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A Presentation of the Elements of the Quotient Sheaves Ω_r^k/Θ_r^k in Variational Sequences

Nong Quoc Chinh

Thai Nguyen University, Thai Nguyen City, Vietnam

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Abstract In this paper we give a concrete presentation of the elements of the quotient sheaves Ω_r^k/Θ_r^k in the variational sequence

$$0 \to R \to \Omega_r^0 \xrightarrow{E_0} \Omega_r^1/\Theta_r^1 \xrightarrow{E_1} \Omega_r^2/\Theta_r^2 \xrightarrow{E_2} ... \xrightarrow{E_{P-2}} \Omega_r^{P-1}/\Theta_r^{P-1} \to$$
$$\xrightarrow{E_{P-1}} \Omega_r^P/\Theta_r^P \xrightarrow{E_P} \Omega_r^{P+1} \xrightarrow{E_{P+1}} ... \to \Omega_r^N \to 0.$$

1. Introduction

The notion of variational bicomplexes was introduced in studying the problem of characterizing the kernel and image of Euler-Lagrange mapping in the calculus of variations. This problem has been considered by Anderson [1], Duchamp [2], Dedecker [4], Tulczyvew [7], Takens [6] and Krupka [5]. The variational bicomplex was mainly studied on the infinite jet prolongation $J^\infty Y$ of the fibered manifold Y with the base X and the projection $\pi : Y \to X$, where $\dim X = n$, $\dim Y - n = m$. The variational bicomplexes contain Euler-Lagrange mapping as one of its morphisms. Then its developments in the theory of variational bicomplexes have made an important role in many problems in caculus of variations on manifolds, in diffrential geometry, in the theory of differential equations and in mathematical physics.

Krupka [5] studied the sheaves of differential forms on finite r-jet prolongations J^rY. Then he constructed the sequence of quotient sheaves Ω_r^k/Θ_r^k. This quotient sequence is called *the variational sequence of order r over Y*. It is an acyclic resolution of the constant sheaf R over Y

$$0 \to R \to \Omega_r^0 \xrightarrow{E_0} \Omega_r^1/\Theta_r^1 \xrightarrow{E_1} \Omega_r^2/\Theta_r^2 \xrightarrow{E_2} ... \xrightarrow{E_{P-2}} \Omega_r^{P-1}/\Theta_r^{P-1} \to$$
$$\xrightarrow{E_{P-1}} \Omega_r^P/\Theta_r^P \xrightarrow{E_P} \Omega_r^{P+1} \xrightarrow{E_{P+1}} ... \to \Omega_r^N \to 0. \quad (1)$$

In the sequence (1), E_n is the Euler-Lagrange mapping and E_{n+1} is the Helmholtz-Sonin mapping. In the study of variational sequence, it is very important to give a concrete presentation of the elements $[\varrho] \in \Omega_r^k/\Theta_r^k$. It also has been shown in [5] that $[\varrho] = p_{r,0}^k \varrho$ for all positive integers k satisfying $1 \leq k \leq n$, $\varrho \in \Omega_r^k$, where $p_{r,0}^k \varrho$ is the horizontal component of k-form ϱ. Then Krupka [5] gave a concrete presentation of $[\varrho]$ for $1 \leq k \leq n$.

The main purpose of this paper is to give a concrete presentation of elements $[\varrho] \in \Omega_r^k/\Theta_r^k$ for all positive integers k satisfying $n+1 \leq k \leq P$. For simplicity, we will solve this problem in the case of $r = 1$ and $r = 2$. Then the other cases follow by the same method.

2. Notations and Premilinaries

Throughout this paper, the following notations will be used: (Y, π, X) is a fibered manifold with base X and the projection $\pi : Y \to X$, where $\dim X = n$, $\dim Y - n = m$. J^rY is the finite r-jet prolongation of the fibered manifold (Y, π, X). $\pi_r : J^rY \to X, \pi_{r,s} : J^rY \to J^sY$ are the canonical jet projections. (V, Ψ) is a fiber chart on Y, where $\Psi = (x^i, y^\sigma), 1 \leq i \leq n, 1 \leq \sigma \leq m$. (V_r, Ψ_r) is the fiber chart on r-jet prolongation J^rY associated with (V, Ψ), where $V_r = \pi_{r,0}^{-1}(V), \Psi_r = (x^i, y^\sigma, y_{j_1}^\sigma, ..., y_{j_1 j_2 \ldots j_r}^\sigma), 1 \leq i \leq n, 1 \leq \sigma \leq m, 1 \leq j_1 \leq ... \leq j_r \leq n$. Ω_r^k is the sheave of k-forms over J^rY and $\pi_{r+1,r}^*$ is the pull-back of the mapping $\pi_{r+1,r}$.

We put

$$\omega_0 = dx^1 \wedge dx^2 \wedge ... \wedge dx^n,$$
$$\omega_i = (-1)^{i-1} dx^1 \wedge ... \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n, 1 \leq i \leq n,$$
$$\omega_{j_1 j_2 \ldots j_k}^\sigma = dy_{j_1 j_2 \ldots j_k}^\sigma - y_{j_1 j_2 \ldots j_k i}^\sigma dx^i, 1 \leq k \leq r-1.$$

Note that the forms

$$dx^i, \omega_{j_1 j_2 \ldots j_k}^\sigma, dy_{j_1 j_2 \ldots j_r}^\sigma,$$

for $0 \leq k \leq r-1$, define a basis of the space of linear forms on V_r.

Obviously we have

$$d\omega_{j_1 j_2 \ldots j_k}^\sigma \wedge \omega_i = -\omega_{j_1 j_2 \ldots j_k i}^\sigma \wedge \omega_0, \text{ for } 0 \leq k \leq r-2,$$
$$d\omega_{j_1 j_2 \ldots j_{r-1}}^\sigma \wedge \omega_i = -dy_{j_1 j_2 \ldots j_{r-1} i}^\sigma \wedge \omega_0.$$

Let $N = n + m \begin{pmatrix} n+r \\ n \end{pmatrix}$, $M = m \begin{pmatrix} n+r-1 \\ n \end{pmatrix}$, $P = m \begin{pmatrix} n+r-1 \\ n \end{pmatrix} + 2n - 1$.

