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Cayley Graphs of Abelian Groups Which Are Not Normal Edge-Transitive

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Abstract. For a group G, and a subset S of G such that $1_G \notin S$, let $\Gamma = Cay(G, S)$ be the corresponding Cayley graph. Then Γ is said to be normal edge transitive, if $N_{Aut(\Gamma)}(G)$ is transitive on edges. In this paper we determine all connected, undirected edge-transitive Cayley graphs of finite abelian groups with valency at most five, which are not normal edge transitive. This is a partial answer to a question of Praeger.

1. Introduction

Let G be a finite abelian group and S a subset of G such that $1_G \notin S, |S| \leq 5$, and $\langle S \rangle = G$. The corresponding Cayley digraph, denoted by $\Gamma = Cay(G, S)$ is the digraph with vertex set G and arcs (x, y) such that $yx^{-1} \in S$. The digraph is also assumed to be undirected that is $S^{-1} = S$, (and in this case each unordered pair $\{x, y\}$ such that (x, y) and (y, x) are arcs is an edge of the corresponding undirected graph).

The graph $Cay(G, S)$ is vertex-transitive since it admits G, acting by right multiplication, as a subgroup of automorphisms. Thus $G \leq Aut(Cay(G, S))$ and this action of G is regular on vertices, that is, G is transitive on vertices and only the identity element of G fixes a vertex. A graph Γ is (isomorphic to) a Cayley graph for some group if and only if its automorphism group $\mathrm{Aut}(\Gamma)$ has a subgroup which is regular on vertices, (see [2, Lemma 16.3]). For small values of n, the vast majority of undirected vertex-transitive graphs with n vertices are Cayley graphs (see [5, Table 1]).

A Cayley graph $\Gamma = Cay(G, S)$ is said to be edge-transitive if $Aut(\Gamma)$ is transitive on edges. Also, if Γ is undirected, then an unordered pair of edges $\{(x,y),(y,x)\}$ is called an unordered edge, and Γ is said to be edge-transitive as an undirected graph if $Aut(\Gamma)$ is transitive on unordered edges. In this paper we present an approach to studying the family of Cayley graphs for a given finite group G, which focuses attention on those graphs Γ for which $N_{Aut(\Gamma)}(G)$ is transitive on edges, and those undirected graphs Γ for which $N_{Aut(\Gamma)}(G)$ is transitive on unordered edges. Such a graph is said to be normal edge-transitive, or normal edge-transitive as an undirected graph, respectively. Not every edge-transitive Cayley graph is normal edge-transitive. This can be seen by considering the complete graphs K_n, on n vertices.

Example 1.1. The complete graph K_n, is an undirected Cayley graph for any group of order n, and its automorphism group S_n acts transitively on edges, and hence also on unordered edges. However K_n is normal edge-transitive (and also normal edge-transitive as an undirected graph) if and only if n is a prime power. If $n = p^a$ (p a prime and $a \geq 1$), then taking $G = Z_p^a$ we have $K_n \cong Cay(G, G\backslash\{1\})$ and $N_{S_n}(G) = AGL(a,p)$ is transitive on edges (and on undirected edges).

However in most situations, it is difficult to find the full automorphism group of a graph. Although we know that a Cayley graph $Cay(G,S)$ is vertex-transitive, simply because of its definition, in general it is difficult to decide whether it is edge-transitive. On the other hand we often have sufficient information about the group G to determine $N = N_{Aut(Cay(G,S))}(G)$; for N is the semidirect product $N = G.Aut(G,S)$, where $Aut(G,S) = \{\sigma \in Aut(G) | S^\sigma = S\}$.

Thus it is often possible to determine whether $Cay(G,S)$ is normal edge-transitive.

Independently of our investigation, and as another attempt to study the structure of finite Cayley graphs, Xu [7] defined a Cayley graph $\Gamma = Cay(G,S)$ to be normal if G is a normal subgroup of the full automorphism group $Aut(\Gamma)$. Xu's concept of normality for a Cayley graph is a very strong condition. For example, K_n is normal if and only if $n \leq 4$. However any edge-transitive Cayley graph which is normal, in the sense of Xu's definition, is automatically normal edge-transitive.

Praeger posed the following question in [6]: What can be said about the structure of Cayley graphs which are edge-transitive but not normal edge-transitive? In the next theorem we will identify all edge-transitive Cayley graph of an abelian group which are not normal edge-transitive and have valency at most 5. This is a partial answer to Question 5 of [6].

