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# On the Continuity of Julia Sets and Hausdorff Dimension of Polynomials

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**Abstract.** We study the continuity of Julia set of some rational maps and the stability of the Hausdorff dimension of the Julia set of polynomials $z^d + c$ $(d \geq 2)$ for some semi-hyperbolic parameters $c$ in the boundary of the generalized Mandelbrot set.

## 1. Introduction and Main Results

We say rational maps $f_n$ converge to $f$ *algebraically* if $\deg f_n = \deg f$ and, when $f_n$ is expressed as the quotient of two polynomials, the coefficients can be chosen to converge to those of $f$. Equivalently, $f_n \to f$ uniformly in the spherical metric.

Recall that compact sets $K_n \to K$ in the Hausdorff topology if

(1) Every neighborhood of a point $x \in K$ meets all but finitely many $K_n$; and
(2) If every neighborhood of $x$ meets infinitely many $K_n$, then $x \in K$.

We define $\liminf K_n$ as the largest set satisfying (1), and $\limsup K_n$ as the smallest set satisfying (2). Then $K_n \to K$ is equivalent to $\limsup K_n = \liminf K_n = K$.

$J_c$ denotes the Julia sets of polynomials $P_c = z^d + c$ and let $M_d = \{c | J_c$ is connected$\}$ be the connectedness locus; for $d = 2$ it is called the Mandelbrot set and $HD$ denotes the Hausdorff dimension.

In this paper we study the stability of the Hausdorff dimension of polynomials $P_c = z^d + c$, $d \geq 2$, such that the critical point 0 is not recurrent and $0 \in J_c$. These polynomials are *semi-hyperbolic* in the sense of [3].

Let $J(f)$ be the Julia sets of rational maps $f$. Recall that the equilibrium measure $\mu(f)$ of $f$ supported in $J(f)$ depends continuously on $f$; see [7]. $n \gg 0$

means for all $n$ sufficiently large. We have the following theorems:

**Theorem 1.** *Let $f_n \to f$ algebraically. Assume that $f$ has no Siegel disc, Herman ring nor parabolic cycles, then*

$$J(f_n) \to J(f)$$

*in the Hausdorff topology.*

**Theorem 2.** *Let $P_{c_0} = z^d + c_0$ ($d \geq 2$) be semi-hyperbolic. If there is $B_1 = B_1(c_0) > 0$ such that for a sequence $c_n \to c_0$ from the interior of $M_d$,*

$$dist\,(c_n, \partial M_d) \geq B_1 |c_n - c_0|^{1+1/d},$$

*then $HD(J_{c_n}) \to HD(J_{c_0})$.*

## 2. Preliminaries

**Definition 2.1.** *The definition of conformal measures for rational maps was first given by Sullivan (see [6]) as a modification of the Patterson measures for limit sets of Fuchsian groups. Let $t > 0$. A probability measure $m$ on $J(f)$ is called $t$-conformal for $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ if*

$$m(f(A)) = \int_A |f^{'}|^t dm$$

*for every Borel set $A \subset J(f)$ such that $f|_A$ is injective.*

A more general definition, showing the connection to ergodic theory, has been given by Denker and Urbański earlier (see [9]). It follows from topological exactness of $f|_{J(f)}$ that a conformal measure $m$ is positive on non-empty sets and therefore

$$M(r) = \inf\{m(B(z,r)) : z \in J(f)\} > 0$$

for every $r > 0$; see [8].

Consider $c_0 \in \partial M_d$ so that $P_{c_0}$ is semihyperbolic, then all (finite) periodic points of $P_{c_0}$ are repelling. Moreover, the set $\omega(0)$ of accumulation points of the orbit of 0 is a hyperbolic set of $P_{c_0}$. Thus by the expansive property, there is $m > 1$ such that $P_{c_0}^m(0) \in \omega(0)$. We suppose $m > 1$ is the least integer with this property, usually set $z_0 = P_{c_0}^m(0)$.

In [3] it is proved that there are constants $\varepsilon > 0$, $B_0 > 0$ and $\theta \in (0,1)$ such that for all $x \in J_{c_0}$ and any connected component $V$ of $P_{c_0}^{-n}(V_\varepsilon(x))$ for $n \geq 0$, the map $P_{c_0}^n : V \to V_\varepsilon(x)$ has degree at most $d$ and diam$(V) < B_0\theta^n$. Moreover the complement of $J_{c_0}$ is a *John domain*; this means that $J_{c_0}$ is locally connected and there is $\delta > 0$ such that if $z \in J_{c_0}$ and $w$ belongs to a ray landing at $z$, then $V_{\delta|z-w|}(w) \cap J_{c_0} = \emptyset$. In particular, by Carathéodory's theorem, the map $\phi_{c_0}^{-1}$, defined in $\overline{\mathbb{C}} - \overline{D}$, extends continuously to $\partial D$, so every ray lands at some point in $J_{c_0}$.