It is clear that $N = \dim J^rY$ and M is the number of linear independent forms $\omega_{j_1 j_2 \ldots j_k}^\sigma$, where $0 \leq k \leq r-1$.

For any function f $:V_r \to R$ we have $h(df) = \sum_{i=1}^n d_i f.dx^i$, where

$$d_i f = \frac{\partial f}{\partial x^i} + \sum_{\sigma=1}^m \sum_{k=0}^r \frac{\partial f}{\partial y^\sigma_{j_1\ldots j_k}} y^\sigma_{j_1\ldots j_k}.$$

For any k-form $\rho \in \Omega^k_r$, we denote by ρ_0 the horizontal component of ρ, and ρ_q, $1 \le q \le k$, is the q-contact component of ρ. Then for any $\rho \in \Omega^k_r$ there exists a unique decomposition

$$\pi^*_{r+1,r}\rho = \rho_0 + \rho_1 + \cdots + \rho_k.$$

We denote by $p^k_{r,q} : \Omega^k_r \to \Omega^k_{r+1}$ the morphism of sheaves defined by $p^k_{r,q}\rho = \rho_q$, for $0 \le q \le k$. $\Omega^k_{r(c)} = ker\ p^k_{r,0}$, $1 \le k \le n$, is the sheave of contact k-forms. $\Omega^k_{r(c)} = ker\ p^k_{r,k-n}$, $n+1 \le k \le N$, is the sheave of strongly contact k-forms.

Let $\Theta^k_r = d\Omega^{k-1}_{r(c)} + \Omega^k_{r(c)}$. In [3] and [5] the authours proved the softness of the sheaves Θ^k_r, considered the quotient sheaves Ω^k_r/Θ^k_r, and they obtained the following short exact sequence

$$0 \to \Theta^k_r \xrightarrow{i^k_r} \Omega^k_r \xrightarrow{\tau^k_r} \Omega^k_r/\Theta^k_r \to 0,$$

where i^k_r is the canonical injective, τ^k_r is the canonical quotient mapping.

Especially, Krupka [5] constructed the following variational sequence of order r over Y

$$0 \to R \to \Omega^0_r \xrightarrow{E_0} \Omega^1_r/\Theta^1_r \xrightarrow{E_1} \Omega^2_r/\Theta^2_r \xrightarrow{E_2} \cdots \xrightarrow{E_{P-2}} \Omega^{P-1}_r/\Theta^{P-1}_r \to$$
$$\xrightarrow{E_{P-1}} \Omega^P_r/\Theta^P_r \xrightarrow{E_P} \Omega^{P+1}_r \xrightarrow{E_{P+1}} \cdots \to \Omega^N_r \to 0,$$

where the sheaf morphism $E_k : \Omega^k_r/\Theta^k_r \to \Omega^{k+1}_r/\Theta^{k+1}_r$ is defined by the formula $E_k([\varrho]) = [d\varrho]$.

He proved that, the variational sequence of order r is an acyclic resolution of the constant sheaf R over Y, and $[\varrho] = p^k_{r,0}\varrho$ for all $1 \le k \le n$ and for all $\varrho \in \Omega^k_r$. This is the horizontal component of k-form ϱ. Let $1 \le k \le n-1$ and $\varrho \in \Omega^k_r$, then

$$[\varrho] = \frac{1}{k!} f_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \tag{2}$$

where $f_{i_1\ldots i_k}$ are some functions on V_r.

Let $k = n$ and $\varrho \in \Omega^n_r$. Then

$$[\varrho] = f\omega_0, \tag{3}$$

where f is some function on V_r. In this case $[\varrho]$ is called the *Lagrange class* of ϱ.

Let $n+1 \le k \le P$. For every $s > r$, Krupka [5] proved that

$$\Omega^k_r/\Theta^k_r \approx Im(\tau^k_s.\pi^*_{s,r+1}.p^k_{r,k-n}),$$

and this implies that for every $\varrho \in \Omega^k_r$

$$[\varrho] = \tau^k_s(p^k_{s-1,k-n}.\pi^*_{s-1,r}\varrho). \tag{4}$$

Let $k = n+1$. For each element $\varrho \in \Omega_r^{n+1}$, $[\varrho]$ is called the *Euler-Lagrange class* of $(n+1)$-forms ϱ.

Let $k = n+2$. For each element $\varrho \in \Omega_r^{n+2}$, $[\varrho]$ is called the *Helmiholtz-Sonin class* of $(n+2)$-forms ϱ.

Below we give a concrete presentation of the elements in Ω_r^k/Θ_r^k for all positive integer k satisfying $n+1 \leq k \leq P$ in the case of $r = 1$ and $r = 2$.

3. The Case of $r = 1$

Theorem 1.

a) *Let $\varrho \in \Omega_1^{n+1}$ be a germ. Suppose that in the fiber chart $(V, \psi), \psi = (x^i, y^\sigma)$,*

$$p_{1,1}^{n+1}\varrho = f_\sigma w^\sigma \wedge w_0 + f_\sigma^i w_i^\sigma \wedge w_0, \tag{5}$$

where $1 \leq \sigma \leq m, 1 \leq i \leq n$, and f_σ , f_σ^i are functions defined on $V_2 \subset J^2Y$. Then we have

$$[\varrho] = (f_\sigma - d_i f_\sigma^i) w^\sigma \wedge w_0. \tag{6}$$

b) *Let $\varrho \in \Omega_1^{n+2}$ be a germ. Suppose that in the fiber chart $(V, \psi), \psi = (x^i, y^\sigma)$,*

$$p_{1,2}^{n+2}\varrho = f_{\sigma\nu} w^\sigma \wedge w^\nu \wedge w_0 + f_{\sigma\nu}^i w_i^\sigma \wedge w^\nu \wedge w_0 + f_{\sigma\nu}^{ij} w_i^\sigma \wedge w_j^\nu \wedge w_0, \tag{7}$$

where $1 \leq \sigma, \nu \leq m, 1 \leq i, j \leq n$, and $f_{\sigma\nu}, f_{\sigma\nu}^i, f_{\sigma\nu}^{ij}$ are functions defined on $V_2 \subset J^2Y$. Then