Theorem 1.2. *Let G be an abelian group and let S be a subset of G not containing the identity element 1_G. Suppose $\Gamma = Cay(G,S)$ is a connected undirected Cayley graph of G relative to S and $|S| \leq 5$. If Γ is an edge-transitive Cayley graph and is not normal edge-transitive as an undirected graph, then Γ, G satisfy one of $(1)-(13)$ follows:*

(1) $G = Z_4, S = G \setminus \{1\}, \Gamma = K_4$.

(2) $G = Z_4 \times Z_2 = \langle a \rangle \times \langle b \rangle, S = \{a, a^{-1}, b\}, \Gamma = Q_3$ *the cube.*

(3) $G = Z_6 = \langle a \rangle, S = \{a, a^3, a^5\}, \Gamma = K_{3,3}$.
(4) $G = Z_4 \times Z_2 = \langle a \rangle \times \langle b \rangle, S = \{a, a^{-1}, a^2b, b\}, \Gamma = K_{4,4}$.
(5) $G = Z_4 \times Z_2^2 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle, S = \{a, a^{-1}, b, c\}, \Gamma = Q_4$, *the 4-dimensional cube.*
(6) $G = Z_m \times Z_2 = \langle a \rangle \times \langle b \rangle$, $m \geq 3$, *and* m *is odd* $S = \{a, ab, a^{-1}, a^{-1}b\}, \Gamma = C_m[2k_1]$.
(7) $G = Z_4 \times Z_2^3 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle \times \langle d \rangle, S = \{a, a^{-1}, b, c, d\}, \Gamma = K_2 \times Q_4 = Q_5$.
(8) $G = Z_4 \times Z_2^2 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle$ $S = \{a, a^{-1}, ab, a^{-1}b, c\}, \Gamma = K_2 \times C_4[2K_1]$.
(9) $G = Z_4^2 \times Z_2 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle, S = \{a, a^{-1}, b, b^{-1}, c\}, \Gamma = C_4 \times Q_3$.
(10) $G = Z_4 \times Z_2^2 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle, S = \{a, a^{-1}, b, c, a^2bc\}, \Gamma = Q_4^d$.
(11) $G = Z_6 = \langle a \rangle, S = \{a, a^2, a^3, a^4, a^5\}, \Gamma = C_3[K_2] = K_6$.
(12) $G = Z_{10} = \langle a \rangle, S = \{a, a^3, a^7, a^9, a^5\}, \Gamma = K_{5,5}$.
(13) $G = Z_6 \times Z_2 = \langle a \rangle \times \langle b \rangle, S = \{a, a^{-1}, a^2b, a^{-2}b, b\}, \Gamma = K_{6,6} - 6K_2$.

Corollary 1.3.
(1) *All edge transitive connected Cayley graphs with valency at most* 5 *of a finite abelian group of odd order are normal edge-transitive.*
(2) *All edge-transitive connected Cayley graph with valency at most* 5 *of a finite cyclic group are normal edge-transitive except for*

$G = Z_4$ *and* $\Gamma = K_4$, *or* $G = Z_6$ *and* $\Gamma = K_{3,3}$ *or* $G = Z_6$ *and* $\Gamma = K_6$ *or* $G = Z_{10}$ *and* $\Gamma = K_{5,5}$.

Our work is entirely dependent on two papers [1] and [3] that classify the graphs Γ as in the first paragraph for which $Aut(\Gamma) \neq N_{Aut(\Gamma)}(G)$.

Suppose that Γ is as in the first paragraph above, and that Γ is edge-transitive but not normal edge-transitive. Then it follows that $Aut(\Gamma) \neq N_{Aut(\Gamma)}(G)$, and hence that Γ is one of the graphs classified in [1] or [3]. There are exactly 15 individual pairs (Γ, G) and 8 infinite families of pairs (Γ, G) in the classification in [1, 3]. Our task is to examine these lists. We will determine which graphs in the lists are edge-transitive and which graphs in the lists are normal edge-transitive.

The proof of Theorem 1.2 is in Secs. 3 and 4. We consider the Cayley graph of abelian groups with valency at most four in Sec 3 and with valency 5 in Sec. 4.

2. Primary Analysis

For a graph Γ, we denote the automorphism group of Γ by $Aut(\Gamma)$. The following propositions are basic.

Proposition 2.1. [4] *Let* $\Gamma = Cay(G, S)$ *be a Cayley graph of group* G *relative on* S.
(1) $Aut(\Gamma)$ *contains the right regular permutation of* G, *so* Γ *is vertex- transitive.*
(2) Γ *is connected if and only if* $G =< S >$.
(3) Γ *is undirected if and only if* $S^{-1} = S$.