We construct a Markov partition for $\omega(0)$ with puzzles; a *puzzle* is a set bounded by a finite number of (closures of ) rays and an equipotential. Recall that by [3] all rays land at some point in $J_{c_0}$. Puzzles are homeomorphic to a disc. We have the following propositions; see [2].

**Proposition 2.1.** (Markov Partitions) *There is a Markov partition for $\omega(0)$ with puzzles. That is, there is a finite collection of disjoint puzzles $U_i$,$(i = 1, 2, ...)$, that cover $\omega(0)$ so that $P_{c_0}$ is univalent in $U_i$ and if $U_i \cap P_{c_0}(U_j) \neq \emptyset$, then $U_i \subset P_{c_0}(U_j)$.*

Consider the Markov partition $U_i$,$(i = 1, 2, ...)$, given by the Proposition 2.1. For $n \geq 0$, the preimages of the sets $U_i$ under $P^n_{c_0}$ that intersect $\omega(0)$ are called the *nth step pieces* of the Markov partition. Note that for $n \geq 1$ the collection of all the $n$th step pieces is a Markov partition; we call it a *refinement* of the Markov partition $U_i$,$(i = 1, 2, ...)$.

**Proposition 2.2.** (Bounded Distortion Property). *For any $k \geq 0$ the distortion of $P^k_{c_0}$ in each of the kth step pieces of the Markov partition is bounded by some constant $K$, independent of $k$.*

Since $P_{c_0}$ is uniformly expanding in $\omega(0)$ there is a holomorphic motion $j : B_\sigma(c_0) \times \omega(0) \to \mathbb{C}$, for some $\sigma > 0$, which is compatible with dynamics; see [4]. This means that for each $c \in B_\sigma(c_0)$ the map$j_c : \omega(0) \to \mathbb{C}$ is injective and for each $z \in \omega(0)$ the function $c \to j_c(z)$ is holomorphic. Being compatible with dynamics means that for every $c \in B_\sigma(c_0)$ the map $j_c$ conjugates $P_{c_0}$ on $\omega(0)$ to $P_c$ on $j_c(\omega(0))$.

**Proposition 2.3.** *Consider a Markov partition $U_a$, $a \in A$, of $\omega(0)$. Then there is $\sigma > 0$ and a holomorphic motion $j : B_\sigma(c_0) \times \bigcup_{a\in A} U_a \to \mathbb{C}$ compatible with dynamics. Moreover there is $R > 0$ such that $j(B_\sigma(c_0) \times \bigcup_{a\in A} U_a) \subset B_R(0)$.*

## 3. Proof of the Theorems

In this section we prove the theorems; the proof is divided into 2 parts.

Let $f$, $f_n$ be rational functions. We have the following lemma; see [1].

**Lemma 3.1.** *If $f_n \to f$ algebraically, then $J(f) \subset \liminf J(f_n)$.*

*Proof of Theorem 1.* By assumption, $f$ has no Siegel disc, Herman ring nor parabolic cycles. Since $f_n \to f$ algebraically , it follows by Lemma 3.1 that $J(f) \subset \liminf J(f_n)$. So to prove $J(f_n) \to J(f)$, we need only show $\limsup J(f_n) \subset J(f)$. This amounts to showing, for each $x \in F(f)$ (the Fatou set of $f$), there exists a neighborhood $U$ of $x$ such that $U \subset F(f_n)$, $\forall n \gg 0$. Since the Fatou set is totally invariant, we can replace $x$ with a finite iteration $f^i(x)$ at any stage of the argument.

For each $x \in F(f)$, under iterating $f^i(x)$ converge to an attracting (super-attracting) fixed-point $d$ of $f$.

Suppose $d$ is attracting (super-attracting). Then this behavior persists under algebraic perturbation of $f$. In fact there is a small neighborhood $U$ of $d$ such that $f_n(U) \subset U$, for all $n \gg 0$. Thus $U \subset F(f_n)$. Choosing $i$ such that $f^i(x) \in U$, from [1] there exists a neighborhood of $x$ persisting in the Fatou set for large $n$.

