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Regularity and Isomorphism Theorems of Generalized Order - Preserving Transformation Semigroups

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Abstract. The full order-preserving transformation semigroup $OT(X)$ on a poset X has long been studied. In this paper, we study the semigroup $(OT(X,Y),\theta)$ where X and Y are chains, $OT(X,Y)$ is the set of all order-preserving maps from X into Y, $\theta \in OT(Y,X)$ and the operation $*$ is defined by $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in OT(X,Y)$. We characterize when $(OT(X,Y),\theta)$ is regular, $(OT(X,Y),\theta) \cong OT(X)$ and $(OT(X,Y),\theta) \cong OT(Y)$.

1. Introduction

The full transformation semigroup on a set X is denoted by $T(X)$ and for $\alpha \in T(X)$, let ran α denote the range of α. It is well-known that $T(X)$ is regular for any set X, that is, for every $\alpha \in T(X)$, $\alpha = \alpha\beta\alpha$ for some $\beta \in T(X)$.

Next, let X and Y be posets. A map $\alpha : X \to Y$ is said to be *order-preserving* if for $x_1, x_2 \in X$, $x_1 \le x_2$ implies $x_1\alpha \le x_2\alpha$. A bijection $\varphi : X \to Y$ is called an *order-isomorphism* if φ and φ^{-1} are order-preserving. It is clear that if both X and Y are chains and $\varphi : X \to Y$ is an order-preserving bijection, then φ is an order-isomorphism. We say that X and Y are *order-isomorphic* if there is an order-isomorphism from X onto Y. Naturally, X and Y are said to be *anti-order-isomorphic* if there exists a bijection $\varphi : X \to Y$ such that for $x_1, x_2 \in X, x_1 \le x_2$ if and only if $x_2\varphi \le x_1\varphi$. Let $OT(X)$ denote the subsemigroup of $T(X)$

consisting of all order-preserving transformations $\alpha : X \to X$. The semigroup $OT(X)$ may be called the *full order-preserving transformation semigroup* on X(see [6]). The full order-preserving transformation semigroup on a poset has long been studied. For examples, see [5, Theorem V.8.9], [2, (Exercise 6.1.7), 3, 7, 1, 4]. Theorem V.8.9 of [5] gives an interesting isomorphism theorem as follows: For posets X and Y, $OT(X) \cong OT(Y)$ if and only if X and Y are order-isomorphic or anti- order-isomorphic. Let $\mathbb{Z}$ and $\mathbb{R}$ denote the set of integers and the set of real numbers, respectively. In [3], Kemprasit and Changphas characterized when $OT(X)$ is regular where X is a nonempty subset of $\mathbb{Z}$ or X is a nonempty interval of $\mathbb{R}$ with their natural order as follows:

Theorem 1.1 [4]. *For any nonempty subset X of $\mathbb{Z}$, $OT(X)$ is regular.*

Theorem 1.2 [4]. *For a nonempty interval X of $\mathbb{R}$, $OT(X)$ is regular if and only if X is closed and bounded.*

In this paper, the semigroup $OT(X)$ is replaced by the semigroup $(OT(X,Y), \theta)$ where $OT(X,Y)$ is the set of all order-preserving maps $\alpha : X \to Y$, $\theta \in OT(Y,X)$ and the operation $*$ is defined by $\alpha * \beta = \alpha\theta\beta$ for all $\alpha, \beta \in OT(X,Y)$. Note that $OT(X) = (OT(X,X), 1_X)$ where 1_X is the identity map on X. We confine our attention to study the semigroup $(OT(X,Y), \theta)$ when X and Y are chains. In this paper we characterize the regularity of the semigroup $(OT(X,Y), \theta)$. Further we provide necessary and sufficient conditions for this semigroup to be isomorphic to $OT(X)$ and, respectively, isomorphic to $OT(Y)$. Our main results are Theorems 3.1, 3.5 and 3.6. From now on we assume that X and Y are chains and $\theta \in OT(Y,X)$.

2. Lemmas

The following series of the lemmas is required to obtain our main results.