Proposition 2.2. [2] *A graph $\Gamma = (V, E)$ is a Cayley graph of a group if and only if $Aut\Gamma$ contains a regular subgroup.*

Let $\Gamma = Cay(G, S)$ be a Cayley graph of G on S, and let

$$Aut(G, S) = \{\alpha \in Aut(G) | S^\alpha = S\}.$$

Obviously, $Aut(\Gamma) \geq G.Aut(G, S)$. Writing $A = Aut(\Gamma)$, we have.

Proposition 2.3.

(1) $N_A(G) = G.Aut(G, S)$.

(2) $A = G.Aut(G, S)$ *is equivalent to* $G \lhd A$.

Proof. Since the normalizer of G in the symmetric group $Sym(G)$ is the holomorph of G, that is $GAut(G)$, we have $N_A(G) = GAut(G) \cap A = G(Aut(G) \cap A)$.

Obviously, $Aut(G) \cap A = Aut(G, S)$. Thus (1) holds. (2) is an immediate consequence of (1). ■

Proposition 2.4. [6] *Let $\Gamma = Cay(G, S)$ be a Cayley graph for a finite group G with $S \neq \phi$. Then Γ is normal edge-transitive if and only if $Aut(G, S)$ is either transitive on S or has two orbits in S which are inverses of each other.*

Let X and Y be two graphs. The direct product $X \times Y$ is defined as the graph with vertex set $V(X \times Y) = V(X) \times V(Y)$ such that for any two vertices $u = [x_1, y_1]$ and $v = [x_2, y_2]$ in $V(X \times Y), [u, v]$ is an edge in $X \times Y$ whenever $x_1 = x_2$ and $[y_1, y_2] \in E(Y)$ or $y_1 = y_2$ and $[x_1, x_2] \in E(X)$. Two graphs are called relatively prime if they have no nontrivial common direct factor. The lexicographic product $X[Y]$ is defined as the graph vertex set $V(X[Y]) = V(X) \times V(Y)$ such that for any two vertices $u = [x_1, y_1]$ and $v = [x_2, y_2]$ in $V(X[Y])$, $[u, v]$ is an edge in $X[Y]$ whenever $[x_1, x_2] \in E(X)$ or $x_1 = x_2$ and $[y_1, y_2] \in E(Y)$. Let $V(Y) = \{y_1, y_2, ..., y_n\}$. Then there is a natural embedding nX in $X[Y]$, where for $1 \leq i \leq n$, the ith copy of X is the subgraph induced on the vertex subset $\{(x, y_i)| x \in V(X)\}$ in $X[Y]$. The deleted lexicographic product $X[Y] - nX$ is the graph obtained by deleting all the edges of (this natural embedding of) nX from $X[Y]$.

3. The Cayley Graph of Abelian Groups with Valency at Most Four

Let $\Gamma = Cay(G, S)$ be a connected undirected Cayley graph of an abelian group G on S, with the valency of Γ being at most four. Then we will give proof of our main theorem. If an edge-transitive Cayley graph is normal, then that is automatically normal edge-transitive. Thus this implies that we first must consider non-normal graphs.

By using [1, Theorem 1.2] all non- normal Cayley graphs of an abelian group are as follows.

(1) $G = Z_4, S = G \backslash \{1\}, \Gamma = K_4$.

(2) $G = Z_4 \times Z_2 = \langle a \rangle \times \langle b \rangle, S = \{a, a^{-1}, b\}, \Gamma = Q_3$ the cube.

(3) $G = Z_6 = \langle a \rangle, S = \{a, a^3, a^5\}, \Gamma = K_{3,3}$.
(4) $G = Z_2^3 = \langle u \rangle \times \langle v \rangle \times \langle w \rangle, S = \{w, wu, wv, wuv\}, \Gamma = K_{4,4}$.
(5) $G = Z_4 \times Z_2 = \langle a \rangle \times \langle b \rangle, S = \{a, a^2, a^3, b\}, \Gamma = Q_3^c$, the complement of the cube.
(6) $G = Z_4 \times Z_2^2 = \langle a \rangle \times \langle b \rangle, S = \{a, a^{-1}, a^2 b, b\}, \Gamma = K_{4,4}$.
(7) $G = Z_4 \times Z_2^2 = \langle a \rangle \times \langle b \rangle \times \langle c \rangle, S = \{a, a^{-1}, b, c\}, \Gamma = Q_4$, the 4-dimensional cube.
(8) $G = Z_6 \times Z_2 = \langle a \rangle \times \langle b \rangle, S = \{a, a^{-1}, a^3, b\}, \Gamma = K_{3,3} \times K_2$.
(9) $G = Z_4 \times Z_4 = \langle a \rangle \times \langle b \rangle, S = \{a, a^{-1}, b, b^{-1}\}, \Gamma = C_4 \times C_4$.
(10) $G = Z_m \times Z_2 = \langle a \rangle \times \langle b \rangle, m \geq 3, S = \{a, ab, a^{-1}, a^{-1}b\}, \Gamma = C_m[2k_1]$.
(11) $G = Z_{4m} = \langle a \rangle, m \geq 2, S = \{a, a^{2m+1}, a^{-1}, a^{2m-1}\}, \Gamma = C_{2m}[2k_1]$.
(12) $G = Z_5, S = G \backslash \{1\}, \Gamma = K_5$.
(13) $G = Z_{10} = \langle a \rangle, S = \{a, a^3, a^7, a^9\}, \Gamma = K_{5,5} - 5K_2$.