Therefore the original sequence satisfies

$$J(f_n) \to J(f).$$

The proof of Theorem 1 is complete. ∎

*Proof of Theorem 2.* Since $P_{c_0}$ is semi-hyperbolic, it follows by [2] that there exists exactly one conformal measure $\mu$ in $J_{c_0}$ which has exponent $d_{c_0} = HD(J_{c_0})$ or is atomic, supported in $\{P_{c_0}^{-n}(0)\}_{n\geq 0}$. ($\mu$ is not atomic if the measure $\mu$ of a point is zero). For $c_n \in M_d$, there is a unique conformal probability measure $\mu_{c_n}$ for $P_{c_n}$ supported in $J_{c_n}$ which has exponent $d_{c_n} = HD(J_{c_n})$ or is an atomic measure living on the inverse orbit of the critical point if 0 is in $J_{c_n}$; see [1, 6]. Thus to prove that

$$\lim_{n\to\infty} HD(J_{c_n}) = HD(J_{c_0})$$

it is enough to prove that

$$\lim_{r\to 0} \lim_{n\to\infty} \mu_{c_n}(V_r(0)) = 0.$$

In fact, if $\mu$ is any weak limit of $\{\mu_{c_n}\}_{n\geq 1}$, then $\mu$ is a conformal probability measure supported in $J_{c_0}$, by convergence of Julia sets. The previous limit implies that the measure $\mu$ is not atomic at 0, so it has exponent $d_{c_0}$ and it follows that $d_{c_n} \to d_{c_0}$.

Consider a Markov partition $U_i$ as in Sec. 2 and consider a holomorphic motion $j : V_\sigma(c_0) \times \bigcup U_i \to \mathbb{C}$ given by Proposition 2.3. Taking $\sigma > 0$ small if necessary we may assume that there are constants $B_0 > 0$ and $\theta_0 \in (0, 1)$ such that for all $m \geq 1$, all $c \in V_\sigma(c_0)$ and all $w \in j_c(\omega(0))$, we have $|(P_c^m)'(w)|^{-1} \leq B_0\theta_0^m$. We suppose that there is a uniform bounded distortion property: *There is a constant $K > 1$ so that for every $c \in V_\sigma(c_0)$, every $k \geq 1$ and every kth step piece $W$ of the Markov partition $j_c(U_i)$, the distortion of $P_c^k$ in $W$ is bounded by $K$.*

Recall that $U_n$ is the *nth* step piece containing $P_{c_0}^m(0) \in \omega(0)$ and $Y_n$ is the pull-back of $U_n$ by $P_{c_0}^m$ containing 0. It follows that for $r > 0$ small there is $n = n(r) \to \infty$, as $r \to 0$ so that $V_r(0) \subset Y_n^c$ for all $c$ sufficiently close $c_0$. So we only need to prove that

$$\lim_{n\to\infty} \lim_{s\to\infty} \mu_{c_s}(Y_n^{c_s}) = 0.$$

Let $D$ be a disc containing 0, small enough so that for $c \in V_\sigma(c_0)$, $P_c^m|_D$ is at most of degree $d$. Refining the Markov partition if necessary, suppose that $U_1^c \subset P_c^m(D)$ for all $c \in V_\sigma(c_0)$.

For all $n \geq 1$, we have

$$\mu_c(Y_n^c) = \sum_{l \geq n} \mu_c(Y_l^c - Y_{l+1}^c).$$

From the Appendix 2 of [2], we have that $z(c) = j_c(c_0)$ is the dynamical continuation of the critical value $c_0$ and $z'(c_0) \neq 1$. The function $z$ is defined in $V_\sigma(c_0)$, it satisfies $P_c^{m-1}(z(c)) = j_c(P_{c_0}^{m-1}(c_0))$. For $c \in B_\sigma(c_0)$ let $\xi(c) = j_c(P_{c_0}^{m-1}(c_0)) = P_c^{m-1}(z(c))$ and put $\beta_c = P_c^m(0)$. For $l \geq 1$ we have

$$\mu_c(Y_l^c - Y_{l+1}^c) \leq d\mu_c(U_l^c - U_{l+1}^c) \inf_{(Y_l^c - Y_{l+1}^c) \cap J_c} |(P_c^m)'(z)|^{-d_c}.$$

By the uniform Bounded Distortion Property and considering that $\mu_c$ is a probability measure, we have

$$\mu_c(U_l^c - U_{l+1}^c) \leq K^{d_c} |(P_c^l)'(\xi(c))|^{-d_c}.$$

On the other hand there is $B_1 > 0$ such that for all $c \in V_\sigma(c_0)$ and $z \in Y_1^c$,

$$|(P_c^m)'(z)| > B_1 |P_c^m(z) - \beta_c|^{(d-1)/d}. \tag{*}$$

Let $k = k(c)$ be the greatest integer such that $\beta_c \in U_k^c$. Let $l \geq 1$. Then there are the following cases:
(1): $k - 1 \leq l \leq k + 1$. By the uniform Bounded Distortion Property and Appendix 2 of [2], we have

$$|(P_c^l)'(\xi(c))|^{-1} \sim |\xi(c) - \beta_c| \sim |z(c) - c| \sim |c - c_0|,$$

with implicit constants independent of $c \in V_\sigma(c_0)$. Hence $|(P_c^l)'(\xi(c))|^{-1} \leq B_2|c - c_0|$ for some $B_2 > 0$ independent of $c$. It follows by [2] that $\partial M_d$ and $J_{c_0}$ are similar near $c_0$; this implies that the local structure of $\partial M_d$ and $J_{c_0}$ are similar near $c_0$. On the other hand

$$\text{dist}(\beta_c, (U_l^c - U_{l+1}^c) \cap J_c) \geq \text{ dist}(\beta_c, J_c) \geq B_3 \text{ dist}(c, J_c) \geq B_3' \text{ dist}(c, J_{c_0})$$

$\geq B_4 \text{ dist}(c, \partial M_d)$.