$$[\varrho] = ((f_{\sigma\nu} w^\sigma + (f_{\sigma\nu}^i - d_i f_{\sigma\nu}^{ij}) w_i^\sigma) - f_{\sigma\nu}^{ij} w_{ij}^\sigma) \wedge w^\nu \wedge w_0. \tag{8}$$

c) *Let $n+3 \leq k \leq P$ and $\varrho \in \Omega_1^k$ be a germ. Suppose that in the fiber chart $(V, \psi), \psi = (x^i, y^\sigma)$,*

$$\begin{aligned} p_{1,k-n}^k \varrho = {} & f_{\sigma_1 \ldots \sigma_{k-n}} w^{\sigma_1} \wedge w^{\sigma_2} \wedge \ldots \wedge \ldots \wedge w^{\sigma_{k-n}} \wedge w_0 \\ & + f_{\sigma_1 \ldots \sigma_{k-n}}^{i_1} w_{i_1}^{\sigma_1} \wedge w^{\sigma_2} \wedge \ldots \wedge \ldots \wedge w^{\sigma_{k-n}} \wedge w_0 \\ & + \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + f_{\sigma_1 \ldots \sigma_{k-n}}^{i_1 \ldots i_{k-n-1}} w_{i_1}^{\sigma_1} \wedge w_{i_2}^{\sigma_2} \wedge \ldots \wedge w_{i_{k-n-1}}^{\sigma_{k-n-1}} \wedge w^{\sigma_{k-n}} \wedge w_0 \\ & + f_{\sigma_1 \ldots \sigma_{k-n}}^{i_1 \ldots i_{k-n}} w_{i_1}^{\sigma_1} \wedge w_{i_2}^{\sigma_2} \wedge \ldots \wedge w_{i_{k-n-1}}^{\sigma_{k-n-1}} \wedge w_{i_{k-n}}^{\sigma_{k-n}} \wedge w_0, \end{aligned} \tag{9}$$

where $1 \leq \sigma_1, \ldots, \sigma_{k-n} \leq m, 1 \leq i_1, \ldots i_{k-n} \leq n$ and each function defined on $V_2 \subset J^2Y$. Then we have

$$\begin{aligned}[\varrho] = \Big(&\sum_{h=0}^{k-n-2} f^{i_1\dots i_h}_{\sigma_1\dots\sigma_{k-n}} w^{\sigma_1}_{i_1} \wedge \dots \wedge w^{\sigma_h}_{i_h} \wedge w^{\sigma_{h+1}} \wedge \dots \wedge w^{\sigma_{k-n-1}} \\ &+ (f^{i_1\dots i_{k-n-1}}_{\sigma_1\dots\sigma_{k-n}} - d_{i_{k-n}} f^{i_1\dots i_{k-n}}_{\sigma_1\dots\sigma_{k-n}}) w^{\sigma_1}_{i_1} \wedge \dots \wedge w^{\sigma_{k-n-1}}_{i_{k-n-1}} \\ &- \sum_{t=1}^{k-n-1} f^{i_1\dots i_{k-n}}_{\sigma_1\dots\sigma_{k-n}} w^{\sigma_1}_{i_1} \wedge \dots \wedge w^{\sigma_{t-1}}_{i_{t-1}} \wedge w^{\sigma_t}_{i_t i_{k-n}} \wedge w^{\sigma_{t+1}}_{i_{t+1}} \wedge \dots \\ &\dots \wedge w^{\sigma_{k-n-1}}_{i_{k-n-1}} \Big) \wedge w^{\sigma_{k-n}} \wedge w_0,\end{aligned} \tag{10}$$

where each function is well defined on $V_3 \subset J^3Y$.

Proof.
a) Considering all the factors in (5) which contains w^σ_i, we get

$$f^i_\sigma w^\sigma_i \wedge w_0 = -f^i_\sigma d(w^\sigma \wedge w_i) = -d(f^i_\sigma w^\sigma \wedge w_i) + df^i_\sigma \wedge w^\sigma \wedge w_i. \tag{11}$$

Since $f^i_\sigma w^\sigma \wedge w_i \in \Theta^n_2$ and

$$\pi^*_{3,2}(df^i_\sigma \wedge w^\sigma \wedge w_i) = h(df^i_\sigma) \wedge w^\sigma \wedge w_i + p(df^i_\sigma) \wedge w^\sigma \wedge w_i,$$

we have

$$\begin{aligned}[\varrho] &= \tau^{n+1}_3 . p^{n+1}_{2,1} . \pi^*_{2,1}(\varrho) = f_\sigma w^\sigma \wedge w_0 + h(df^i_\sigma) \wedge w^\sigma \wedge w_i \\ &= (f_\sigma - d_i f^i_\sigma) w^\sigma \wedge w_0.\end{aligned}$$

b) Considering all the factors in (7) which contains $w^\sigma_i \wedge w^\nu_j$, we get

$$\begin{aligned}f^{ij}_{\sigma\nu} w^\sigma_i \wedge w^\nu_j \wedge w_0 &= -f^{ij}_{\sigma\nu} w^\sigma_i \wedge d(w^\nu \wedge w_j) = \\ &= f^{ij}_{\sigma\nu} d(w^\sigma_i \wedge w^\nu \wedge w_j) - f^{ij}_{\sigma\nu} dw^\sigma_i \wedge w^\nu \wedge w_j \\ &= d(f^{ij}_{\sigma\nu} w^\sigma_i \wedge w^\nu \wedge w_j) - df^{ij}_{\sigma\nu} \wedge w^\sigma_i \wedge w^\nu \wedge w_j + \\ &\quad + \sum_{l=1}^{n} f^{ij}_{\sigma\nu} dy^\sigma_{il} \wedge dx^l \wedge w^\nu \wedge w_j.\end{aligned} \tag{12}$$

Since $f^{ij}_{\sigma\nu} w^\sigma_i \wedge w^\nu \wedge w_j \in \Theta^{n+1}_2$, we have

$$\begin{aligned}[\varrho] =& \tau^{n+2}_3 . p^{n+2}_{2,2} . \pi^*_{2,1}(\varrho) \\ &= f_{\sigma\nu} w^\sigma \wedge w^\nu \wedge w_0 + f^i_{\sigma\nu} w^\sigma_i \wedge w^\nu \wedge w_0 \\ &- d_j f^{ij}_{\sigma\nu} w^\sigma_i \wedge w^\nu \wedge w_0 - f^{ij}_{\sigma\nu} w^\sigma_{ij} \wedge w^\nu \wedge w_0.\end{aligned} \tag{13}$$