Lemma 3.1. *The graphs* Γ *in cases* (4), (9), (10)[*for* m *even*], (11), (12), *and* (13) *from the list above are normal edge transitive.*

Proof. We will apply Proposition 2.4, and will show in each case that $Aut(G, S)$ is transitive on S.

In the case (4) G may be regarded as a vector space and elements of $Aut(G)$ are determined by their action on the basis u, v, w. We define three maps f, g, h as follows, that they lie in $Aut(G)$, and that the subgroup they generate is transitive on (S). [f maps $u- > v, v- > u, w- > w$, g maps $u- > u, v- > uv, w- > w$, and h maps $u- > u, v- > v, w- > wu$].

In the case (9), elements of $Aut(G)$ are determined by their action on the generators a, b. We define two maps α, β as follows, that they lie in $Aut(G, S)$, and that the subgroup they generate is transitive on S. [α maps $a- > a^{-1}, b- > b^{-1}$, β maps $a- > b, b- > a$].

In the case (10), elements of $Aut(G)$ are determined by their action on the generators a, b. We define three maps α, β, γ as follows, that they lie in $Aut(G, S)$, and that the subgroup they generate is transitive on S. [α maps $a- > a^{-1}, b- > b^{-1}$, β maps $a- > a^{-1}b, b- > b$, γ maps $a- > ab, b- > b$].

In the case (11), elements of $Aut(G)$ are determined by their action on the generator a. We define three maps α, β, γ as follows, that they lie in $Aut(G, S)$, and that the subgroup they generate is transitive on S.[α maps $a- > a^{-1}$, β maps $a- > b^{2m-1}$, γ maps $a- > a^{2m+1}$].

In the case (12), $G = Z_5, S = G - \{1\}$ we conclude by Example 1.1.

In the case 13, $G = Z_{10}, S = \{a, a^3, a^7, a^9\}$ we have $Aut(G, S) = Aut(G)$ and $Aut(G)$ is transitive on S. Then we conclude by Proposition 2.4. ∎

Lemma 3.2. *The graphs* Γ *in cases* (5)*, and* (8) *from the list above are not edge transitive.*

Proof. In the case (5), $\Gamma = Q_3^c$ and so $Aut(\Gamma) = Aut(Q_3) = AGL(3, 2)$. The graph Q_3 has vertex set Z_2^3 and $x = (x_1, x_2, x_3)$ is joined to $y = (y_1, y_2, y_3)$ by an edge if and only if $x - y$ has exactly 1 non-zero entry. Hence x is adjacent to y in Γ if and only if $x - y$ has two or three non-zero entries. The group $Aut(\Gamma)$

has two orbits on edges, namely edges x, y where $x - y$ has two non-zero entries, and pairs x, y where $x - y$ has three non-zero entries.

In the case $(8), \Gamma = K_{3,3} \times K_2$ forms of two complete bipartite graphs $K_{3,3}, K'_{3,3}$ with $V(K_{3,3}) = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and $V(K'_{3,3}) = \{x'_1, x'_2, x'_3, y'_1, y'_2, y'_3\}$ that $(x_i, y_j) \in E(K_{3,3}), 1 \leq i, j \leq 3$, and $(x'_i, y'_j) \in E(K'_{3,3}), 1 \leq i, j \leq 3$ and also $(t, t') \in E(K_{3,3} \times K_2)$ for $t \in \{x_1, x_2, x_3, y_1, y_2, y_3\}$ there is no automorphism f such that $f(x_1, y_1) = (x_1, x'_1)$, because the arc (x_1, y_1) lie on five circuits of length 4, but the arc (x_1, x'_1) lies on three circuits of length 4.

