Vietnam Journal of Mathematics **33**:3 (2005) 291–308

Vietnam Journal
of
MATHEMATICS

The Existence of Solutions to Generalized Bilevel Vector Optimization Problems

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Received April 29, 2004
Revised October 6, 2005

Abstract. Generalized bilevel vector optimization problems are formulated and some sufficient conditions on the existence of solutions for generalized bilevel weakly, Pareto and ideal problems are shown. As special case, we obtain results on the existence of solutions to generalized bilevel programming problems given by Lignola and Morgan. These also include a large number of results concerning variational and quasi-variational inequalities, equilibrium and quasi-equilibrium problems.

1. Introduction

Let D be a subset of a topological vector space X and R be the space of real numbers. Given a real function f from D into R, the problem of finding $\bar{x} \in D$ such that

$$f(\bar{x}) = \min_{x \in D} f(x)$$

plays a central role in the optimization theory. There is a number of books on optimization theory for linear, convex, Lipschitz and, in general, continuous problems. Today this problem is also formulated for vector multi-valued mappings. One developed the optimization theory concerning multi-valued mappings

*The author was partially supported by the Fritz-Thyssen Foundation from Germany for the three months stay at the Institute of Mathematics of the Humboldt University in Berlin and the Institute of Mathematics of the Cologne University.

with the methodology and the applications similar to the ones with scalar functions. Given a cone C in a topological vector space Y and a subset $A \subset Y$, one can define efficient points of A with respect to C by different senses as: Ideal, Pareto, Properly, Weakly, ... (see Definition 2 below). The set of these efficient points is denoted by $\alpha \operatorname{Min}(A/C)$ for the case of ideal, Pareto, properly, weakly efficient points, respectively. By 2^Y we denote the family of all subsets in Y. For a given multi-valued mapping $F : D \to 2^Y$, we consider the problem of finding $\bar{x} \in D$ such that

$$F(\bar{x}) \cap \alpha \operatorname{Min}(F(D)/C) \neq \emptyset. \qquad (GVOP)_\alpha$$

This is called a general vector α optimization problem corresponding to D and F. The set of such points $\bar{x}$ is denoted by $\alpha\mathcal{S}(D, F; C)$ and is called the solution set of $(GVOP)_\alpha$. The elements of $\alpha \operatorname{Min}(F(D)/C)$ are called optimal values of $(GVOP)_\alpha$. These problems have been studied by many authors, for examples, Corley [6], Luc [14], Benson [1], Jahn [11], Sterna-Karwat [21],...

Now, let X, Y and Z be topological vector spaces, $D \subset X, K \subset Z$ be nonempty subsets and $C \subset Y$ be a cone. Given the following multi-valued mappings

$$S : D \to 2^D,$$
$$T : D \to 2^K,$$
$$F : D \times K \times D \to 2^Y,$$

we are interested in the problem of finding $\bar{x} \in D, \bar{z} \in K$ such that

$$\bar{x} \in S(\bar{x}),$$
$$\bar{z} \in T(\bar{x}) \qquad (GVQOP)_\alpha$$

and

$$F(\bar{x}, \bar{z}, \bar{x}) \cap \alpha \operatorname{Min}(F(\bar{x}, \bar{z}, S(\bar{x})) \neq \emptyset.$$

This is called a general vector α quasi-optimization problem (α is one of the words: "ideal", "Pareto", "properly", "weakly", ..., respectively). Such a couple $(\bar{x}, \bar{z})$ is said to be the solution of $(GVQOP)_\alpha$. The set of such solutions is said to be the solution set of $(GVQOP)_\alpha$ and denoted by $\alpha\mathcal{S}(D, K, S, T, F, C)$. The above multi-valued mappings S, T and F are called a constraint, potential and utility mapping, respectively.

These problems contain as special cases, for example, quasi-equilibrium problems, quasi-variational inequalities, fixed point problems, complementarity problems, as well as different others that have been considered by many mathematicians as: Park [20], Chan and Pang [5], Parida and Sen [19], Fu [9] for quasi-equilibrium problems, by Blum and Oettli [3], Minh and Tan [16], Browder and Minty [17], Ky-Fan [7],..., for equilibrium and variational inequality problems and by some others for vector optimization problems.

Let Y_0 be another topological vector space with a cone C_0 and $f : D \times K \to 2^{Y_0}$, we are interested in the problem of finding $(x^*, z^*) \in \alpha\mathcal{S}(D, K, S, T, F, C)$ such that

$$f(x^*, z^*) \cap \gamma \operatorname{Min} f(\alpha\mathcal{S}(D, K, S, T, F, C))/C) \neq \emptyset. \qquad (1)_{(\alpha,\gamma)}$$

This is called an (α, γ) bilevel vector optimization problem. Such a couple (x^*, z^*) is said to be a solution of $(1)_{(\alpha,\gamma)}$. The set of such solutions is said to be the solution set of $(1)_{(\alpha,\gamma)}$ and denoted by $\alpha\mathcal{S}_2(D, K, S, T, F, f, C)$. These problems ($\alpha, \gamma$ is one of the words: "ideal", "Pareto", "properly","weakly", ..., respectively) contain, as a special case, the generalized bilevel problem given in [12] and some others in the literature therein.

2. Preliminaries and Definitions

Throughout this paper, as in the introduction, by X, Y, Z and Y_0 we denote real locally convex topological vector spaces. Given a subset $D \subset X$, we consider a multi-valued mapping $F : D \to 2^Y$. The definition domain and the graph of F are denoted by

$$\operatorname{dom} F = \{x \in D / F(x) \neq \emptyset\}$$
$$Gr(F) = \{(x, y) \in D \times Y / y \in F(x)\},$$

respectively. We recall that F is said to be a closed mapping if the graph $Gr(F)$ containing F is a closed subset in the product space $X \times Y$ and it is said to be a compact mapping if the closure $\overline{F(D)}$ of its range $F(D)$ is compact in Y. A nonempty topological space is said to be acyclic if all its reduced Čech homology group over rational vanish. Note that any convex, star-shaped, contractible set (see, for example, Definition 3.1, Chapter 6 in [14]) of a topological vector space is acyclic. The following definitions can be found in [2]. A multi-valued mapping $F : D \to 2^Y$ is said to be upper semi-continuous (u.s.c) at $\bar{x} \in D$ if for each open set V containing $F(\bar{x})$, there exists an open set U containing $\bar{x}$ such that for each $x \in U$, $F(x) \subset V$. F is said to be u.s.c on D if it is u.s.c at all $x \in D$. And, F is said to be lower semi-continuous (l.s.c) at $\bar{x} \in D$ if for any open set V with $F(\bar{x}) \cap V \neq \emptyset$, there exists an open set U containing $\bar{x}$ such that for each $x \in U$, $F(x) \cap V \neq \emptyset$; F is said to be l.s.c on D if it is l.s.c at all $x \in D$. F is said to be continuous on D if it is at the same time u.s.c and l.s.c on D. F is said to be acyclic if it is u.s.c with compact acyclic values. And, F is said to be a compact acyclic mapping if it is a compact mapping and an acyclic mapping simultaneously.