Therefore we have

$$[\varrho] = ((f_{\sigma\nu} w^\sigma + (f^i_{\sigma\nu} - d_i f^{ij}_{\sigma\nu}) w^\sigma_i) - f^{ij}_{\sigma\nu} w^\sigma_{ij}) \wedge w^\nu \wedge w_0.$$

c) In the formula (9), we consider all the factors containing $w^{\sigma_{k-n}}_{i_{k-n}}$. They are $f^{i_1\dots i_{k-n}}_{\sigma_1\dots\sigma_{k-n}} w^{\sigma_1}_{i_1} \wedge \dots \wedge w^{\sigma_{k-n}}_{i_{k-n}} \wedge w_0$. We get

$$
\begin{aligned}
& f^{i_1\ldots i_{k-n}}_{\sigma_1\ldots\sigma_{k-n}} w^{\sigma_1}_{i_1} \wedge \ldots \wedge w^{\sigma_{k-n}}_{i_{k-n}} \wedge w_0 \\
= & - f^{i_1\ldots i_{k-n}}_{\sigma_1\ldots\sigma_{k-n}} w^{\sigma_1}_{i_1} \wedge \ldots \wedge w^{\sigma_{k-n-1}}_{i_{k-n-1}} \wedge d(w^{\sigma_{k-n}} \wedge w_{i_{k-n}}) \\
= & (-1)^{k-n} f^{i_1\ldots i_{k-n}}_{\sigma_1\ldots\sigma_{k-n}} d(w^{\sigma_1}_{i_1} \wedge \ldots \wedge w^{\sigma_{k-n-1}}_{i_{k-n-1}} \wedge w^{\sigma_{k-n}} \wedge w_{i_{k-n}}) \\
& + \sum_{t=1}^{k-n-1} (-1)^{k-n+t} f^{i_1\ldots i_{k-n}}_{\sigma_1\ldots\sigma_{k-n}} w^{\sigma_1}_{i_1} \wedge \ldots \wedge w^{\sigma_{t-1}}_{i_{t-1}} \wedge dw^{\sigma_t}_{i_t} \wedge \\
& \quad \wedge w^{\sigma_{t+1}}_{i_{t+1}} \wedge \ldots \wedge w^{\sigma_{k-n-1}}_{i_{k-n-1}} \wedge w^{\sigma_{k-n}} \wedge w_{i_{k-n}} \qquad (14) \\
= & (-1)^{k-n} d(f^{i_1\ldots i_{k-n}}_{\sigma_1\ldots\sigma_{k-n}} w^{\sigma_1}_{i_1} \wedge \ldots \wedge w^{\sigma_{k-n-1}}_{i_{k-n-1}} \wedge w^{\sigma_{k-n}} \wedge w_{i_{k-n}}) \\
& - (-1)^{k-n} df^{i_1\ldots i_{k-n}}_{\sigma_1\ldots\sigma_{k-n}} \wedge w^{\sigma_1}_{i_1} \wedge \ldots \wedge w^{\sigma_{k-n-1}}_{i_{k-n-1}} \wedge w^{\sigma_{k-n}} \wedge w_{i_{k-n}} \\
& + \sum_{t=1}^{k-n-1} \sum_{l=1}^{n} (-1)^{k-n+t+1} f^{i_1\ldots i_{k-n}}_{\sigma_1\ldots\sigma_{k-n}} w^{\sigma_1}_{i_1} \wedge \ldots \wedge w^{\sigma_{t-1}}_{i_{t-1}} \wedge dy^{\sigma_t}_{i_t l} \wedge dx^l \wedge \\
& \quad \wedge w^{\sigma_{t+1}}_{i_{t+1}} \wedge \ldots \wedge w^{\sigma_{k-n-1}}_{i_{k-n-1}} \wedge w^{\sigma_{k-n}} \wedge w_{i_{k-n}}.
\end{aligned}
$$

Since $f^{i_1\ldots i_{k-n}}_{\sigma_1\ldots\sigma_{k-n}} w^{\sigma_1}_{i_1} \wedge \ldots \wedge w^{\sigma_{k-n-1}}_{i_{k-n-1}} \wedge w^{\sigma_{k-n}} \wedge w_{i_{k-n}} \in \Theta^{k-1}_2$, we have

$$
\begin{aligned}
& \tau^k_3 . p^k_{2,k-n} (f^{i_1\ldots i_{k-n}}_{\sigma_1\ldots\sigma_{k-n}} w^{\sigma_1}_{i_1} \wedge \ldots \wedge w^{\sigma_{k-n}}_{i_{k-n}} \wedge w_0) \\
= & - d_{i_{k-n}} f^{i_1\ldots i_{k-n}}_{\sigma_1\ldots\sigma_{k-n}} w^{\sigma_1}_{i_1} \wedge \ldots \wedge w^{\sigma_{k-n-1}}_{i_{k-n-1}} \wedge w^{\sigma_{k-n}} \wedge w_0 \\
- & \sum_{t=1}^{k-n-1} f^{i_1\ldots i_{k-n}}_{\sigma_1\ldots\sigma_{k-n}} w^{\sigma_1}_{i_1} \wedge \ldots \wedge w^{\sigma_{t-1}}_{i_{t-1}} \wedge w^{\sigma_t}_{i_t i_{k-n}} \wedge w^{\sigma_{t+1}}_{i_{t+1}} \wedge \ldots \wedge w^{\sigma_{k-n-1}}_{i_{k-n-1}} \wedge w^{\sigma_{k-n}} \wedge w_0.
\end{aligned}
\qquad (15)
$$