Lemma 3.3. *The graphs* Γ *in cases* $(1), (2), (3), (6), (7)$*, and* (10) [*for* m *odd*] *from the list above satisfy the conditions of Theorem* 1.2.

Proof. By using Proposition 2.4, since in normal edge-transitive Cayley graphs, all elements of S have same order, hence these graphs are not normal edge-transitive. Since the complete graph K_n and complete bipartite graph $K_{n,n}$ are edge transitive, hence it is sufficient to show that graphs $\Gamma = Q_3, \Gamma = Q_4$ and $\Gamma = C_m[2K_1]$ are edge-transitive.

By $[2$, chapter $20, 20a]$ the cube graph $\Gamma = Q_k$ is distance-transitive, that is, for all vertices u, v, x, y of Γ such that $d(u, v) = d(x, y)$ there is an automorphism α in $Aut(\Gamma)$ satisfying $\alpha(u) = x$ and $\alpha(v) = y$. Hence Q_k is edge-transitive.

In the final case for graph $\Gamma = C_m[2K_1]$ let $V(C_m) = \{x_0, x_1, ..., x_{m-1}\}$ and $V(2K_1) = \{y_1, y_2\}$. The graph C_m is edge-transitive and the automorphism group $Aut(\Gamma)$ contains $C_2 wr D_{2m}$ and permutation $\sigma = ((x_0, y_1, (x_0, y_2))$ on $V(\Gamma)$. By combination of automorphisms the subgroup $H = \langle C_2 wr D_{2m}, \sigma \rangle$ of $Aut(\Gamma)$ is transitive on $E(\Gamma)$. Hence $\Gamma = C_m(2K_1)$ is edge-transitive. By Lemmas 3.1, 3.2, and 3.3 we conclude Theorem 2.1 for $|S| \leq 4$. ∎

4. Edge-Transitive Cayley Graph of Abelian Groups with Valency Five

Our purpose in this section is to show all edge-transitive Cayley graphs of abelian groups with valency five which are not normal edge-transitive. As in Sec. 3, we first consider all non-normal Cayley graphs with the above condition.

Let Γ be a graph and α a permutation $V(\Gamma)$ and C_n a circuit of length n. The twisted product $\Gamma \times_\alpha C_n$ of Γ by C_n with respect to α is defined by

$$\begin{aligned} &V(\Gamma \times_\alpha C_n) = V(\Gamma) \times V(C_n) = \{(x, i) \mid x \in V(\Gamma), i = 0, 1, ..., n-1\} \\ &E(\Gamma \times_\alpha C_n) = \{[(x, i), (x, i+1)] | x \in V(\Gamma), i = 0, 1, ..., n-2\} \cup \{[(x, n-1), \\ &\quad (x^\alpha, 0)] \mid x \in V(\Gamma)\} \cup \{[(x, i), (y, i)] | [x, y] \in E(\Gamma), i = 0, 1, ..., n-1\}. \end{aligned}$$

Now we introduce some graphs which appears in our main theorem. The graph Q_4^d denotes the graph obtained by connecting all long diagonals of 4- cube Q_4, that is connecting all vertex u and v in Q_4 such that $d(u, v) = 4$. The graph $K_{m,m} \times_c C_n$ is the twisted product of $K_{m,m}$ by C_n such that c is a cycle permutation on each part of the complete bipartite graph $K_{m,m}$. The graph $Q_3 \times_d C_n$

is the twisted product of Q_3 by C_n such that d transposes each pair elements on long diagonals of Q_3. The graph $C^d_{2m}[2K_1]$ is defined by:

$$V(C^d_{2m}[2K_1]) = V(C_{2m}[2K_1])$$
$$E(C^d_{2m}[2K_1]) = E(C_{2m}[2K_1]) \cup \{[(x_i, y_j), (x_{i+m}, y_j)] \mid i = 0, 1, ..., m-1, j = 1, 2\}$$

where $V(C_{2m}) = \{x_0, x_1, ..., x_{2m-1}\}$ and $V(2K_1) = \{y_1, y_2\}$.