We also recall that a nonempty subset D of a topological space X is said to be admissible if for every compact subset Q of D and every neighborhood V of the origin in X, there is a continuous mapping $h : Q \to D$ such that $x - h(x) \in V$ for all $x \in Q$ and $h(Q)$ is contained in a finite dimensional subspace L of X.

Further, let Y be a topological vector space with a cone C. We denote $l(C) = C \cap (-C)$. If $l(C) = 0$ we say that C is a pointed cone. We recall the following definitions (see, for example, Definition 2.1, Chapter 2 in [14]).

Definition 1. *Let A be a nonempty subset of Y. We say that:*

1. *$x \in A$ is an ideal efficient (or ideal minimal) point of A with respect to C if $y - x \in C$ for every $y \in A$.*
   The set of ideal minimal points of A is denoted by $I\ Min\ (A/C)$.
2. *$x \in A$ is an efficient (or Pareto–minimal, or nondominant) point of A w.r.t. C if there is no $y \in A$ with $x - y \in C \setminus l(C)$.*
   The set of efficient points of A is denoted by $P\ Min\,(A/C)$.
3. *$x \in A$ is a (global) properly efficient point of A w.r.t. C if there exists a convex cone $\tilde{C}$ which is not the whole space and contains $C \setminus l(C)$ in its interior so that $x \in P\ Min\ \left(A/\tilde{C}\right)$.*
   The set of properly efficient points of A is denoted by $Pr\ Min\ (A/C)$.
4. *Supposing that $int\ C$ is nonempty, $x \in A$ is a weakly efficient point of A w.r.t. C if $x \in P\ Min\ (A/\{0\} \cup\ int\,C)$.*
   The set of weakly efficient points of A is denoted by $W\ Min\,(A/C)$.

We use $\alpha\,\mathrm{Min}\,(A/C)$ to denote one of $I\,\mathrm{Min}\,(A/C), P\,\mathrm{Min}\,(A/C), \ldots$. The notions of $I\,\mathrm{Max}\,(A/C)$, $P\,\mathrm{Max}\,(A/C)$, $Pr\,\mathrm{Max}\,(A/C)$, $W\,\mathrm{Max}\,(A/C)$ are defined dually.

We have the following inclusions:

$$I\,\mathrm{Min}\,,(A/C) \subset Pr\,\mathrm{Min}\,(A/C) \subset P\,\mathrm{Min}\,(A/C) \subset W\,\mathrm{Min}\,(A/C).$$

Moreover, if $I\,\mathrm{Min}\,(A/C) \neq \emptyset$, then $I\,\mathrm{Min}\,(A/C) = P\,\mathrm{Min}\,(A/C)$ and it is a point whenever C is pointed (see Proposition 2.2, Chapter 2 in [14]).

Now, we introduce new definitions of the C-continuities of a multi-valued mapping $F : D \to 2^Y$.

Definition 2.

1. *F is said to be upper (lower) C–continuous at $\bar{x} \in\ dom\,F$ if for any neighborhood V of the origin in Y there is a neighborhood U of $\bar{x}$ such that:*

$$F(x) \subset F(\bar{x}) + V + C$$
$$\big(F(\bar{x}) \subset F(x) + V - C, \quad respectively\big)$$

   holds for all $x \in U \cap\ dom\,F$.
2. *If F is upper C–continuous and lower C–continuous at $\bar{x}$ simultaneously, we say that it is C–continuous at $\bar{x}$.*
3. *If F is upper, lower, ..., C–continuous at any point of $dom\,F$, we say that it is upper, lower,... continuous.*

In the sequel if $C = \{0\}$ we shall say that F is upper, lower, ..., continuous instead of upper, lower, ..., $\{0\}$–continuous.

Remark 1.

a) If $C = \{0\}$ and $F(\bar{x})$ is compact, then it is easy to see that the above definitions of continuities coincide with the ones given by Berge [2].
b) If F is upper continuous with $F(x)$ closed for any $x \in D$, then F is closed.

c) If F is compact and $F(x)$ closed for each $x \in D$, then F is upper continuous if and only if F is closed.
d) If $F(\bar{x})$ is compact, the the above definitions coincide with the ones in [14] (Definition 7.1, Chapter 1).

In the sequel, we give some necessary and sufficient conditions on the upper and the lower $C-$ continuities .

Proposition 1. *Let $F : D \to 2^Y$ and $C \subset Y$ be a closed cone.*
1) *If F is upper C–continuous at $x_0 \in \operatorname{dom} F$ with $F(x_0) + C$ closed, then for any net $x_\beta \to x_0, y_\beta \in F(x_\beta) + C, y_\beta \to y_0$ imply $y_0 \in F(x_0) + C$.*
Conversely, if F is compact and for any net $x_\beta \to x_0, y_\beta \in F(x_\beta)+C, y_\beta \to y_0$ imply $y_0 \in F(x_0) + C$, then F is upper C–continuous at x_0.
2) *If F is compact and lower C–continuous at $x_0 \in \operatorname{dom} F$, then for any net $x_\beta \to x_0, y_0 \in F(x_0) + C$, there is a net $\{y_\beta\}, y_\beta \in F(x_\beta)$, which has a convergent subnet $\{y_{\beta_\gamma}\}, y_{\beta_\gamma} - y_0 \to c \in C$(i.e. $y_{\beta_\gamma} \to y_0 + c \in y_0 + C$).*
Conversely, if $F(x_0)$ is compact and for any net $x_\beta \to x_0, y_0 \in F(x_0) + C$, there is a net $\{y_\beta\}, y_\beta \in F(x_\beta)$, which has a convergent subnet $\{y_{\beta_\gamma}\}, y_{\beta_\gamma} - y_0 \to c \in C$, then F is lower C–continuous at x_0.

Proof.
1) Assume first that F is upper C–continuous at $x_0 \in \operatorname{dom} F$ and $x_\beta \to x_0, y_\beta \in F(x_\beta) + C, y_\beta \to y_0$. We suppose on the contrary that $y_0 \notin F(x_0) + C$. We can find a convex and closed neighborhood V_0 of the origin in Y such that

$$(y_0 + V_0) \cap (F(x_0) + C) = \emptyset,$$

or,

$$(y_0 + V_0/2) \cap (F(x_0) + V_0/2 + C) = \emptyset.$$

Since $y_\beta \to y_0$, one can find $\beta_1 \geq 0$ such that $y_\beta - y_0 \in V_0/2 \quad$ for all $\beta \geq \beta_1$. Therefore, $y_\beta \in y_0 + V_0/2$ and F is upper C–continuous at x_0, it follows that one can find a neighborhood U of x_0 such that

$$F(x) \subset (F(x_0) + V_0/2 + C) \quad \text{for all} \quad x \in U \cap \operatorname{dom} F.$$