Then $[\varrho]$ can be presented in the following form

$$
\begin{aligned}
[\varrho] = \Bigg(& \sum_{h=0}^{k-n-2} f^{i_1\ldots i_h}_{\sigma_1\ldots\sigma_{k-n}} w^{\sigma_1}_{i_1} \wedge \ldots \wedge w^{\sigma_h}_{i_h} \wedge w^{\sigma_{k+1}} \wedge \ldots \wedge w^{\sigma_{k-n-1}} \\
& + (f^{i_1\ldots i_{k-n-1}}_{\sigma_1\ldots\sigma_{k-n}} - d_{i_{k-n}} f^{i_1\ldots i_{k-n}}_{\sigma_1\ldots\sigma_{k-n}}) w^{\sigma_1}_{i_1} \wedge \ldots \wedge w^{\sigma_{k-n-1}}_{i_{k-n-1}} \\
& - \sum_{t=1}^{k-n-1} f^{i_1\ldots i_{k-n}}_{\sigma_1\ldots\sigma_{k-n}} w^{\sigma_1}_{i_1} \wedge \ldots \wedge w^{\sigma_{t-1}}_{i_{t-1}} \wedge w^{\sigma_t}_{i_t i_{t-1}} \wedge w^{\sigma_{t+1}}_{i_{t+1}} \wedge \ldots \\
& \ldots \wedge w^{\sigma_{k-n-1}}_{i_{k-n-1}} \Bigg) \wedge w^{\sigma_{k-n}} \wedge w_0,
\end{aligned}
$$

where each function is defined on $V_3 \subset J^3 Y$. ■

4. The Case of $r = 2$

Theorem 2.

a) *Let* $\varrho \in \Omega^{n+1}_2$ *be a germ. Suppose that in the fiber chart* $(V, \psi), \psi = (x^i, y^\sigma)$,

$$
p^{n+1}_{2,1} \varrho = f_\sigma w^\sigma \wedge w_0 + f^i_\sigma w^\sigma_i \wedge w_0 + f^{ij}_\sigma w^\sigma_{ij} \wedge w_0, \qquad (16)
$$

where $1 \le \sigma \le m, 1 \le i,j \le m, f_\sigma, f^i_\sigma, f^{ij}_\sigma$ *are functions well defined on* $V_3 \subset J^3Y$. *Then we have*

$$[\varrho] = (f_\sigma - d_i f^i_\sigma + d_i d_j f^{ij}_\sigma) w^\sigma \wedge w_0, \tag{17}$$

where each function is well defined on $V_5 \subset J^5Y$.

b) *Let* $\varrho \in \Omega^{n+2}_2$ *be a germ. Suppose that in the fiber chart* $(V,\psi), \psi = (x^i, y^\sigma)$,

$$p^{n+2}_{2,2}\varrho = (f^J_{\sigma\nu} w^\sigma_J \wedge w^\nu \wedge w_0 + f^{Ji}_{\sigma\nu} w^\sigma_J \wedge w^\nu_i \wedge w_0 + f^{Jij}_{\sigma\nu} w^\sigma_J \wedge w^\nu_{ij} \wedge w_0, \tag{18}$$

where $0 \le |J| \le 2, 1 \le \sigma,\nu \le m, 1 \le i,j \le n,$ *and functions* $f^J_{\sigma\nu}, f^{Ji}_{\sigma\nu}, f^{Jij}_{\sigma\nu}$ *are well defined on* $V_3 \subset J^3Y$. *Then we have*

$$\begin{aligned}[\varrho] = ((f^J_{\sigma\nu} - d_i f^{Ji}_{\sigma\nu} + d_i d_j f^{Jij}_{\sigma\nu}) w^\sigma_J \\ +(-f^{Ji}_{\sigma\nu} + 2 d_j f^{Jij}_{\sigma\nu}) w^\sigma_{Ji} + f^{Jij}_{\sigma\nu} w^\sigma_{Jij}) \wedge w^\nu \wedge w_0,\end{aligned} \tag{19}$$

where each function is well defined on $V_5 \subset J^5Y$.

c) *Let* $n+3 \le k \le P$ *and* $\varrho \in \Omega^k_2$ *be a germ. Suppose that in the fiber chart* $(V,\psi), \psi = (x^i, y^\sigma)$ *we have*

$$p^k_{2,k-n}\varrho = \sum_{q=0}^{2} f^{J_1\dots J_{k-n-1} j_1 \dots j_q}_{\sigma_1 \dots \sigma_{k-n}} w^{\sigma_1}_{J_1} \wedge \dots \wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}} \wedge w^{\sigma_{k-n}}_{j_1 .. j_q} \wedge w_0, \tag{20}$$

where $0 \le |J_1|, \dots, |J_{k-n-1}| \le 2, 1 \le \sigma_1, \dots, \sigma_{k-n} \le m, 1 \le j_1, \dots, j_q \le n,$ *and every function* $f^{J_1 \dots J_{k-n-1} j_1 \dots j_q}_{\sigma_1 \dots \sigma_{k-n}}$ *is well defined on* $V_3 \subset J^3Y$. *Then we have*

$$\begin{aligned}[\varrho] = \Big(& (f^{J_1\dots J_{k-n-1}}_{\sigma_1\dots\sigma_{k-n}} - d_{j_1} f^{J_1\dots J_{k-n-1} j_1}_{\sigma_1\dots\sigma_{k-n}} + d_{j_1} d_{j_2} f^{J_1\dots J_{k-n-1} j_1 j_2}_{\sigma_1\dots\sigma_{k-n}}) w^{\sigma_1}_{J_1} \wedge \dots \\ & \dots \wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}} + \sum_{t=1}^{k-n-1} (-f^{J_1\dots J_{k-n-1} j_1}_{\sigma_1\dots\sigma_{k-n}} + 2 d_{j_2} f^{J_1\dots J_{k-n-1} j_1 j_2}_{\sigma_1\dots\sigma_{k-n}}) w^{\sigma_1}_{J_1} \wedge \dots \\ & \dots \wedge w^{\sigma_{t-1}}_{J_{t-1}} \wedge w^{\sigma_t}_{J_t j_1} \wedge w^{\sigma_{t+1}}_{J_{t+1}} \wedge \dots \wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}} \\ & + \sum_{\substack{h=1 \\ h \ne t}}^{k-n-1} \sum_{t=1}^{k-n-1} f^{J_1\dots J_{k-n-1} j_1 j_2}_{\sigma_1\dots\sigma_{k-n}} w^{\sigma_1}_{J_1} \wedge \dots \wedge w^{\sigma_{h-1}}_{J_{h-1}} \wedge w^{\sigma_h}_{J_h j_1} \wedge w^{\sigma_{h+1}}_{J_{h+1}} \wedge \dots \\ & \dots \wedge w^{\sigma_{t-1}}_{J_{t-1}} \wedge w^{\sigma_t}_{J_t j_2} \wedge w^{\sigma_{t+1}}_{J_{t+1}} \wedge \dots \wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}} \\ & + \sum_{t=1}^{k-n-1} f^{J_1\dots J_{k-n-1} j_1 j_2}_{\sigma_1\dots\sigma_{k-n}} w^{\sigma_1}_{J_1} \wedge \dots \wedge w^{\sigma_{t-1}}_{J_{t-1}} \wedge w^{\sigma_t}_{J_t j_1 j_2} \wedge w^{\sigma_{t+1}}_{J_{t+1}} \wedge \dots \\ & \dots \wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}} \Big) \wedge w^{\sigma_{k-n}} \wedge w_0,\end{aligned} \tag{21}$$

