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Subclasses of Uniformly Starlike and Convex Functions Defined by Certain Integral Operator

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Abstract. In this paper, we consider a class of uniformly starlike functions defined by certain integral operator. We determine a sufficient condition for a function f to be uniformly starlike function that is also necessary when f has negative coefficients. Similar results for corresponding subclasses of uniformly convex functions are also obtained.

1. Introduction

Let $\mathcal{S}$ denote the class of functions f which are analytic and univalent in $D = \{z : 0 < |z| < 1\}$ and given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0. \tag{1}$$

A function $f \in \mathcal{S}$ is called a uniformly starlike function if and only if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} \geq \left|\frac{zf'(z)}{f(z)} - 1\right|, \quad z \in D.$$

We denote this class by $\mathcal{S}_p$.

A function $f \in \mathcal{S}$ is called a uniformly convex function if and only if

$$\text{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} \geq \left|\frac{zf''(z)}{f'(z)}\right|, \quad z \in D.$$

We denote this class by $\mathcal{UCV}$.

Rønning [3] generalized the class $\mathcal{S}_p$ and $\mathcal{UCV}$ by introducing a parameter α in the following way.

Definition 1. [4] *A function $f \in \mathcal{S}_p(\alpha),\ 0 \leqslant \alpha \leqslant 1$, if f satisfies the analytic characterization*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leqslant Re\left\{\frac{zf'(z)}{f(z)}\right\} - \alpha$$

and $f \in \mathcal{UCV}(\alpha)$ if and only if $zf' \in \mathcal{S}_p(\alpha)$.

In [1], Bharati et al. obtained coefficient characterization for some subclasses of $\mathcal{S}_p(\alpha)$ and $\mathcal{UCV}(\alpha)$.

Definition 2. [5] *Let $\mathcal{T}$ be the subclass of $\mathcal{S}$ consisting of functions f of the form*

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0. \tag{2}$$

Also, Bharati et al. in [1] obtained coefficient characterization for some subclasses of $\mathcal{S}_p(\alpha)$ and $\mathcal{UCV}(\alpha)$ for $f \in \mathcal{T}$.

Recently, Jung et al. [2] introduced the following one-parameter families of integral operator for functions $f \in \mathcal{S}$:

$$Q_\beta^\alpha f(z) = \left(\frac{\alpha+\beta}{\beta}\right)\frac{\alpha}{z^\beta}\int_0^z \left(1-\frac{t}{z}\right)^{\alpha-1} t^{\beta-1} f(t)dt, \quad (\alpha > 0, \beta > -1) \tag{3}$$

and

$$J_\alpha f(z) = \frac{\alpha+1}{z^\alpha}\int_0^z t^{\alpha-1} f(t)dt, \quad (\alpha > -1). \tag{4}$$

They showed that

$$Q_\beta^\alpha f(z) = z + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} a_n z^n, \quad (\alpha > 0, \beta > -1) \tag{5}$$

and

$$J_\alpha f(z) = z + \sum_{n=2}^{\infty}\left(\frac{\alpha+1}{\alpha+n}\right) a_n z^n, \quad (\alpha > -1). \tag{6}$$

By virtue (5) and (6), we see that

$$J_\alpha f(z) = Q^1_\alpha f(z), \quad (\alpha > -1). \tag{7}$$

For $f \in \mathcal{T}$, the operator in (5) and (6) becomes

$$Q^\alpha_\beta f(z) = z - \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} a_n z^n, \quad (\alpha > 0, \beta > -1) \tag{8}$$

and

$$J_\alpha f(z) = z - \sum_{n=2}^{\infty} \Big(\frac{\alpha+1}{\alpha+n}\Big) a_n z^n, \quad (\alpha > -1). \tag{9}$$

Using equations (3) and (4), we introduce the following new subclasses of $\mathcal{S}_p(\alpha)$.