Since $x_\beta \to x_0$, one can find $\beta_2 \geq 0$ such that $x_\beta \in U$ and

$$y_\beta \in F(x_\beta) + C \subset (F(x_0) + V_0/2 + C) \quad \text{for all} \quad x \in U \cap \operatorname{dom} F.$$

This implies that

$$y_\beta \in (y_0 + V_0/2) \cap (F(x_0) + V_0/2 + C) = \emptyset \quad \text{for all} \quad \beta \geq \max\{\beta_1, \beta_2\}.$$

and we have a contradiction. Thus, we conclude $y_0 \in F(x_0) + C$. Now, assume that F is compact and for any net $x_\beta \to x_0, y_\beta \in F(x_\beta) + C, y_\beta \to y_0$ imply $y_0 \in F(x_0) + C$. On the contrary, we assume that F is not upper C–continuous at x_0. This implies that there is a neighborhood V of the origin in Y such that for any neighborhood U_β of x_0 one can find $x_\beta \in U_\beta$ such that

$$F(x_\beta) \not\subset (F(x_0) + V + C).$$

We can choose $y_\beta \in F(x_\beta)$ with $y_\beta \notin (F(x_0)+V+C)$. Since $\overline{F(D)}$ is compact, we can assume, without loss of generality, that $y_\beta \to y_0$, and hence $y_0 \in F(x_0)+C$. On the other hand, since $y_\beta \to y_0$, there is $\beta_0 \geq 0$ such that $y_\beta - y_0 \in V$ for all $\beta \geq \beta_0$. Consequently,

$$y_\beta \in y_0 + V \subset (F(x_0) + V + C), \quad \text{for all} \quad \beta \geq \beta_0$$

and we have a contradiction.

2) Assume that F is compact and lower C–continuous at $x_0 \in \operatorname{dom} F$, and $x_\beta \to x_0, y_0 \in F(x_0)$. For any neighborhood V of the origin in Y there is a neighborhood U of x_0 such that

$$F(x_0) \subset (F(x) + V - C), \quad \text{for all} \quad x \in U \cap \operatorname{dom} F.$$

Since $x_\beta \to x_0$, there is $\beta_0 \geq 0$ such that $x_\beta \in U$ and then

$$F(x_0) \subset (F(x_\beta) + V - C), \quad \text{for all} \quad \beta \geq \beta_0.$$

For $y_0 \in F(x_0)$, we can write

$$y_0 = y_\beta + v_\beta - c_\beta \quad \text{with} \quad y_\beta \in F(x_\beta) \subset \overline{F(D)}, v_\beta \in V, c_\beta \in C.$$

Since $\overline{F(D)}$ is compact, we can choose $y_{\beta_\gamma} \to y^*, v_{\beta_\gamma} \to 0$. Therefore, $c_{\beta_\gamma} = y_{\beta_\gamma} + v_{\beta_\gamma} - y_0 \to y^* - y_0 \in C$, or $y_{\beta_\gamma} \to y^* \in y_0 + C$. Thus, for any $x_\beta \to x_0, y_0 \in F(x_0)$, one can find $y_{\beta_\gamma} \in F(x_{\beta_\gamma})$ with $y_{\beta_\gamma} \to y^* \in y_0 + C$.

Now, we assume that $\overline{F(x_0)}$ is compact and for any net $x_\beta \to x_0, y_0 \in F(x_0) + C$, there is a net $\{y_\beta\}, y_\beta \in F(x_\beta)$ which has a convergent subnet $y_{\beta_\gamma} - y_0 \to c \in C$. On the contrary, we suppose that F is not lower C–continuous at x_0. This implies that there is a neighborhood V of the origin in Y such that for any neighborhood U_β of x_0 one can find $x_\beta \in U_\beta$ such that

$$F(x_0) \not\subset (F(x_\beta) + V - C).$$

We can choose $z_\beta \in F(x_0)$ with $z_\beta \notin (F(x_\beta) + V - C)$. Since $F(x_0)$ is compact, we can assume, without loss of generality, that $z_\beta \to z_0 \in F(x_0)$, and hence $z_0 \in F(x_0) + C$. We may assume that $x_\beta \to x_0$. Therefore, there is a net $\{y_\beta\}, y_\beta \in F(x_\beta)$ which has a convergent subnet $\{y_{\beta_\gamma}\}, y_{\beta_\gamma} - z_0 \to c \in C$. Without loss of generality, we suppose $y_\beta \to y^* \in z_0 + C$. This implies that there is $\beta_1 \geq 0$ such that $z_\beta \in z_0 + V/2, y_\beta \in y^* + V/2$ and $z_0 \in y_\beta + V/2 - C$ for all $\beta \geq \beta_1$.

Consequently,

$$z_\beta \in y_\beta + V/2 + V/2 - C \subset F(x_\beta) + V - C, \quad \text{for all} \quad \beta \geq \beta_1$$

and we have a contradiction. ∎

Definition 3. *A multi-valued mapping $F : D \to 2^Y$ is said to be subcontinuous on D if for any net $\{x_\alpha\}$ converging in D, every net $\{y_\alpha\}$ such that $y_\alpha \in F(x_\alpha)$ has a convergent subnet.*

We recall the following definitions.

Definition 4. *Let F be a multi-valued mapping from D to 2^Y. We say that:*

1. *F is upper (lower) C–convex on D if for any $x_1,\ x_2 \in D,\ t \in [0,1]$,*

$$tF(x_1) + (1-t)F(x_2) \subset F(tx_1 + (1-t)x_2) + C$$
$$\big(F(tx_1 + (1-t)x_2) \subset tF(x_1) + (1-t)F(x_2) - C, \quad \textit{respectively}\big)$$

   holds.
   If F is both upper C–convex and lower C–convex, we say that F is C–convex.

2. (i) *F is upper C-quasi-convex on D if for any $x_1,\ x_2 \in D,\ t \in [0,1]$, either*

$$F(x_1) \subset F(tx_1 + (1-t)x_2) + C$$
$$\textit{or,} \quad F(x_2) \subset F(tx_1 + (1-t)x_2) + C,$$

   holds.
   (ii) *F is lower C-quasi-convex on D if for any $x_1, x_2 \in D, t \in [0,1]$, either*

$$F(tx_1 + (1-t)x_2) \subset F(x_1) - C$$
$$\textit{or,} \quad F(tx_1 + (1-t)x_2) \subset F(x_2) - C,$$

   holds.
   If F is both upper C-quasi-convex and lower C-quasi-convex, we say that F is C-quasi-convex.

3. *Let F be a single-valued mapping. F is said to be strictly C–quasi-convex on D, when $\mathit{int}\, C \neq \emptyset$, if for $y \in Y, x_1, x_2 \in D, x_1 \neq x_2, t \in (0,1)$ and $F(x_i) \in y - C, i = 1,2$, implies $F(tx_1 + (1-t)x_2) \in y - \mathit{int}\, C$.*

Remark 2. It is clear that for $Y = R$(the space of real numbers),$C = R_+, F : X \to R$ is (strictly) R_+-convex if and only if it is convex (strictly convex, respectively) in the usual sense and any convex(strictly convex) function is quasi-convex(strictly quasi-convex). But, in general, a mapping may be upper (lower) C–convex and not upper (lower)C–quasi-convex, and conversely (see, for instance, Ferro [8]).