where each function is well defined on $V_5 \subset J^5Y$.

Proof.
a) In the formula (16) we consider the factors containing w_i^σ. They are$f_\sigma^i w_i^\sigma \wedge w_0$. We get

$$f_\sigma^i w_i^\sigma \wedge w_0 = -f_\sigma^i d(w^\sigma \wedge w_i) = -d(f_\sigma^i w^\sigma \wedge w_i) + df_\sigma^i \wedge w^\sigma \wedge w_i. \tag{22}$$

Since $f_\sigma^i w^\sigma \wedge w_i \in \Omega_3^n = \Theta_3^n$ and

$$\pi_{4,3}^*(df_\sigma^i \wedge w^\sigma \wedge w_i) = h(df_\sigma^i) \wedge w^\sigma \wedge w_i + p(df_\sigma^i) \wedge w^\sigma \wedge w_i,$$

it implies that

$$\tau_4^{n+1}.p_{3,1}^{n+1}(f_\sigma^i w_i^\sigma \wedge w_0) = -d_i f_\sigma^i \wedge w^\sigma \wedge w_0. \tag{23}$$

Now we consider all the factors containing w_{ij}^σ in the formula (16). Then we have

$$f_\sigma^{ij} w_{ij}^\sigma \wedge w_0 = -d(f_\sigma^{ij} w_i^\sigma \wedge w_j) + df_\sigma^{ij} \wedge w_i^\sigma \wedge w_j. \tag{24}$$

This implies that

$$\begin{aligned} \tau_4^{n+1}.p_{3,1}^{n+1}(f_\sigma^{ij} w_{ij}^\sigma \wedge w_0) &= -d_j f_\sigma^{ij} w_i^\sigma \wedge w_0 \\ = d(d_j f_\sigma^{ij}.w^\sigma \wedge w_i) &- d(d_j f_\sigma^{ij}) \wedge w^\sigma \wedge w_i. \end{aligned} \tag{25}$$

Therefore we have

$$\tau_5^{n+1}.p_{4,1}^{n+1}.\pi_{4,3}^*(f_\sigma^{ij} w_{ij}^\sigma \wedge w_0) = d_i d_j f_\sigma^{ij} w^\sigma \wedge w_0. \tag{26}$$

Since (16), (23) and (26) we get

$$[\varrho] = \tau_5^{n+1}.p_{4,1}^{n+1}.\pi_{4,2}^*(\varrho) = (f_\sigma - d_i f_\sigma^i + d_i d_j f_\sigma^{ij}) w^\sigma \wedge w_0,$$

where each function is defined on $V_5 \subset J^5 Y$.
b) In the formula (18) we consider all the factors containing w_i^ν. Then we get

$$\begin{aligned} f_{\sigma\nu}^{Ji} w_J^\sigma \wedge w_i^\nu \wedge w_0 &= -f_{\sigma\nu}^{Ji} w_J^\sigma \wedge d(w^\nu \wedge w_i) \\ &= f_{\sigma\nu}^{Ji} d(w_J^\sigma \wedge w^\nu \wedge w_i) - f_{\sigma\nu}^{Ji} dw_J^\sigma \wedge w^\nu \wedge w_i \\ &= d(f_{\sigma\nu}^{Ji} w_J^\sigma \wedge w^\nu \wedge w_i) - df_{\sigma\nu}^{Ji} \wedge w_J^\sigma \wedge w^\nu \wedge w_i \\ &+ \sum_{l=1}^n df_{\sigma\nu}^{Ji} dy_{Jl}^\sigma \wedge dx^l \wedge w^\nu \wedge w_i. \end{aligned}$$

Therefore

$$\begin{aligned} \tau_4^{n+2}.p_{3,2}^{n+2}(f_{\sigma\nu}^{Ji} w_J^\sigma \wedge w_i^\nu \wedge w_0) = &- d_i f_{\sigma\nu}^{Ji} w_J^\sigma \wedge w^\nu \wedge w_0 \\ &- f_{\sigma\nu}^{Ji} w_{Ji}^\sigma \wedge w^\nu \wedge w_0. \end{aligned} \tag{27}$$

Now we consider all the factors containing w_{ij}^ν in the formula (16). Then we have

$$
\begin{aligned}
& f^{Jij}_{\sigma\nu} w^\sigma_J \wedge w^\nu_{ij} \wedge w_0 = -f^{Jij}_{\sigma\nu} w^\sigma_J \wedge d(w^\nu_i \wedge w_j) \\
&= d(f^{Jij}_{\sigma\nu} w^\sigma_J \wedge w^\nu_i \wedge w_j) - df^{Jij}_{\sigma\nu} \wedge w^\sigma_J \wedge w^\nu_i \wedge w_j \\
&+ \sum_{l=1}^{n} f^{Jij}_{\sigma\nu} dy^\sigma_{Jl} dx^l \wedge w^\nu_i \wedge w_j \\
& \xrightarrow{\tau_4^{n+2}.p_{3,2}^{n+2}} -d_j f^{Jij}_{\sigma\nu} w^\sigma_J \wedge w^\nu_i \wedge w_0 - f^{Jij}_{\sigma\nu} w^\sigma_{Jj} \wedge w^\nu_i \wedge w_0 \\
& \xrightarrow{\tau_5^{n+2}.p_{4,2}^{n+2}} d_i d_j f^{Jij}_{\sigma\nu} w^\sigma_J \wedge w^\nu \wedge w_0 + d_j f^{Jij}_{\sigma\nu} w^\sigma_{Ji} \wedge w^\nu \wedge w_0 \\
&+ d_i f^{Jij}_{\sigma\nu} w^\sigma_{Jj} \wedge w^\nu \wedge w_0 + f^{Jij}_{\sigma\nu} w^\sigma_{Jij} \wedge w^\nu \wedge w_0,
\end{aligned} \tag{28}
$$