Lemma 2.1. *Let $a,b \in X$ and $c, d \in Y$ be such that $a < b, c < d$ and $c\theta = d\theta$. If $\alpha : X \to Y$ is defined by*

$$x\alpha = \begin{cases} c & \text{if } x < b, \\ d & \text{if } x \geq b, \end{cases}$$

then $\alpha \in OT(X,Y), |ran\ \alpha| = 2$ and $|ran(\alpha\theta)| = 1$.

Proof. It is clear that $\alpha \in OT(X,Y)$, ran $\alpha = \{c, d\}$ and $\text{ran}(\alpha\theta) = (\text{ran}\ \alpha)\theta = \{c\theta, d\theta\} = \{c\theta\}$. ∎

Lemma 2.2. *If $|X| > 1$ and $(OT(X,Y), \theta)$ is regular, then θ is one-to-one.*

Proof. Let $|X| > 1$ and assume that θ is not one-to-one. Then there are $a, b \in X$ and $c, d \in Y$ such that $a < b, c < d$ and $c\theta = d\theta$. Define $\alpha : X \to Y$ as in Lemma 2.1. By Lemma 2.1, $\alpha \in OT(X,Y)$, $|\text{ran}\ \alpha| = 2$ and $|\text{ran}(\alpha\theta)| = 1$. Since for $\beta \in OT(X,Y)$, $|\text{ran}(\alpha\theta\beta\theta\alpha)| \leq |\text{ran}(\alpha\theta)| = 1$, it follows that $\alpha \neq \alpha\theta\beta\theta\alpha$. Hence

α is not a regular element of $(OT(X,Y),\theta)$, so $(OT(X,Y),\theta)$ is not a regular semigroup. ∎

Lemma 2.3. *If $(OT(X,Y),\theta)$ has an identity η, then θ is one-to-one and ran $\eta = Y$.*

Proof. By assumption $\eta\theta\lambda = \lambda\theta\eta = \lambda$ for all $\lambda \in OT(X,Y)$. For $y \in Y$, let X_y denote the constant map with domain X and range $\{y\}$. Then $X_y \in OT(X,Y)$ for all $y \in Y$, and hence

$$X_y\theta\eta = X_y \text{ for all } y \in Y$$

which implies that

$$y(\theta\eta) = yX_y\theta\eta = yX_y = y \text{ for all } y \in Y.$$

Therefore $\theta\eta = 1_Y$, the identity map on Y. We then deduce that θ is one-to-one and ran $\eta = Y$. ∎

Lemma 2.4. *Let $e,f \in Y$ be such that $e < f$ and $a \in X$.*
(i) *If $x < a$ for all $x \in$ ran θ and $\alpha : X \to Y$ is defined by*

$$x\alpha = \begin{cases} e & \text{if } x < a, \\ f & \text{if } x \geq a, \end{cases}$$

then $\alpha \in OT(X,Y), |\text{ran } \alpha| = 2$ and $|\text{ ran}(\theta\alpha)| = 1$.
(ii) *If $x > a$ for all $x \in$ ran θ and $\beta : X \to Y$ is defined by*

$$x\beta = \begin{cases} e & \text{if } x \leq a, \\ f & \text{if } x > a, \end{cases}$$

then $\beta \in OT(X,Y), |\text{ran } \beta| = 2$ and $|\text{ran}(\theta\beta)| = 1$.

Proof.
(i) We clearly have that $\alpha \in OT(X,Y)$, ran $\alpha = \{e,f\}$ and $\text{ran}(\theta\alpha) = (\text{ran } \theta)\alpha = \{e\}$.
(ii) It is also clear that $\beta \in OT(X,Y)$, ran $\beta = \{e,f\}$ and $\text{ran}(\theta\beta) = (\text{ran } \theta)\beta = \{f\}$. ∎

Lemma 2.5. *If $|Y| > 1$ and $(OT(X,Y),\theta)$ is regular, then for every $x \in X, y \leq x \leq z$ for some $y,z \in$ ran θ.*

Proof. Let $e,f \in Y$ be such that $e < f$ and assume that it is not true that for every $x \in X, y \leq x \leq z$ for some $y,z \in$ ran θ. Then there is an element $a \in X$ such that $a > x$ for all $x \in$ ran θ or $a < x$ for all $x \in$ ran θ.