By using [3, Theorem 1.1] all non-normal Cayley graphs of an abelian group with valency five are as follows:

(1) $G = Z_2^4 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle, S = \{a, b, c, d, abc\}$ and $\Gamma = K_2 \times K_{4,4}$.
(2) $G = Z_4 \times Z_2^2 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle, S = \{a, a^{-1}, a^2, b, c\}$ and $\Gamma = C_4 \times K_4$.
(3) $G = Z_4 \times Z_2^2 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle, S = \{a, a^{-1}, b, c, a^2b\}$ and $\Gamma = K_2 \times K_{4,4}$.
(4) $G = Z_4 \times Z_2^3 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle \times \langle d\rangle, S = \{a, a^{-1}, b, c, d\}$ and $\Gamma = K_2 \times Q_4 = Q_5$.
(5) $G = Z_6 \times Z_2^2 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle, S = \{a, a^{-1}, a^3, b, c\}$ and $\Gamma = K_{3,3} \times C_4$.
(6) $G = Z_m \times Z_2^2 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ with $m \geq 3, S = \{a, a^{-1}, ab, a^{-1}b, c\}$ and $\Gamma = K_2 \times C_m[2K_1]$.
(7) $G = Z_{4m} \times Z_2 = \langle a\rangle \times \langle b\rangle$ with $m \geq 3, S = \{a, a^{-1}, a^{2m-1}, a^{2m+1}, b\}$ and $\Gamma = K_2 \times C_m[2K_1]$.
(8) $G = Z_{10} = \langle a\rangle, S = \{a^2, a^4, a^6, a^8, a^5\}$ and $\Gamma = K_2 \times K_5$.
(9) $G = Z_{10} \times Z_2 = \langle a\rangle \times \langle b\rangle, S = \{a, a^{-1}, a^3, a^7, b\}, \Gamma = K_2 \times (K_{5,5} - 5K_2)$.
(10) $G = Z_m \times Z_4 = \langle a\rangle \times \langle b\rangle$ with $m \geq 3, S = \{a, a^{-1}, b, b^{-1}b^2\}$ and $\Gamma = C_m \times K_4$.
(11) $G = Z_m \times Z_6 = \langle a\rangle \times \langle b\rangle$ with $m \geq 3, S = \{a, a^{-1}, b, b^{-1}, b^3\}$ and $\Gamma = C_m \times K_{3,3}$.
(12) $G = Z_m \times Z_4 \times Z_2 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle$ with $m \geq 3, S = \{a, a^{-1}, b, b^{-1}, c\}$ and $\Gamma = C_m \times Q_3$.
(13) $G = Z_2^3 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle, S = \{a, b, c, ab, ac\}$ and $\Gamma = K_2[2K_2]$.
(14) $G = Z_4 \times Z_2 = \langle a\rangle \times \langle b\rangle, S = \{a, a^{-1}, b, a^2, a^2b\}$ and $\Gamma = K_2[2K_2]$.
(15) $G = Z_4 \times Z_2^2 = \langle a\rangle \times \langle b\rangle \times \langle c\rangle, S = \{a, a^{-1}, b, c, a^2bc\}$ and $\Gamma = Q_4^d$.
(16) $G = Z_{2m} = \langle a\rangle$ with $m \geq 3, S = \{a, a^{-1}, a^{m+1}, a^{m-1}, a^m\}$ and $\Gamma = C_m[K_2]$.
(17) $G = Z_{2m} \times Z_2 = \langle a\rangle \times \langle b\rangle$ with $m \geq 2, S = \{a, a^{-1}, ab, a^{-1}b, b\}$ and $\Gamma = C_{2m}[K_2]$.
(18) $G = Z_{2m} \times Z_2 = \langle a\rangle \times \langle b\rangle$ with $m \geq 2, S = \{a, a^{-1}, ab, a^{-1}ba^m\}$ and $\Gamma = C^d_{2m}[2K_1]$.
(19) $G = Z_{10} = \langle a\rangle, S = \{a, a^3, a^7, a^9, a^5\}$ and $\Gamma = K_{5,5}$.
(20) $G = Z_6 \times Z_2 = \langle a\rangle \times \langle b\rangle, S = \{a, a^{-1}, a^2b, a^{-2}b, b\}$ and $\Gamma = K_{6,6} - 6K_2$.
(21) $G = Z_{2m} \times Z_4 = \langle a\rangle \times \langle b\rangle$ with $m \geq 2, S = \{a, a^{-1}, b, b^{-1}, a^mb^2\}$ and $\Gamma = Q_3 \times C_m$.
(22) $G = Z_{6m} = \langle a\rangle$ with m odd and $m \geq 3, S = \{a^2, a^{-2}, a^m, a^{5m}, a^{3m}\}$ and $\Gamma = K_{3,3} \times_c C_m$.
(23) $G = Z_{6m} \times Z_2 = \langle a\rangle \times \langle b\rangle$ with $m \geq 2, S = \{a, a^{-1}, ba^m, ba^{-m}, ba^{3m}\}$ and $\Gamma = K_{3,3} \times_c C_{2m}$.

We want to show that some of the above mentioned cases satisfy Theorem 1.2.