For a cone C, we define:

$$C' = \{\xi \in Y' : \xi(x) \geq 0, \quad \text{for all} \quad x \in C\}.$$

$C^{'}$ is said to be a polar cone of C.

3. The Main Results

Let X, Y, Y_0 and Z be locally convex topological vector spaces, $D \subset X$, $K \subset Z$ be nonempty subsets, $C \subset Y, C_0 \subset Y_0$ be closed cones. Let multi-valued mappings S, T, F and f be as in Introduction. First of all, we prove the following theorem.

Theorem 1. *Let $G : D \to 2^{Y_0}$ be an upper C_0-continuous multi-valued mapping with nonempty compact values on $D \times K$. Then for any nonempty compact subset A of $D \times K$ there is $x^* \in A$ such that*

$$G(x^*) \cap P\, Min\, G(A/C) \neq \emptyset.$$

Proof. Since A is a nonempty compact set and f is an upper C_0-continuous multi-valued mapping with $f(x)$ nonempty compact, then $G(A)$ is also C_0-compact in Y_0 (see Theorem 7.2, Chapter 1, in [14]) and hence C_0-complete (see Lemma 3.5, Chapter 1, in [14]). Since $G(A)$ is C_0-compact, then for any $z \in Y_0$ the set $G(A) \cap (z - C_0)$ is also C_0-compact and so C_0-complete. Applying Theorem 3.3, Chapter 2 in [14], we conclude $P\,\mathrm{Min}\,(G(A)/C_0) \neq \emptyset$. This means that there is $x^* \in A$ such that

$$G(x^*) \cap P\,\mathrm{Min}\; G(A/C) \neq \emptyset.$$

■

We assume that the pairing $\langle ., . \rangle$ between elements of Y and its dual $Y^{'}$ is a continuous function from the product topology of the topology in Y and the weak* topology in Y'. The cone C is supposed to be nonempty, convex and closed and its polar cone have weakly* base B. The following Theorem 2 and Corollaries 1,2 are proved in [22]

Theorem 2. *Let D and K be nonempty convex and closed subsets of locally convex Hausdorff topological vector spaces X and Z, respectively. Let $C \subset Y$ be a closed convex cone and $C^{'}$ have a weak* compact base B. Let $S : D \to 2^D$ be a compact continuous mapping with $S(x) \neq \emptyset$, closed and convex for each $x \in D$, $T : D \to 2^K$ be a compact acyclic mapping with $T(x) \neq \emptyset$ for all $x \in D$, $F : D \times K \times D \to 2^Y$ be an upper C–continuous and lower $(-C)$–continuous mapping with $F(x, y, z)$ nonempty and compact convex for any $(x, y, z) \in D \times K \times D$. In addition, assume that for each $(x, y) \in D \times K$ the multi-valued mapping $F(x, y, \cdot) : D \to 2^Y$ is upper C–quasi-convex. Then there is $(\bar{x}, \bar{y}) \in D \times K$ such that:*

$$\bar{x} \in S(\bar{x}), \bar{y} \in T(\bar{x})$$

and

$$F(\bar{x}, \bar{y}, x) \subset F(\bar{x}, \bar{y}, \bar{x}) + C, \quad \text{for all} \quad x \in S(\bar{x}). \tag{1}$$

Corollary 1. *Let D, K, C, S, T and F be as in Theorem 2. In addition, assume that $F(x, y, x) \subset C$ for all $(x, y) \in D \times K$.Then there is $(\bar{x}, \bar{y}) \in D \times K$ such that:*

$$\bar{x} \in S(\bar{x}), \bar{y} \in T(\bar{x})$$

and

$$F(\bar{x}, \bar{y}, x) \subset C, \quad \text{for all} \quad x \in S(\bar{x}).$$

Corollary 2. *Let D, K, S, T and F be as in Theorem 1 and $I\,Min\,(F(x,y,x) \neq \emptyset$ for all $(x,y) \in D \times K$. Then $(\bar{x}, \bar{y})$ satisfies (1) if and only if it is a solution of $(GVQOP)_I$.*

Further, let O be a subset of D and f be a multi-valued mapping from D into 2^Y. We denote

$$\alpha\mathcal{S}(O, f; C) = \{x \in O / f(x) \cap \alpha\,\mathrm{Min}\,(f(O)/C) \neq \emptyset\}.$$

We have

Corollary 3. *Let O be a nonempty convex compact subset of D and $f : D \to 2^Y$ be an upper C-quasi-convex, upper C-continuous and lower $(-C)$-continuous multi-valued mapping with nonempty convex and compact values and $I\,Min\,(f(x)/C) \neq \emptyset$ for any $x \in O$. Then $I\,Min\,(f(O)/C)$ is a nonempty closed subset and $I\mathcal{S}(O, f; C)$ is a nonempty convex and compact subset.*

Proof. Let Z be an arbitrary topological vector space and $K \subset Z$ be a nonempty convex compact set. We define the multi-valued mappings $S : D \to 2^D, T : D \to 2^K$ and $F : D \times K \times D \to 2^Y$ by

$$S(x) = O,$$
$$T(x) = K \;\text{ for }\; x \in D,$$
$$F(x,y,z) = f(z) \quad \text{for} \quad (x,y,z) \in D \times K \times D.$$

Applying Theorem 2, we conclude

$$f(x) \subset f(\bar{x}) + C, \quad \text{for all} \quad x \in O. \tag{2}$$

For $v^* \in I\,\mathrm{Min}\ f(\bar{x})$,

$$f(\bar{x}) \subset v^* + C.$$

Together with (2), we have

$$f(x) \subset f(\bar{x}) + C \subset v^* + C, \quad \text{for all} \quad x \in O.$$

This shows that $v^* \in I\,\mathrm{Min}\,(f(O)/C)$.

Further, we verify that the set $I\,\mathrm{Min}\,(f(O)/C)$ is closed. Indeed, let $v_n \in I\,\mathrm{Min}\,(f(O)/C)$ and $v_n \to v$. Let V be an arbitrary neighborhood of the origin in Y. One can find n_0 such that $v_n \in v + V$, for $n \geq n_0$. On the other hand, we have

$$f(O) \subset v_n + C.$$

Therefore,

$$f(O) \subset v + V + C$$

and then

$$f(O) \subset v + C.$$

Consequently, $v \in I\,\mathrm{Min}\,(f(O)/C)$. Further, we claim that the set $I\mathcal{S}(O, f; C)$ is nonempty convex and compact. Since $I\,\mathrm{Min}\,(f(O) \neq \emptyset$, then $I\mathcal{S}(O, f; C) \neq \emptyset$.