Case 1. $a > x$ for all $x \in$ ran θ. Define $\alpha : X \to Y$ as in Lemma 2.4(i). Then $\alpha \in OT(X,Y), |\text{ran } \alpha| = 2$ and $|\text{ran}(\theta\alpha)| = 1$. Thus $|\text{ran}(\alpha\theta\lambda\theta\alpha)| = 1$ for all $\lambda \in OT(X,Y)$, so $\alpha \neq \alpha\theta\lambda\theta\alpha$ for every $\lambda \in OT(X,Y)$. Hence α is not regular in $(OT(X,Y),\theta)$.

Case 2. $a < x$ for all $x \in$ ran θ. Define $\beta : X \to Y$ as in Lemma 2.4(ii). Then $\beta \in OT(X,Y), |\text{ran } \beta| = 2$ and $|\text{ran}(\theta\beta)| = 1$ which implies that $|\text{ran}(\beta\theta\lambda\theta\beta)| =$

1 for every $\lambda \in OT(X,Y)$. Thus $\beta \neq \beta\theta\lambda\theta\beta$ for every $\lambda \in OT(X,Y)$, so β is not a regular element in $(OT(X,Y),\theta)$. ■

Lemma 2.6. *If $|Y| > 1$ and $(OT(X,Y),\theta)$ has an identity, then for every $x \in X, y \leq x \leq z$ for some $y, z \in \operatorname{ran}\theta$.*

Proof. Let η be the identity of $(OT(X,Y),\theta)$. Then

$$\eta\theta\lambda = \lambda\theta\eta = \lambda \text{ for all } \lambda \in OT(X,Y).$$

Suppose the conclusion is false. Then there exists an element $a \in X$ such that either $a > x$ for all $x \in \operatorname{ran}\theta$ or $a < x$ for all $x \in \operatorname{ran}\theta$. By Lemma 2.4, there exists an element $\gamma \in OT(X,Y)$ such that $|\operatorname{ran}\gamma| = 2$ but $|\operatorname{ran}(\theta\gamma)| = 1$. It then follows that $\eta\theta\gamma \neq \gamma$, a contradiction. ■

Lemma 2.7. *Let $a \in X \setminus \operatorname{ran}\theta$ be such that $b < a < c$ for some $b, c \in \operatorname{ran}\theta$ and $e, f, g \in Y$ such that $e < f < g$. If $\alpha : X \to Y$ is defined by*

$$x\alpha = \begin{cases} e & \text{if } x < a, \\ f & \text{if } x = a, \\ g & \text{if } x > a, \end{cases}$$

then $\alpha \in OT(X,Y)$, $|\operatorname{ran}\alpha| = 3$ and $|\operatorname{ran}(\theta\alpha)| = 2$.

Proof. Obviously, $\alpha \in OT(X,Y)$ and $\operatorname{ran}\alpha = \{e,f,g\}$. Since $a \notin \operatorname{ran}\theta$, $\operatorname{ran}(\theta\alpha)=(\operatorname{ran}\theta)\alpha=(\{x \in \operatorname{ran}\theta \mid x < a\} \cup \{x \in \operatorname{ran}\theta \mid x > a\})\alpha = \{e,f\}$. ■

Lemma 2.8. *Let $|Y| > 2$. If $(OT(X,Y),\theta)$ is regular or $(OT(X,Y),\theta)$ has an identity, then* $\operatorname{ran}\theta = X$.

Proof. Let $e, f, g \in Y$ be such that $e < f < g$. Suppose that $\operatorname{ran}\theta \neq X$. Let $a \in X \setminus \operatorname{ran}\theta$. If $a < x$ for all $x \in \operatorname{ran}\theta$ or $a > x$ for all $x \in \operatorname{ran}\theta$, then by Lemmas 2.5 and 2.6, $(OT(X,Y),\theta)$ is not regular and $(OT(X,Y),\theta)$ has no identity, respectively. Next, assume that $b < a < c$ for some $b, c \in \operatorname{ran}\theta$. Define $\alpha : X \to Y$ as in Lemma 2.7. Then $\alpha \in OT(X,Y)$, $|\operatorname{ran}\alpha| = 3$ and $|\operatorname{ran}(\theta\alpha)| = 2$. Hence for every $\lambda \in OT(X,Y)$, $|\operatorname{ran}(\alpha\theta\lambda\theta\alpha)| \leq |\operatorname{ran}(\lambda\theta\alpha)| \leq |\operatorname{ran}(\theta\alpha)| = 2$, so $\alpha \neq \alpha\theta\lambda\theta\alpha$ and $\alpha \neq \lambda\theta\alpha$ for every $\lambda \in OT(X,Y)$. Thus α is not a regular element of $(OT(X,Y),\theta)$ and for every $\lambda \in OT(X,Y)$, λ is not an identity of $(OT(X,Y),\theta)$.