Lemma 4.1. *If* Γ *is in one of the cases* $(1)-(23)$*, from the list above, but is not in cases* $(4), (6)$ *(for* $m = 4$*)*$, (12)($ *for* $m = 4), (15), (16)($ *for* $m = 3), (19), (20)$ *or* $(21)($ *for* $m = 4)$*, then* Γ *is not edge-transitive.*

Proof. If $Aut(\Gamma)$ is edge-transitive, then since $Aut(\Gamma)$ is vertex transitive and has odd valency, $Aut(\Gamma)$ must be transitive on arcs. We show that in each case $Aut(\Gamma)$ is not transitive on arcs. In the cases (1) and (3), let $V(K_2) = \{y_1, y_2\}$ and $V(K_{4,4}) = \{x_1, x_2, x_3, x_4, x_1^{'}, x_2^{'}, x_3^{'}, x_4^{'}\}$ such that $(x_i, x_j^{'}) \in E(K_{4,4})$ for $1 \leq i, j \leq 4$. There is no automorphism f such that $f([(y_1, x_1), (y_2, x_1)]) = [(y_1, x_1), (y_1, x_1^{'})]$, because the arc $((y_1, x_1), (y_2, x_1))$ lies on four circuits of length 4, but the arc $((y_1, x_1), (y_1, x_1^{'}))$ lies on nine circuits of length 4.

In the cases (2) and (10), let $V(C_m) = \{1, 2, 3, ..., m\}$ and $V(K_4) = \{x_1, x_2, x_3, x_4\}$. There is no automorphism f such that $f((2, x_1), (2, x_4)) = ((2, x_1), (3, x_1))$, because if $m \neq 3$, the arc of $((2, x_1), (2, x_4))$ lies on some circuit of length 3, but the arc $((2, x_1), (3, x_1))$ does not lie on any circuit of length 3. If $m = 3$, the arc $((2, x_1), (3, x_1))$ lies on one circuit of length 3, but the arc $((2, x_1), (2, x_4))$ lies on two circuits of length 3.

In the cases (5) and (11), let $V(K_{3,3}) = \{x_1, x_2, x_3, x_1^{'}, x_2^{'}, x_3^{'}\}$ and $V(C_m) = \{1, 2, ..., m\}$. We have $(x_i, x_j^{'}) \in E(K_{3,3})$, for $1 \leq i, j \leq 3, (k, k+1) \in E(C_m), 1 \leq k \leq m-1$ and $(m, 1) \in E(C_m)$. There is no automorphism f such that $f((x_1, 1), (x_1^{'}, 1)) = ((x_1, 1), (x_1, 2))$, because if $m = 3$, the arc $((x_1, 1), (x_1, 2))$ lies on some circuit of length 3, but the arc $((x_1, 1), (x_1^{'}, 1))$ does not lie on any circuit of length 3. If $m = 4$, the arc $((x_1, 1), (x_1, 2))$ lies on four circuits of length 4, but the arc $((x_1, 1), (x_1^{'}, 1))$ lies on six circuits of length 4. If $m > 4$, the arc $((x_1, 1), (x_1, 2))$ lies on three circuits of length 4, but the arc $((x_1, 1), (x_1^{'}, 1))$ lies on six circuits of length 4.

In the cases $(6)(m \neq 4), (7), \Gamma = K_2 \times C_m[2K_1]$, suppose that $Aut(\Gamma)$ is edge-transitive. Then since $Aut(\Gamma)$ is vertex-transitive and has odd valency, $Aut(\Gamma)$ must be transitive on arcs, and so for a vertex x, the stabiliser $Aut(\Gamma)_x$ is transitive on 5 vertices adjacent to x. However if $m = 3$ then the subgraph induced on these 5 vertices is $K_1 \bigcup C_4$ which is not vertex-transitive. If $m \geq 5$ there is one vertex $x^{'}$ at distance 2 from x that is joined to exactly 4 of the 5 vertices joined to x. $Aut(\Gamma)_x$ fixes the unique vertex joined to x but not joined to $x^{'}$. This is a contradiction to the fact that $Aut(\Gamma)$ is arc-transitive.

In the case (8), let $V(K_5) = \{1, 2, 3, 4, 5\}$ and $V(K_2) = \{x_1, x_2\}$. The arc $((1, x_1), (2, x_1))$ lies on some circuit of length 3, but the arc $((1, x_1), (1, x_2))$ does not lie on any circuit of length 3.