Let $x_1, x_2 \in I\mathcal{S}(O, f; C)$ and $t \in [0, 1]$. We have

$$f(x_i) \cap I\,\mathrm{Min}\,(f(O)/C) \neq \emptyset, i = 1, 2.$$

Since f is upper C-quasi-convex, it follows either

$$f(x_1) \subset f(tx_1 + (1-t)x_2) + C, \tag{3}$$

or

$$f(x_2) \subset f(tx_1 + (1-t)x_2) + C. \tag{4}$$

If (3) holds, then we conclude

$$(f(tx_1 + (1-t)x_2) + C) \cap I\,\mathrm{Min}\,(f(O)/C) \neq \emptyset.$$

Take v from the left side, we obtain

$$f(x) \subset v + C, \quad \text{for all} \quad x \in O. \tag{5}$$

On the other hand, we can write $v = v_1 + c$, with $v_1 \in f(tx_1 + (1-t)x_2), c \in C$. Then, (5) gives

$$f(x) \subset v_1 + C, \quad \text{for all} \quad x \in O.$$

This implies $v_1 \in I\,\mathrm{Min}\,(f(O)/C)$ and hence

$$f(tx_1 + (1-t)x_2) \cap I\,\mathrm{Min}\,(f(O)/C) \neq \emptyset.$$

Therefore,$(tx_1 + (1-t)x_2) \in I\mathcal{S}(O, f; C)$. If (4) holds, we also obtain $(tx_1 + (1-t)x_2) \in I\mathcal{S}(O, f; C)$. Thus, the set $I\mathcal{S}(O, f; C)$ is convex. To complete the proof, it remains to show that this set is closed. Indeed, let $x_n \in I\mathcal{S}(O, f; C)$ and $x_n \to x^*$. We have

$$f(x_n) \cap I\,\mathrm{Min}\,(f(O)/C) \neq \emptyset.$$

The upper C-continuity of f implies that to any neighborhood V of the origin in Y one can find a neighborhood U of x^* and n_0 such that $x_n \in U$ and

$$f(x_n) \subset f(x^*) + V + C, \quad \text{for all} \quad n \geq n_0.$$

This implies

$$(f(x^*) + V + C) \cap I\,\mathrm{Min}\,(f(O)/C) \neq \emptyset.$$

Since V is arbitrary, $f(x^*)$ is compact, this yields

$$(f(x^*) + C) \cap I\,\mathrm{Min}\,(f(O)/C) \neq \emptyset,$$

and then

$$f(x^*) \cap I\,\mathrm{Min}\,(f(O)/C) \neq \emptyset.$$

Consequently, $x^* \in I\mathcal{S}(O, f; C)$ and so this set is closed. ■

Remark 3. It is obvious that if $F(x, y, x)$ is a point (instead of a set) for any $(x, y) \in D \times K$, then $I\,\mathrm{Min}\,(F(x, y, x)/C) = \{F(x, y, x)\} \neq \emptyset$.

Corollary 4. *Let D, K, C, S, T and F be as in Theorem 2. In addition, assume that there exists a convex cone $\tilde{C}$ which is not the whole space and contains $C \setminus \{0\}$ in its interior. Then the problem $(GVQOP)_{Pr}$ has a solution.*

Proof. Since C has the above mentioned property, then any compact set A in Y has $Pr\,\mathrm{Min}\,(A/C) \neq \emptyset$ (by using the cone $C^* = \{0\} \cup \text{ int } \tilde{C}$ one can verify $P\,\mathrm{Min}\,(A/C^*) \neq \emptyset$, see, for example, Corollary 3.15, Chapter 2 in [14]). We then apply Theorem 2 to obtain $(\bar{x}, \bar{y}) \in D \times K$ such that:

$$\bar{x} \in S(\bar{x}), \bar{y} \in T(\bar{x})$$

and

$$F(\bar{x}, \bar{y}, x) \subset F(\bar{x}, \bar{y}, \bar{x}) + C, \quad \text{for all} \quad x \in S(\bar{x}). \tag{6}$$

Due to $F(\bar{x}, \bar{y}, \bar{x})$ is a compact set, it follows that $Pr\,\mathrm{Min}\,(F(\bar{x}, \bar{y}, \bar{x})/C) \neq \emptyset$. Take $\bar{v} \in Pr\,\mathrm{Min}\,(F(\bar{x}, \bar{y}, \bar{x})/C)$, we show that $\bar{v} \in Pr\,\mathrm{Min}\,(F(\bar{x}, \bar{y}, S(\bar{x}))/C)$. By contrary, we suppose that $\bar{v} \notin Pr\,\mathrm{Min}\,(F(\bar{x}, \bar{y}, S(\bar{x}))/C)$. Then, there is $v^* \in F(\bar{x}, \bar{y}, S(\bar{x}))$ such that

$$\bar{v} - v^* \in C^* \setminus l(C^*). \tag{7}$$

Assume that $v^* \in F(\bar{x}, \bar{y}, x^*)$, for some $x^* \in S(\bar{x})$. It follows from (6) that there exists $v^o \in F(\bar{x}, \bar{y}, \bar{x})$ such that $v^* - v^o = c \in C$. If $c = 0$, then $v^* = v^o$ and then $\bar{v} - v^o \in C^* \setminus l(C^*)$. If $c \neq 0$, using (7), we conclude

$$\bar{v} - v^o = \bar{v} - v^* + v^* - v^o \in C^* \setminus l(C^*) + C \setminus \{0\} \subset C^* \setminus l(C^*).$$

So, in any case, we get $\bar{v} - v^o \in C^* \setminus l(C^*)$. Remarking $\bar{v} \in Pr\,\mathrm{Min}\,(F(\bar{x}, \bar{y}, \bar{x})/C)$ and $v^o \in F(\bar{x}, \bar{y}, \bar{x})$, we have a contradiction. Therefore,

$$F(\bar{x}, \bar{y}, \bar{x}) \cap Pr\,\mathrm{Min}\,(F(\bar{x}, \bar{y}, S(\bar{x}))/C) \neq \emptyset$$

and $(\bar{x}, \bar{y})$ is a solution of the problem $(GVQOP)_{Pr}$. ∎

Corollary 5. *Assume that there exists a convex cone $\tilde{C}$ which is not the whole space and contains $C \setminus \{0\}$ in its interior. Let O be a nonempty convex compact subset of D. Let $f : D \to 2^Y$ be an upper C-quasi-convex, upper C and lower $(-C)$-continuous multi-valued mapping with nonempty convex and compact values and $Pr\ Min\,(f(x)/C)$ is nonempty and closed for any $x \in O$. Then $Pr\ Min\,(f(O)/C)$ is a nonempty closed subset and $Pr\mathcal{S}(O, f; C)$ is a nonempty convex and compact subset.*

Proof. Let Z be an arbitrary topological vector space and $K \subset Z$ be a nonempty convex compact set. We define the multi-valued mappings $S : D \to 2^D, T : D \to 2^K$ and $F : D \times K \times D \to 2^Y$ as in the proof of Corollary 3. Applying Theorem 1, we conclude

$$f(x) \subset f(\bar{x}) + C, \quad \text{for all} \quad x \in O. \tag{8}$$

Since C has the above property, we deduce that $Pr\,\mathrm{Min}\,(f(\bar{x})/C) \neq \emptyset$. Take $v^* \in Pr\,\mathrm{Min}\,(f(\bar{x})/C)$, and proceed the proof exactly as the one in Corollary 4, we show that $v^* \in Pr\,\mathrm{Min}\,(f(O)/C)$. Therefore, this set is not empty.

