mathbb{R}^d)$ such that

$$\gamma = \int f \, dT > 0.$$

For given $\epsilon > 0$ let

$$G = \{x \in \mathbb{R}^d : d(x, \text{supp } T) < \epsilon\}.$$

Then supp T and $\mathbb{R}^d \setminus G$ are disjoint closed sets, by the Urysohn-Tietze Theorem we can find a continuous function $f : \mathbb{R}^d \to [0, 1]$ such that

$$f(x) = \begin{cases} 1 & \text{if } x \in \text{supp } T \\ 0 & \text{if } x \in \mathbb{R}^d \setminus G. \end{cases}$$

Note that the compactness of supp T implies the compactness of supp f, hence $f \in C_0^+(\mathbb{R}^d)$. Since $T(\text{supp } T) = 1$, we have

$$\gamma = \int f \, dT = \int_0^1 T(\{x : f(x) \geq t\}) dt \geq \int_0^1 T(\text{supp } T) dt = 1 > 0.$$

Since $f(x_n) \to \gamma > 0$, for $0 < \delta < \gamma$ there is $n_0 \in \mathbb{N}$ such that

$$|f(x_n) - \gamma| < \delta \text{ for all } n \geq n_0.$$

That means

$$x_n \in \text{supp } f \text{ for all } n \geq n_0.$$

By the compactness of supp f there is a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x \in \mathbb{R}^d$.

If $x_n \nrightarrow x$, then there exists a subsequence $\{x'_{n_k}\} \subset \{x_n\}$ such that

$$|x'_{n_k} - x| \geq \delta > 0 \text{ for all } k \in \mathbb{N}. \tag{4.2}$$

Again, by the compactness of supp f, there is a subsequence $\{x''_{n_k}\} \subset \{x'_{n_k}\}$ such that $x''_{n_k} \to x' \in \mathbb{R}^d$. Then we have

$$T_{x''_{n_k}} \to T_{x'} \text{ and } T_{x_{n_k}} \to T_x.$$

Since $\{T_{x''_{n_k}}\}, \{T_{x_{n_k}}\} \subset \{T_{x_n}\}$ and $T_{x_n} \to T$, we get $T_{x'} = T_x = T$, and so $x = x'$. From (4.2) we obtain a contradiction. Hence $x_n \to x$. Consequently V^{-1} is continuous. ∎

Remark 4.2. By Theorem 4.1 we can identify $\mathbb{R}^d$ with the closed subset $V(\mathbb{R}^d)$ of $\tilde{\mathcal{C}}$. Therefore, the space $\tilde{\mathcal{C}}$ contains $\mathbb{R}^d$ topologically.

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