where $\tau_4^{n+2}.p_{3,2}^{n+2}$(resp. $\tau_5^{n+2}.p_{4,2}^{n+2}$) are morphisms of sheaves. Since the formulas (18), (27), (28) and the symmetry of indexes i, j we have

$$
\begin{aligned}
[\varrho] = \tau_5^{n+2}.p_{4,2}^{n+2}.\pi^*_{4,2}(\varrho) &= ((f^J_{\sigma\nu} - d_i f^{Ji}_{\sigma\nu} + d_i d_j f^{Jij}_{\sigma\nu}) w^\sigma_J \\
&+ (-f^{Ji}_{\sigma\nu} + 2 d_j f^{Jij}_{\sigma\nu}) w^\sigma_{Ji} + f^{Jij}_{\sigma\nu} w^\sigma_{Jij}) \wedge w^\nu \wedge w_0,
\end{aligned}
$$

where each function is defined on $V_5 \subset J^5Y$.

c) We consider the factors containing $w^{\sigma_{k-n}}_{J_1}$ in the formula (20). We have

$$
\begin{aligned}
& f^{J_1\dots J_{k-n-1}j_1}_{\sigma_1\dots\sigma_{k-n}} w^{\sigma_1}_{J_1} \wedge \dots \wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}} \wedge w^{\sigma_{k-n}}_{j_1} \wedge w_0 \\
&= -f^{J_1\dots J_{k-n-1}j_1}_{\sigma_1\dots\sigma_{k-n}} w^{\sigma_1}_{J_1} \wedge \dots \wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}} \wedge d(w^{\sigma_{k-n}} \wedge w_{j_1}) \\
&= (-1)^{k-n} f^{J_1\dots J_{k-n-1}j_1}_{\sigma_1\dots\sigma_{k-n}} d(w^{\sigma_1}_{J_1} \wedge \dots \wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}} \wedge w^{\sigma_{k-n}} \wedge w_{j_1}) \\
&+ \sum_{t=1}^{k-n-1} (-1)^{k-n+t} f^{J_1\dots J_{k-n-1}j_1}_{\sigma_1\dots\sigma_{k-n}} w^{\sigma_1}_{J_1} \wedge \dots \wedge w^{\sigma_{t-1}}_{J_{t-1}} \wedge \\
& \wedge dw^{\sigma_t}_{J_t} \wedge w^{\sigma_{t+1}}_{J_{t+1}} \wedge \dots \wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}} \wedge w^{\sigma_{k-n}} \wedge w_{j_1} \\
&= (-1)^{k-n} d(f^{J_1\dots J_{k-n-1}j_1}_{\sigma_1\dots\sigma_{k-n}} w^{\sigma_1}_{J_1} \wedge \dots \wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}} \wedge w^{\sigma_{k-n}} \wedge w_{j_1}) \\
&- (-1)^{k-n} df^{J_1\dots J_{k-n-1}j_1}_{\sigma_1\dots\sigma_{k-n}} \wedge w^{\sigma_1}_{J_1} \wedge \dots \wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}} \wedge w^{\sigma_{k-n}} \wedge w_{j_1} \\
&- \sum_{t=1}^{k-n-1} \sum_{l=1}^{n} (-1)^{k-n+t} f^{J_1\dots J_{k-n-1}j_1}_{\sigma_1\dots\sigma_{k-n}} w^{\sigma_1}_{J_1} \wedge \dots \wedge w^{\sigma_{t-1}}_{J_{t-1}} \wedge \\
& \wedge dy^{\sigma_t}_{J_t l} \wedge dx^l \wedge w^{\sigma_{t+1}}_{J_{t+1}} \wedge \dots \wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}} \wedge w^{\sigma_{k-n}} \wedge w_{j_1} \\
& \xrightarrow{\tau_4^k.p_{3,k-n}^k} -d_{j_1} f^{J_1\dots J_{k-n-1}}_{\sigma_1\dots\sigma_{k-n}} w^{\sigma_1}_{J_1} \wedge \dots \wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}} \wedge w^{\sigma_{k-n}} \wedge w_0 \\
&- \sum_{t=1}^{k-n-1} f^{J_1\dots J_{k-n-1}}_{\sigma_1\dots\sigma_{k-n}} w^{\sigma_1}_{J_1} \wedge \dots \wedge w^{\sigma_{t-1}}_{J_{t-1}} \wedge w^{\sigma_t}_{J_t j_1} \wedge \\
& \qquad \wedge w^{\sigma_{t+1}}_{J_{t+1}} \wedge \dots \wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}} \wedge w^{\sigma_{k-n}} \wedge w_0.
\end{aligned} \tag{29}
$$