Now, let $v_n \in Pr\,\mathrm{Min}\,(f(O)/C), v_n \to v^*$. For any n there is $x_n \in O$ such that

$$v_n \in f(x_n) \subset f(\bar{x}) + C \quad \text{for all} \quad n.$$

Therefore,

$$v_n = {v_n}^1 + c_n, \quad {v_n}^1 \in f(\bar{x}), c_n \in C.$$

If $c_n \neq 0$, then

$$v_n - {v_n}^1 \in C \setminus \{0\} \subset \text{ int } \tilde{C} \subset \tilde{C} \setminus l(\tilde{C}),$$

(if $\text{int } \tilde{C} \not\subset \tilde{C} \setminus l(\tilde{C})$, there is a point $a \in \text{int } \tilde{C} \cap l(\tilde{C})$. This implies that one can find a neighborhood U of 0 such that $U \subset \tilde{C} - a \subset \tilde{C} + \tilde{C} = \tilde{C}$ and so $0 \in \text{int } \tilde{C}$. It is impossible). We then have a contradiction. This implies $v_n = {v_n}^1$ for all n. Consequently, $v_n \in f(\bar{x})$ for all n. And, moreover, $v_n \in Pr\,\mathrm{Min}\,(f(\bar{x})/C)$. The closedness of $Pr\,\mathrm{Min}\,(f(\bar{x})/C)$ and $v_n \to v^*$ imply that $v^* \in Pr\,\mathrm{Min}\,(f(\bar{x})/C)$. Assume that $v^* \notin Pr\,\mathrm{Min}\,(f(O)/C)$. Then, there exists $v^1 \in f(O)$ such that

$$v^* - v^1 \in \tilde{C} \setminus l(\tilde{C}).$$

Since (8) holds, it follows that $v^1 \in f(\bar{x}) + C$ and so $v^1 = v^2 + c$ with $v^2 \in f(\bar{x})$. We have

$$v_n - v^2 = v_n - v^* + v^* - v^1 + v^1 - v^2 \in v_n - v^* + v^1 - v^2 + \tilde{C} \setminus l(\tilde{C}),$$

If $v^1 \neq v^2$, then $v^1 - v^2 \in C \setminus \{0\} \subset \text{ int } \tilde{C}$. Together with the fact $v_n \to v^*$, we conclude that $v_n - v^2 \in \tilde{C} \setminus l(\tilde{C})$ for n large enough. This contradicts $v_n \in Pr\,\mathrm{Min}\,(f(\bar{x})/C)$. If $v^1 = v^2$, then $v^2 \in f(\bar{x})$ and $v^* - v^2 \in \tilde{C} \setminus l(\tilde{C})$. This contradicts $v^* \in Pr\,\mathrm{Min}\,(f(\bar{x})/C)$. Thus, $Pr\,\mathrm{Min}\,(f(O)/C)$ is a closed subset. Further, by the same arguments as in the proof of Corollary 3, we can verify that $Pr\mathcal{S}(O, f; C)$ is a convex and closed subset. ∎

Let X, Y and Z be topological vector spaces, $D \subset X$, $K \subset Z$ be nonempty subsets, $C \subset Y$ be a closed cone and let be given multi-valued mappings S, T and F as in Introduction. We define the multi-valued mappings $N^\alpha : D \times K \to 2^Y, M^\alpha : D \times K \to 2^D$ by

$$N^\alpha(x, y) = \alpha\,\mathrm{Min}\,F(x, y, S(x)), (x, y) \in D \times K, \tag{9}$$

$$M^\alpha(x, y) = \{u \in S(x) \mid F(x, y, u) \cap \alpha\,\mathrm{Min}\,(F(x, y, S(x))) \neq \emptyset\}. \tag{10}$$

It is clear that if $S(x)$ is compact for any $x \in D$ and $F(x, y, .) : D \to 2^Y$, for any $(x, y) \in D \times K$, is an upper C–continuous multi-valued mapping with nonempty C–compact values, then $N^P(x, y), M^P(x, y)$ are nonempty (see, Theorem 7.2 in [14]). The closedness of M^α will play an important role in our main results. In the sequel, we give some sufficient conditions for the closedness of the mapping M^α.

Lemma 1. *Let C be a closed convex cone in Y and $F : D \times K \times D \to 2^Y$ be an upper C–continuous with $F(x, y, z)$ nonempty and compact for each $(x, y, z) \in$*

$D\times K\times K$. In addition, assume that the multi-valued mapping N^{α} defined in (9) is upper $(-C)$–continuous and $N^{\alpha}(x,y)\neq\emptyset$, compact for each $(x,y)\in D\times K$. Then the mapping M^{α} defined in (10) is closed.

Proof. Indeed, let $(x_\beta,y_\beta,u_\beta)\in GrM^{\alpha}$ and $x_\beta\to x,\ y_\beta\to y,\ u_\beta\to u$. Let V be an arbitrary neighborhood of the origin in Y. Without loss of generality, we may assume that V is balanced. Then there is $\bar\beta$ such that

$$F(x_\beta,y_\beta,u_\beta)\subset F(x,y,u)+V+C$$

and

$$N^{\alpha}(x_\beta,y_\beta)\subset N^{\alpha}(x,y)+V-C \quad \text{for all} \quad \beta\geq\bar\beta$$

These follow from the upper C–continuity of F and the upper $(-C)$–continuity of N^{α}. Therefore, we obtain

$$\emptyset\neq F(x_\beta,y_\beta,u_\beta)\cap N^{\alpha}(x_\beta,y_\beta)\subset(F(x,y,u)+V+C)\cap(N^{\alpha}(x,y)+V-C)\,,$$

and hence

$$F(x,y,u)\cap(N^{\alpha}(x,y)+2V-C)\neq\emptyset. \tag{11}$$

This holds for arbitrary V. Consequently, using the compactness of $N^{\alpha}(x,y)$ and the closedness of C we conclude

$$F(x,y,u)\cap(N^{\alpha}(x,y)-C)\neq\emptyset.$$

Let $a_0\in F(x,y,u)\cap(N^{\alpha}(x,y)-C)$. By contradiction, we assume that $a_0\notin N^{\alpha}(x,y)$. For example, α is "Pareto". Then there exists $b\in F(x,y,S(x))$ with

$$a_0-b\in C\setminus l(C).$$

On the other hand, since $a_0\in N^{\alpha}(x,y)-C$, we can write

$$a_0=a_1-c,\ \text{with}\ c\in C\ \text{and}\ a_1\in N^{\alpha}(x,y).$$

Setting it in (11), we obtain

$$a_1-c-b\in C\setminus l(C),$$

and so $a_1-b\in C\setminus l(C)$. This contradicts $a_1\in N^{\alpha}(x,y)$. Thus, we deduce $a_0\in F(x,y,u)\cap N^{\alpha}(x,y)$ and $u\in M^{\alpha}(x,y)$. For the other case of α, the proof is similar. ■