Hence the lemma is proved. ■

Lemma 2.9. *If $|Y| = 2$ and* $\operatorname{ran}\theta = \{\min X, \max X\}$, *then $(OT(X,Y),\theta)$ is an idempotent semigroup (a band).*

Proof. Let $\alpha \in OT(X,Y)$. Then either $|\operatorname{ran}\alpha| = 1$ or $|\operatorname{ran}\alpha| = 2$. Since $\operatorname{ran}(\alpha\theta\alpha) \subseteq \operatorname{ran}\alpha$, it follows that $\alpha\theta\alpha = \alpha$ if $|\operatorname{ran}\alpha| = 1$. Next, assume that $|\operatorname{ran}\alpha| = 2$. Then $\operatorname{ran}\alpha = Y$. Let $Y = \{e,f\}$ with $e < f$. Thus $X = e\alpha^{-1} \cup f\alpha^{-1}$ which is a disjoint union, $\min X \in e\alpha^{-1}$ and $\max X \in f\alpha^{-1}$. Since $Y\theta = \{e,f\}\theta = \{\min X, \max X\}$ and $e < f$, it follows that $e\theta = \min X$ and $f\theta = \max X$. Consequently,

$$(e\alpha^{-1})\alpha\theta\alpha = \{e\theta\}\alpha = \{\min X\}\alpha = \{e\} = (e\alpha^{-1})\alpha,$$
$$(f\alpha^{-1})\alpha\theta\alpha = \{f\theta\}\alpha = \{\max X\}\alpha = \{f\} = (f\alpha^{-1})\alpha,$$

which implies that $\alpha = \alpha\theta\alpha$, so α is an idempotent of $(OT(X,Y),\theta)$. ∎

Lemma 2.10. *If* $|Y| = 2$, ran $\theta = \{\min X, \max X\}$ *and* $(OT(X,Y),\theta)$ *has an identity, then* $|X| = 2$.

Proof. Let $Y = \{e, f\}$ with $e < f$ and η be the identity of $(OT(X,Y),\theta)$. Since $\theta : Y \to$ ran $\theta = \{\min X, \max X\}$, we deduce that $e\theta = \min X$ and $f\theta = \max X$. But θ is one-to-one from Lemma 2.3, thus min $X < \max X$, and hence $|X| \geq 2$. To show that $|X| = 2$, suppose in the contrary that there is an element a in $X\backslash\{\min X, \max X\}$. Then $\min X < a < \max X$, ran $\eta = Y$ by Lemma 2.3, so $a\eta = e$ or $a\eta = f$. Define $\lambda_1, \lambda_2 : X \to Y$ by

$$x\lambda_1 = \begin{cases} e & \text{if } x < a, \\ f & \text{if } x \geq a, \end{cases} \quad \text{and} \quad x\lambda_2 = \begin{cases} e & \text{if } x \leq a, \\ f & \text{if } x > a. \end{cases}$$

Then $\lambda_1, \lambda_2 \in OT(X,Y)$.

Case 1. $a\eta = e$. Then $a\eta\theta\lambda_1 = (e\theta)\lambda_1 = (\min X)\lambda_1 = e < f = a\lambda_1$.

Case 2. $a\eta = f$. Then $a\eta\theta\lambda_2 = (f\theta)\lambda_2 = (\max X)\lambda_2 = f > e = a\lambda_2$.

These two cases yield a contradiction since η is an identity of $(OT(X,Y),\theta)$. Hence we prove that $|X| = 2$, as required. ∎

Lemma 2.11. *Let* θ *be an order-isomorphism. Then the following statements hold.*

(i) *The map* $\alpha \mapsto \alpha\theta$ *is an isomorphism of* $(OT(X,Y),\theta)$ *onto* $OT(X)$.