We consider all the factors containing $w^{\sigma_{k-n}}_{j_1 j_2}$ in the formula (20). We have

$$\begin{aligned}
&f^{J_1\ldots J_{k-n-1}j_1j_2}_{\sigma_1\ldots\sigma_{k-n}} w^{\sigma_1}_{J_1}\wedge\ldots\wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}}\wedge w^{\sigma_{k-n}}_{j_1j_2}\wedge w_0 \overset{\tau^k_4.p^k_{3,k-n}}{\longrightarrow}\\
&-d_{j_2}f^{J_1\ldots J_{k-n-1}j_1j_2}_{\sigma_1\ldots\sigma_{k-n}} w^{\sigma_1}_{J_1}\wedge\ldots\wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}}\wedge w^{\sigma_{k-n}}_{j_1}\wedge w_0\\
&-\sum_{t=1}^{k-n-1} f^{J_1\ldots J_{k-n-1}j_1j_2}_{\sigma_1\ldots\sigma_{k-n}} w^{\sigma_1}_{J_1}\wedge\ldots\wedge w^{\sigma_{t-1}}_{J_{t-1}}\wedge w^{\sigma_t}_{J_tj_2}\wedge w^{\sigma_{t+1}}_{J_{t+1}}\wedge\\
&\ldots\wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}}\wedge w^{\sigma_{k-n}}_{j_1}\wedge w_0\\
&\overset{\tau^k_5.p^k_{4,k-n}d_{j_1}}{\longrightarrow} d_{j_2}f^{J_1\ldots J_{k-n-1}j_1j_2}_{\sigma_1\ldots\sigma_{k-n}} w^{\sigma_1}_{J_1}\wedge\ldots\wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}}\wedge w^{\sigma_{k-n}}\wedge w_0\\
&+\sum_{h=1}^{k-n-1} d_{j_2}f^{J_1\ldots J_{k-n-1}j_1j_2}_{\sigma_1\ldots\sigma_{k-n}} w^{\sigma_1}_{J_1}\wedge\ldots\wedge w^{\sigma_{h-1}}_{J_{h-1}}\wedge w^{\sigma_h}_{J_hj_1}\wedge w^{\sigma_{h+1}}_{J_{h+1}}\wedge\\
&\ldots\wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}}\wedge w^{\sigma_{k-n}}\wedge w_0\\
&+\sum_{t=1}^{k-n-1} d_{j_1}f^{J_1\ldots J_{k-n-1}j_1j_2}_{\sigma_1\ldots\sigma_{k-n}} w^{\sigma_1}_{J_1}\wedge\ldots\wedge w^{\sigma_{t-1}}_{J_{t-1}}\wedge w^{\sigma_t}_{J_tj_2}\wedge w^{\sigma_{t+1}}_{J_{t+1}}\wedge\\
&\ldots\wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}}\wedge w^{\sigma_{k-n}}\wedge w_0\\
&+\sum_{\substack{h=1\\h\neq t}}^{k-n-1}\sum_{t=1}^{k-n-1} f^{J_1\ldots J_{k-n-1}j_1j_2}_{\sigma_1\ldots\sigma_{k-n}} w^{\sigma_1}_{J_1}\wedge\ldots\wedge w^{\sigma_{h-1}}_{J_{h-1}}\wedge w^{\sigma_h}_{J_hj_1}\wedge w^{\sigma_{h+1}}_{J_{h+1}}\wedge\\
&\ldots\wedge w^{\sigma_{t-1}}_{J_{t-1}}\wedge w^{\sigma_t}_{J_tj_2}\wedge w^{\sigma_{t+1}}_{J_{t+1}}\wedge\ldots\wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}}\wedge w^{\sigma_{k-n}}\wedge w_0\\
&+\sum_{t=1}^{k-n-1} f^{J_1\ldots J_{k-n-1}j_1j_2}_{\sigma_1\ldots\sigma_{k-n}} w^{\sigma_1}_{J_1}\wedge\ldots\wedge w^{\sigma_{t-1}}_{J_{t-1}}\wedge w^{\sigma_t}_{J_tj_1j_2}\wedge w^{\sigma_{t+1}}_{J_{t+1}}\wedge\\
&\ldots\wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}}\wedge w^{\sigma_{k-n}}\wedge w_0.
\end{aligned}\tag{30}$$

Since the formulas (20), (29), (30) and the symmetry of indexes j_1, j_2 we have

$$\begin{aligned}
[\varrho]&=\tau^k_5.p^k_{4,k-n}.\pi^*_{4,2}(\varrho)\\
&=\Bigg((f^{J_1\ldots J_{k-n-1}}_{\sigma_1\ldots\sigma_{k-n}}-d_{j_1}f^{J_1\ldots J_{k-n-1}j_1}_{\sigma_1\ldots\sigma_{k-n}}+d_{j_1}d_{j_2}f^{J_1\ldots J_{k-n-1}j_1j_2}_{\sigma_1\ldots\sigma_{k-n}})w^{\sigma_1}_{J_1}\wedge\\
&\quad\ldots\wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}}+\sum_{t=1}^{k-n-1}(-f^{J_1\ldots J_{k-n-1}j_1}_{\sigma_1\ldots\sigma_{k-n}}+2d_{j_2}f^{J_1\ldots J_{k-n-1}j_1j_2}_{\sigma_1\ldots\sigma_{k-n}})w^{\sigma_1}_{J_1}\wedge\\
&\quad\ldots\wedge w^{\sigma_{t-1}}_{J_{t-1}}\wedge w^{\sigma_t}_{J_tj_1}\wedge w^{\sigma_{t+1}}_{J_{t+1}}\wedge\ldots\wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}}\\
&\quad+\sum_{\substack{h=1\\h\neq t}}^{k-n-1}\sum_{t=1}^{k-n-1} f^{J_1\ldots J_{k-n-1}j_1j_2}_{\sigma_1\ldots\sigma_{k-n}} w^{\sigma_1}_{J_1}\wedge\ldots\wedge w^{\sigma_{h-1}}_{J_{h-1}}\wedge w^{\sigma_h}_{J_hj_1}\wedge w^{\sigma_{h+1}}_{J_{h+1}}\wedge\\
&\quad\ldots\wedge w^{\sigma_{t-1}}_{J_{t-1}}\wedge w^{\sigma_t}_{J_tj_2}\wedge w^{\sigma_{t+1}}_{J_{t+1}}\wedge\ldots\wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}}
\end{aligned}$$

$$+\sum_{t=1}^{k-n-1} f^{J_1\dots J_{k-n-1}j_1j_2}_{\sigma_1\dots\sigma_{k-n}} w^{\sigma_1}_{J_1}\wedge\dots\wedge w^{\sigma_{t-1}}_{J_{t-1}}\wedge w^{\sigma_t}_{J_tj_1j_2}\wedge w^{\sigma_{t+1}}_{J_{t+1}}\wedge$$

$$\left.\dots\wedge w^{\sigma_{k-n-1}}_{J_{k-n-1}}\right)\wedge w^{\sigma_{k-n}}\wedge w_0,$$

where each function is defined on $V_5\subset J^5Y$. ■

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