Lemma 2. *Let C be a closed convex cone. Let $S:D\to 2^D$ be a continuous multi-valued mapping with $S(x)$ nonempty and compact for any $x\in D$ and $F:D\times K\times D\to 2^Y$ be an upper C–continuous and lower $(-C)$–continuous multi-valued mapping with $F(x,y,z)$ nonempty and compact for each $(x,y,z)\in D\times K\times D$. Then the multi-valued mapping M^W defined as in (10) $(\alpha=W)$ is closed.*

Proof. It is clear that $M^W(x,y)\neq\emptyset$ for each $(x,y)\in D\times K$. Let $((x_\beta,y_\beta),u_\beta)\in GrM^W,\ x_\beta\to x,\ y_\beta\to y$ and $u_\beta\to u$. We have to show $((x,y),u)\in GrM^W$. Indeed, for an arbitrary neighborhood V of the origin in Y there is β_0 such that

$$F(x_\beta, y_\beta, u_\beta) \subset F(x, y, u) + V + C, \quad \text{for all} \quad \beta \geq \beta_0. \tag{12}$$

Since $(x_\beta, y_\beta, u_\beta) \in GrM^W$, then we can take

$$z_\beta \in F(x_\beta, y_\beta, u_\beta) \cap W\,\mathrm{Min}\,(F(x_\beta, y_\beta, S(x_\beta))).$$

Using (12) we write

$$z_\beta \in \bar{z}_\beta + V + C \quad \text{with} \quad \bar{z}_\beta \in F(x, y, u). \tag{13}$$

From the compactness of $F(x, y, u)$, we may assume $\bar{z}_\beta \to \bar{z} \in F(x, y, u)$. We claim $\bar{z} \in W\,\mathrm{Min}\,(F(x, y, S(x)))$. By contradiction, we assume $\bar{z} \notin W\,\mathrm{Min}\,(F(x, y, S(x)))$. Then, there is $\hat{z} \in F(x, y, S(x))$ with $\bar{z} - \hat{z} \in \text{ int } C$. Take a convex neighborhood U of the origin in Y such that

$$\bar{z} - \hat{z} + 3U \subset \text{ int } C. \tag{14}$$

Further, we have $\hat{z} \in F(x, y, S(x)), \hat{z} \in F(x, y, \bar{u})$ for some $\bar{u} \in S(x)$. Since S is continuous and $x_\beta \to x$, there is $\bar{u}_\beta \in S(x_\beta)$, with $\bar{u}_\beta \to \bar{u}$. It follows from the lower $(-C)$–continuity of F that there is $\beta_1 \geq \beta_0$ such that

$$F(x, y, \bar{u}) \subset F(x_\beta, y_\beta, \bar{u}_\beta) + U + C, \quad \text{for all} \quad \beta \geq \beta_1.$$

For $\hat{z} \in F(x, y, \bar{u})$, we have

$$\hat{z} \in \hat{z}_\beta + U + C, \quad \text{for some} \quad \hat{z}_\beta \in F(x_\beta, y_\beta, \bar{u}_\beta), \quad \beta \geq \beta_1. \tag{15}$$

It follows from 13), (14) and (15) that:

$$\begin{gathered} z_\beta - \hat{z}_\beta = \hat{z} - \hat{z}_\beta + \bar{z} - \hat{z} + \bar{z}_\beta - \bar{z} + z_\beta - \bar{z}_\beta \in \\ U + C + \bar{z} - \hat{z} + U + U + C \\ \subset \bar{z} - \hat{z} + 3U + C \subset \text{ int } C + C = \text{ int } C. \end{gathered}$$

Since

$$\hat{z}_\beta \in F(x_\beta, y_\beta, u_\beta) \subset F(x_\beta, y_\beta, S(x_\beta))$$

and

$$z_\beta - \hat{z}_\beta \in \text{ int } C,$$

it contradicts the fact $z_\beta \in W\,\mathrm{Min}\,(x, y, S(x))$. So, we conclude $\bar{z} \in F(x, y, u) \cap W\,\mathrm{Min}\,(x, y, S(x))$. This means $u \in M^W(x, y)$ and then M^W is closed. ■

Let D, K, S, T and F be as above. For the sake of simple notations we set

$$\alpha\mathcal{S} = \alpha\mathcal{S}(D, K, S, T, F, C) = \{(\bar{x}, \bar{y}) \in D \times K | (\bar{x}, \bar{y}) \text{ satisfies } 1), 2), 3)\}, \tag{16}$$

where,

$$\begin{aligned} &1) \quad \bar{x} \in S(\bar{x}), \\ &2) \quad \bar{z} \in T(\bar{x}), \end{aligned}$$

and

$$3) \quad F(\bar{x},\bar{z},\bar{x}) \cap \alpha\,\mathrm{Min}\,(F(\bar{x},\bar{z},S(\bar{x}))) \neq \emptyset.$$

Lemma 3. *Let D and K be nonempty closed sets and $F : D\times K\times D \to 2^Y$ be a compact upper C-continuous and lower $(-C)$-continuous multi-valued mapping with nonempty and C-compact values. Let $S : D \to 2^D$ be a compact continuous multi-valued mapping with nonempty closed values and $T : D \to 2^K$ be closed and sub-continuous multi-valued mapping with nonempty values. Then the set $W\mathcal{S}$ defined as in* (16) (*with* $\alpha = W$) *is compact.*

Proof. If $W\mathcal{S} = \emptyset$ then it is obvious. We assume that $W\mathcal{S} \neq \emptyset$. One can easily verify that

$$W\mathcal{S} = \{(x,y) \in D\times K | (x,y) \in M^W(x,y)\times T(x)\}.$$

Let $(x_\beta,y_\beta) \in W\mathcal{S}, x_\beta \in M^W(x_\beta,y_\beta), y_\beta \in T(x_\beta), (x_\beta,y_\beta) \to (x,y)$. Since M^W and T are closed, we conclude that $x \in M^W(x,y)$ and $y \in T(x)$. This shows that $(x,y) \in W\mathcal{S}$ and $W\mathcal{S}$ is a closed set. Now, we prove that any net $(x_\beta,y_\beta) \in W\mathcal{S}$ has a convergent subnet. Indeed, since $x_\beta \in M^W(x_\beta,y_\beta) \subset \overline{S(D)}$, a compact set, without loss of generality, we may assume that $x_\beta \to x$. We have $y_\beta \in T(x_\beta)$ and T is a sub-continuous multi-valued mapping. It follows that $\{y_\beta\}$ has a convergent subnet $\{y_{\beta_\tau}\}, y_{\beta_\tau} \to y$. For $y_{\beta_\tau} \in T(x_{\beta_\tau}), x_{\beta_\tau} \to x, y_{\beta\tau} \to y$ and M^W, T are closed multi-valued mappings, we deduce $(x,y) \in M^W(x,y)\times T(x)$ and then $(x,y) \in W\mathcal{S}$. This implies that $W\mathcal{S}$ is a compact set. ■