In the case (9), let $V(K_{5,5} - 5K_2) = \{x_1, x_2, ..., x_5, x_1^{'}, x_2^{'}, ...x_5^{'}\}, V(K_2) = \{y_1, y_2\}$ such that $(x_i, x_j^{'}) \in E(K_{5,5} - 5K_2)$ for $i \neq j, 1 \leq i, j \leq 5$. There is no automorphism f such that $f((y_1, x_1), (y_1, x_2^{'})) = ((y_1, x_1), (y_2, x_1))$, because the arc $((y_1, x_1), (y_1, x_2^{'}))$ lies on six circuits of length 4, but the arc $((y_1, x_1), (y_2, x_1))$ lies on four circuits of length 4.

In the cases $(12), (21)$ [for $m \neq 4$], let $V(C_m) = \{0, 1, 2, 3, ..., m-1\}$ and Q_3 contains two circuits $C_4, C_4^{'}$ respectively with the sets of vertices $V(C_4) = \{x_1, x_2, x_3, x_4\}$ and $V(C_4^{'}) = \{y_1, y_2, y_3, y_4\}$. In addition, $(x_i, x_i^{'}) \in E(Q_3)$ for $1 \leq i \leq 4$. There is no automorphism f such that $f((x_1, 0), (x_1^{'}, 0)) =$

$((x_1,0),(x_1,1)))$, because if $m=3$, the arc $((x_1,o),(x_1^{'},0))$ does not lie on any circuit of length 3, but the arc $((x_1,0),(x_1,1))$ lies on some circuits of length 3. If $m>4$, the arc $((x_1,0),(x_1^{'},0))$ lies on four circuits of length 4, but the arc $((x_1,0),(x_1,1))$ lies on three circuits of length 4.

In the cases (13) and (14), let $V(K_2)=\{x,y\}$ and $V(2K_2)=\{1,2,3,4\}$, and also $E(2K_2)$ contains two edges $(1,2),(3,4)$. There is no automorphism f such that $f((x,1),(y,1))=((y,1),(y,2))$, because the arc $((x,1),(y,1))$ lies on one circuit of length 3, but the arc $((y,1),(y,2))$ lies on four circuits of length 3.

In the case (16) for $[m\neq 3]$, let $V(C_m)=\{1,2,..,m\}$ and $V(K_2)=\{x,y\}$. There is no automorphism f such that $f((1,y),(1,x))=((1,y),(2,y))$, because the arc $((1,y),(1,x))$ lies on four circuits of length 3, but the arc $((1,y),(2,y))$ lies on two circuits of length 3.

The case (17) is a special case of (16), since $2m\neq 3$.

In the case (18), there is no automorphism f such that $f((x_0,y_2),(x_1,y_2))= ((x_0,y_2),(x_m,y_2))$, because the arc $((x_0,y_2),(x_1,y_2))$ lies on six circuits of length 4, but the arc $((x_0,y_2),(x_m,y_2))$ lies on two circuits of length 4.

In the cases (22) and (23), let $V(C_m)=\{0,1,...,m-1\}, V(K_{3,3})=\{x_1,x_2, x_3,x_1^{'},x_2^{'},x_3^{'}\}$ and also $(x_i,x_j^{'})\in E(K_{3,3})$ for $1\leq i,j\leq 3$. There is no automorphism f such that $f((x_1,0),(x_1^{'},0))=((x_1,0),(x_1,1))$, because the arc $((x_1,0),(x_1^{'},0))$ lies on six circuits of length 4, but the arc $((x_1,0),(x_1,1))$ lies on three circuits of length 4.

Lemma 4.2. *If* Γ *is in one of the cases* (4), (6)(*for* $m=4$), (12)(*for* $m=4$), (15), (16)(*for* $m=3$), (19), (20), (21)(*for* $m=4$) *from the list above, then* Γ *satisfies the conditions of Theorem* 1.2.

Proof. By using Proposition 2.4, since in a normal edge-transitive Cayley graph all elements of the set S have the same order, so these graphs are not normal edge-transitive. We show that these graphs are edge-transitive.

In the cases (4), (12)($m=4$)and (21)($m=4$)$\Gamma\simeq K_2\times Q_4\simeq C_4\times Q_3\simeq Q_5$ and Q_5 is edge transitive.

In the case (6), m=4, $\Gamma=Q_4$, and Q_4 is edge transitive.

In the case (15) we will obtain similarly graph $\Gamma=Q_4$.

In the case (16) for $[m=3]$ we have $\Gamma\simeq K_6$.

The case (19) is obvious and in the case (20),$\Gamma=K_{6,6}-6K_2$, and we will obtain the same result in graph $K_{6,6}$. Thus we conclude Theorem 1.2 for $|S|=5$ by Lemmas 4.1 and 4.2.

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