(ii) *The map* $\alpha \mapsto \theta\alpha$ *is an isomorphism of* $(OT(X,Y),\theta)$ *onto* $OT(Y)$.

Proof. Note that $\theta^{-1} \in OT(X,Y)$. If $\alpha, \beta \in OT(X,Y)$, then

$$(\alpha\theta\beta)\theta = (\alpha\theta)(\beta\theta),\ \theta(\alpha\theta\beta) = (\theta\alpha)(\theta\beta),$$
$$\alpha\theta = \beta\theta \ \Rightarrow\ \alpha = \alpha\theta\theta^{-1} = \beta\theta\theta^{-1} = \beta,$$
$$\theta\alpha = \theta\beta \ \Rightarrow\ \alpha = \theta^{-1}\theta\alpha = \theta^{-1}\theta\beta = \beta.$$

Also, for $\gamma \in OT(X)$ and $\lambda \in OT(Y)$, we have $\gamma\theta^{-1}, \theta^{-1}\lambda \in OT(X,Y)$ and $(\gamma\theta^{-1})\theta = \gamma$ and $\theta(\theta^{-1}\lambda) = \lambda$. Hence (i) and (ii) are proved. ∎

3. Regularity and Isomorphism Theorems

Now we are ready to provide our main resuls.

Theorem 3.1. *The semigroup* $(OT(X,Y),\theta)$ *is regular if and only if one of the following statements holds.*

(i) *$OT(X)$ is regular and θ is an order-isomorphism.*
(ii) $|X| = 1$.
(iii) $|Y| = 1$.
(iv) $|Y| = 2$ *and* $\operatorname{ran}\theta = \{\min X, \max X\}$.

Proof. To prove necessity, assume that $(OT(X,Y),\theta)$ is regular and suppose that (ii),(iii) and (iv) are false. Then

$$|X| > 1,\ |Y| > 1 \text{ and } (|Y| \neq 2 \text{ or } \operatorname{ran}\theta \neq \{\min X, \max X\}).$$

Therefore we have $|X| > 1$ and either $|Y| > 2$ or $|Y| = 2$ and ran $\theta \neq \{\min X, \max X\}$. From that $|X| > 1$, we have by Lemma 2.2 that θ is one-to-one. First suppose that $|Y| = 2$ and ran $\theta \neq \{\min X, \max X\}$. Since $|Y| = 2$ and θ is one-to-one, $|\operatorname{ran}\theta| = 2$. Note that $\min X$ or $\max X$ may not exist. Let ran $\theta = \{e, f\}$ with $e < f$. Then $\{e, f\} \neq \{\min X, \max X\}$.

Case 1. $\min X$ does not exist. Then there exists $a \in X$ such that $a < e$, so $a < e < f$.

Case 2. $\max X$ does not exist. Then $a > f$ for some $a \in X$, so $a > f > e$.

Case 3. $\min X$ and $\max X$ exist. But $\{e, f\} \neq \{\min X, \max X\}$, so $\min X < e$ or $\max X > f$. Then either $\min X < e < f$ or $\max X > f > e$.

From Case 1 - Case 3, we conclude that there exists an element $a \in X$ such that $a < x$ for all $x \in \operatorname{ran}\theta$ or $a > x$ for all $x \in \operatorname{ran}\theta$. It follows from Lemma 2.5 that $(OT(X,Y),\theta)$ is not regular. Hence this case cannot occur. Thus $|Y| > 2$, and by Lemma 2.8, we have ran $\theta = X$. Consequently, θ is an order-isomorphism because X and Y are chains. We then deduce from Lemma 2.11(i) that $(OT(X,Y),\theta) \cong OT(X)$. But $(OT(X,Y),\theta)$ is regular, so $OT(X)$ is regular. Hence (i) holds.

To prove sufficiency, assume that one of (i)-(iv) holds. If (i) is true, then $(OT(X,Y),\theta)$ is regular by Lemma 2.11(i).