Theorem 3. *Let D and K be nonempty admissible convex subsets of topological vector spaces X and Z, respectively. Let $f : D\times K \to Y_0$ be an upper C_0-continuous multi-valued mapping with nonempty compact values. Let $S : D \to 2^D$ be a compact closed multi-valued mapping with $S(x) \neq \emptyset$, convex for each $x \in D$, $T : D \to 2^K$ be a compact acyclic multi-valued mapping with $T(x) \neq \emptyset$ for all $x \in D$, $F : D\times K\times D \to 2^Y$ be an upper C–continuous and lower $(-C)$–continuous multi-valued mapping with nonempty convex and compact values. In addition, assume that for each $(x,y) \in D\times K$ the multi-valued mapping $F(x,y,\cdot) : D \to 2^Y$ is upper C–quasi-convex. Then Problem $(1)_{(P,W)}$ has a solution, i.e. there is $(\bar{x},\bar{z}) \in D\times K$ such that*

$$f(\bar{x},\bar{z}) \cap P\,Min\,(f(W\mathcal{S}) \neq \emptyset$$

with $W\mathcal{S}$ defined as in (16).

Proof. By Lemma 3 $W\mathcal{S} = W\mathcal{S}(D,K,S,T,F;C)$ is a compact set. Therefore, applying Theorem 1 to complete the proof of this theorem, it remains to show that $W\mathcal{S}$ is not empty. Indeed, by Lemma 2, the multi-valued mapping M^W defined as in (10) with $\alpha = W$ is closed. $M^W(x,y)$ is nonempty for all $(x,y) \in D\times K$ because of $N^W(x,y)$ nonempty. Since $M^W(D\times K) \subset \overline{S(D)}$, it follows that M^W is a compact multi-valued mapping. Applying Proposition 1, we conclude that M^W is u.s.c with nonempty compact values. Let $u_1,u_2 \in M^W(x,y)$ and $t \in [0,1]$. Since $S(x)$ is convex, we deduce that $tu_1 + (1-t)u_2 \in S(x)$ and

$$F(x,y,u_1)\cap W\,\mathrm{Min}\,(F(x,y,S(x))/C)\neq\emptyset,$$
$$F(x,y,u_2)\cap W\,\mathrm{Min}\,(F(x,y,S(x))/C)\neq\emptyset.$$

Take $v_i\in F(x,y,u_i)\cap W\,\mathrm{Min}\,(F(x,y,S(x))/C)\neq\emptyset$. The upper C–quasi-convexity of $F(x,y,.)$ implies that there exists $v_t\in F(x,y,tu_1+(1-t)u_2)\subset(F(x,y,S(x))$ such that either $v_1-v_t\in C$ or $v_2-v_t\in C$. If $v_1-v_t\in C$ and $v_1\in F(x,y,u_1)\cap W\,\mathrm{Min}\,(F(x,y,S(x))/C)$, then $v_t\in W\,\mathrm{Min}\,(F(x,y,S(x))/C)$. Otherwise, there is $v\in F(x,y,S(x))$ with $v_t-v\in\ \mathrm{int}\ C$ and then $v_1-v=v_1-v_t+v_t-v\in C+\ \mathrm{int}\ C\subset\ \mathrm{int}\ C$. It is impossible. If $v_2-v_t\in C$, the proof is similar and it is also impossible. This shows that $M^W(x,y)$ is a convex set. Therefore, M^W is a compact acyclic mapping with nonempty compact values. Using Theorem 2 in [10], we conclude that there are $\bar{x}\in D,\bar{y}\in K$ such that

$$\bar{x}\in S(\bar{x}),$$
$$\bar{y}\in T(\bar{x}),$$

and

$$F(\bar{x},\bar{y},\bar{x})\cap W\,\mathrm{Min}\,(F(\bar{x},\bar{y},S(\bar{x})))\neq\emptyset.$$

This shows $W\mathcal{S}\neq\emptyset$. Applying Theorem 1, we conclude that there is $(x^*,z^*)\in W\mathcal{S}$ with

$$f(x^*,z^*)\cap P\,\mathrm{Min}\,(f(W\mathcal{S})/C_0)\neq\emptyset.$$

Thus, (x^*,z^*) is a solution of $(1)_{(W,P)}$. ∎

Theorem 4. *Let D,K,S,T,f be as in Theorem 3 and let $F:D\times K\times D\to Y$ be a compact C and $(-C)$-continuous single-valued mapping. In addition, assume that for any fixed $(x,y)\in D\times K, F(x,y,.)$ is a strictly C-quasi-convex single-valued mapping. Then Problem $(1)_{(P,P)}$ has a solution.*

Proof. Let M^W be defined as in (10) with $\alpha=W$. It has been shown in the proof of the previous theorem, M^W is a compact mapping with nonempty compact values. Since $F(x,y,.)$ is a strictly C-quasi-convex mapping, applying Proposition 5.13, Chapter 2 and Corollary 4.15, Chapter 6 in [14], we conclude that $W\,\mathrm{Min}\,(F(x,y,S(x))/C)=P\,\mathrm{Min}\,(F(x,y,S(x))/C), M^W=M^P$ and $M^P(x,y)$ is a contractible set for all $(x,y)\in D\times K$. This implies that the mapping M^P is a compact acyclic multi-valued mapping with nonempty compact values. Using Theorem 2 in [10] again, we deduce that there are $\bar{x}\in D,\bar{y}\in K$ such that

$$\bar{x}\in S(\bar{x}),$$
$$\bar{y}\in T(\bar{x}),$$

and

$$F(\bar{x},\bar{y},\bar{x})\cap P\,\mathrm{Min}\,(F(\bar{x},\bar{y},S(\bar{x}))\neq\emptyset.$$

Thus, the set $P\mathcal{S}$ defined as in (16) with $\alpha=P$ is nonempty and compact. To complete the proof of the theorem, it remains to apply Theorem 1. ∎

Theorem 5. *Let D, K, S, T, F and C be as in Theorem 3. Let $f : D \times K \to 2^{Y_0}$ be an upper C_0-continuous multi-valued mapping. In addition, assume that $I\,Min\,(F(x, y, x)/C) \neq \emptyset$ for all $(x, y) \in D \times K$. Then Problem $(1)_{(I,P)}$ has a solution.*

Proof. It follows from Theorem 2 and Corollary 2 that $I\mathcal{S} \neq \emptyset$. The mapping M^I defined as in (10) is closed and compact. Consequently, the set $I\mathcal{S}$ is nonempty and compact. Therefore, to complete the proof of the theorem, it remains to apply Theorem 1. ∎

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