So for all $z \in (Y_l^c - Y_{l+1}^c) \cap J_c$

$$|(P_c^m)'(z)| > B_1 B_4^{(d-1)/d} (\text{dist}(c, \partial M_d))^{(d-1)/d},$$

thus

$$\mu_c(Y_l^c - Y_{l+1}^c) \leq B_5 |c - c_0|^{d_c} (\text{dist}(c, \partial M_d))^{-d_c(d-1)/d},$$

where $B_5 = d(KB_2(B_1 B_4^{(d-1)/d})^{-1})^{d_c}$.
(2): $l < k - 1$. Note that

$$\text{dist}(\beta_c, U_l^c - U_{l+1}^c) \geq \text{ dist}(\partial U_{l+1}^c, U_{l+2}^c),$$

thus by the uniform Bounded Property,

$$\text{dist}(\beta_c, U_l^c - U_{l+1}^c) > B_6 |(P_c^l)'(\xi(c))|^{-1}.$$

Hence by above $(*)$ we have

$$|(P_c^m)'(z)| > B_1(\mathrm{dist}(\beta_c, U_l^c - U_{l+1}^c))^{(d-1)/d} \geq B_1 B_6^{(d-1)/d} |(P_c^l)'(\xi(c))|^{-(d-1)/d}.$$

Therefore

$$\mu_c(Y_l^c - Y_{l+1}^c) \leq dK^{d_c} |(P_c^l)'(\xi(c))|^{-d_c} (B_1 B_6^{(d-1)/d})^{-d_c} |(P_c^l)'(\xi(c))|^{d_c(d-1)/d}.$$

Thus

$$\mu_c(Y_l^c - Y_{l+1}^c) \leq B_7 \theta_0^{ld_c/d},$$

where $B_7 = dK^{d_c}(B_1 B_6^{(d-1)/d})^{-d_c} B_0^{d_c/d}$.
(3): $l > k+1$. We have $\mathrm{dist}(\beta_c, U_l^c - U_{l+1}^c) \geq B_8\ \mathrm{dist}(\partial U_{l-1}^c, U_l^c)$. Thus reducing $B_6 > 0$ if necessary, as in case (2), we have

$$\mathrm{dist}(\beta_c, U_l^c - U_{l+1}^c) > B_6 |(P_c^l)'(\xi(c))|^{-1},$$

and $\mu_c(Y_l^c - Y_{l+1}^c) \leq B_7 \theta_0^{ld_c/d}$.

So we have

$$\begin{aligned} \mu_c(Y_n^c) = \sum_{l \geq n} \mu_c(Y_l^c - Y_{l+1}^c) &\leq B_5 |c - c_0|^{d_c} (\mathrm{dist}(c, \partial M_d))^{-d_c(d-1)/d} \\ &+ B_7 \sum_{l \geq n, l \neq k-1, k, k+1} \theta_0^{ld_c/d}. \end{aligned}$$

By our assumption, there are $B_1 > 0$ and a sequence $c_n \to c_0$ from the interior of $M_d$ such that $\mathrm{dist}(c_n, \partial M_d) \geq B_1 |c_n - c_0|^{1+1/d}$, thus,

$$\begin{aligned} B_1(\mathrm{dist}(c_n, \partial M_d))^{-1} &\leq |c_n - c_0|^{(-1-\frac{1}{d})}, \\ B_5 |c - c_0|^{d_c} (\mathrm{dist}(c, \partial M_d))^{-d_c(d-1)/d} &\leq B_1 B_5 |c - c_0|^{d_c} |c - c_0|^{(-1-\frac{1}{d})\frac{d_c(d-1)}{d}} \\ &\leq B_1 B_5 |c - c_0|^{\frac{d_c}{d^2}}. \end{aligned}$$

Since

$$\sum_{l \geq n} \theta_0^{ld_c/d} = \frac{(\theta_0^{d_c/d})^n}{1 - \theta_0^{d_c/d}} \to 0 \quad \text{as} \quad n \to \infty,$$

we conclude that

$$\lim_{n \to \infty} \lim_{s \to \infty} \mu_{c_s}(Y_n^{c_s}) = 0,$$

the proof of theorem 2 is complete. ∎

*Open Question.* I do not know any example satisfying the hypothesis of Theorem 2.

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