If $|X| = 1$, then for $\alpha \in OT(X,Y)$, $|\operatorname{ran}\alpha| = 1$, so $\alpha = \alpha\theta\alpha$ since $\operatorname{ran}(\alpha\theta\alpha) \subseteq \operatorname{ran}\alpha$. If $|Y| = 1$, then $|OT(X,Y)| = 1$. Hence, if (ii) or (iii) holds, then $(OT(X,Y),\theta)$ is regular. If (iv) is true, then by Lemma 2.9 $(OT(X,Y),\theta)$ is an idempotent semigroup, so it is regular. ■

The two following corollaries are directly obtained from Theorems 3.1, 1.1 and 1.2.

Corollary 3.2. *Let X and Y be nonempty subsets of $\mathbb{Z}$. Then $(OT(X,Y),\theta)$ is regular if and only if one of the following statements holds.*
(i) *θ is an order-isomorphism.*
(ii) $|X| = 1$.
(iii) $|Y| = 1$.
(iv) $|Y| = 2$ *and* $\operatorname{ran}\theta = \{\min X, \max X\}$.

Corollary 3.3. *Let X and Y be intervals of $\mathbb{R}$ containing more than one element. Then $(OT(X,Y),\theta)$ is regular if and only if X is closed and bounded and θ is an order-isomorphism.*

Proposition 3.4. *The semigroup $(OT(X,Y),\theta)$ has an identity if and only if $|Y|=1$ or θ is an order-isomorphism.*

Proof. If $|Y|=1$, then $|OT(X,Y)|=1$, so $(OT(X,Y),\theta)$ has an identity. If θ is an order-isomorphism, then by Lemma 2.11(i), $(OT(X,Y),\theta)\cong OT(X)$, so $(OT(X,Y),\theta)$ has an identity since $OT(X)$ does.

For the converse, assume that $(OT(X,Y),\theta)$ has an identity, say η, and $|Y|>1$. By Lemma 2.3, θ is one-to-one. If $|Y|>2$, then we have from Lemma 2.8 that ran $\theta = X$. Next, assume that $|Y|=2$, say $Y=\{e,f\}$ with $e<f$. Then ran $\theta=\{e\theta,f\theta\}$ and $e\theta<f\theta$. We deduce from Lemma 2.6 that $e\theta\le x\le f\theta$ for all $x\in X$. Consequently, $\min X=e\theta$ and $\max X=f\theta$. It then follows from Lemma 2.10 that $|X|=2$. Thus $X=\{e\theta,f\theta\}=$ ran θ. This shows that ran $\theta=X$ for every case of $|Y|\ge 2$. Therefore θ is an order-isomorphism. ∎

Theorem 3.5. *The semigroups $(OT(X,Y),\theta)$ and $OT(X)$ are isomorphic if and only if θ is an order-isomorphism.*

Proof. We deduce from Lemma 2.11(i) that if θ is an order-isomorphism, then $(OT(X,Y),\theta)\cong OT(X)$.

Conversely, assume that $(OT(X,Y),\theta)\cong OT(X)$. Then $|OT(X,Y)|=|OT(X)|$. Since $OT(X)$ has an identity, $(OT(X,Y),\theta)$ has an identity. We have by Proposition 3.4 that $|Y|=1$ or θ is an order-isomorphism. If $|Y|=1$, then $|OT(X,Y)|=1$, and hence $|OT(X)|=1$ which implies that $|X|=1$.

Therefore the theorem is proved. ∎

Theorem 3.6. *The semigroups $(OT(X,Y),\theta)$ and $OT(Y)$ are isomorphic if and only if $|Y|=1$ or θ is an order-isomorphism.*

Proof. If $|Y|=1$, then $|OT(X,Y)|=1=|OT(Y)|$, so $(OT(X,Y),\theta)\cong OT(Y)$. Also, $(OT(X,Y),\theta)\cong OT(Y)$ by Lemma 2.11(ii) if θ is an order-isomorphism.

The converse holds by Proposition 3.4 since $OT(Y)$ has an identity. ∎

Proposition 3.4, Theorems 3.5 and 3.6 yield the following theorem directly.

Theorem 3.7. *If $|Y|>1$, then the following statements are equivalent.*
(i) *$(OT(X,Y),\theta)$ has an identity.*
(ii) *$(OT(X,Y),\theta)\cong OT(X)$.*
(iii) *$(OT(X,Y),\theta)\cong OT(Y)$.*
(iv) *θ is an order-isomorphism.*

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