Definition 3. *Let $\mathcal{T}Q(\alpha,\beta,\sigma)$, $\alpha > 0, \beta > -1$ and $0 \leqslant \sigma \leqslant 1$ be the class of functions $f \in \mathcal{T}$ satisfying the condition*

$$\Big| \frac{z(Q^\alpha_\beta f(z))'}{Q^\alpha_\beta f(z)} - 1 \Big| \leqslant Re \Big\{ \frac{z(Q^\alpha_\beta f(z))'}{Q^\alpha_\beta f(z)} \Big\} - \sigma, \quad z \in D \tag{10}$$

where $Q^\alpha_\beta f$ is defined as in (3).

Definition 4. *Let $\mathcal{T}J(\alpha,\sigma)$, $\alpha > -1$ and $0 \leqslant \sigma \leqslant 1$ be the class of functions $f \in \mathcal{T}$ satisfying the condition*

$$\Big| \frac{z(J_\alpha f(z))'}{J_\alpha f(z)} - 1 \Big| \leqslant Re \Big\{ \frac{z(J_\alpha f(z))'}{J_\alpha f(z)} \Big\} - \sigma, \quad z \in D \tag{11}$$

where $J_\alpha f$ is defined as in (4).

2. Properties of $\mathcal{T}Q(\alpha,\beta,\sigma)$

In this section, we give some results for $\mathcal{T}Q(\alpha,\beta,\sigma)$. We first state a preliminary lemma, required for proving our result.

Lemma 1. *If $Q^\alpha_\beta f \in \mathcal{T}$ then*

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} n a_n \leq 1.$$

Proof. Suppose on the contrary that $\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} n a_n > 1$. We can write

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} n a_n = 1 + \varepsilon, \ (\varepsilon > 0).$$

Then there exists an integer N such that

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{N}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}\, na_n > 1+\frac{\varepsilon}{2}.$$

For $\left(\frac{1}{1+\frac{\varepsilon}{2}}\right)^{\frac{1}{N-1}} < z < 1$, we have

$$\begin{aligned}(Q^{\alpha}_{\beta}f(z))' &= 1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}\, na_n z^{n-1}\\ &\leqslant 1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{N}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}\, na_n z^{n-1}\\ &\leqslant 1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}z^{N-1}\sum_{n=2}^{N}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}\, na_n\\ &< 1-z^{N-1}\Big(1+\frac{\varepsilon}{2}\Big)\\ &< 0.\end{aligned}$$

Since $(Q^{\alpha}_{\beta}f(0))' = 1 > 0$, there exists a real number $z_0,\ 0 < z_0 < \left(\frac{1}{1+\frac{\varepsilon}{2}}\right)^{\frac{1}{N-1}}$, such that $(Q^{\alpha}_{\beta}f(z_0))' = 0$. Hence $Q^{\alpha}_{\beta}f$ is not univalent. ∎

Theorem 1. *Let the functions $f\in\mathcal{T}$. Then*

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}(2n-1-\sigma)a_n \leqslant 1-\sigma \tag{12}$$

for some $\alpha>0,\beta>-1$ and $0\leqslant\sigma\leqslant 1$ if and only if $f\in\mathcal{T}Q(\alpha,\beta,\sigma)$.

Proof. First, we have

$$\begin{aligned}&\Big|\frac{z(Q^{\alpha}_{\beta}f(z))'}{Q^{\alpha}_{\beta}f(z)}-1\Big| - \operatorname{Re}\Big(\frac{z(Q^{\alpha}_{\beta}f(z))'}{Q^{\alpha}_{\beta}f(z)}-1\Big) \leqslant 2\Big|\frac{z(Q^{\alpha}_{\beta}f(z))'}{Q^{\alpha}_{\beta}f(z)}-1\Big|\\ &\leqslant \frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}2(n-1)|a_n||z|^{n-1}}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}|a_n||z|^{n-1}}\\ &\leqslant \frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}2(n-1)a_n}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}a_n},\end{aligned}$$

where

$$1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}|a_n| > 0$$

by Lemma 1. The above expression is bounded by $1-\sigma$ if and only if (12) is satisfied. Consequently, we can write

$$\Big|\frac{z(Q^{\alpha}_{\beta}f(z))'}{Q^{\alpha}_{\beta}f(z)}-1\Big| - \operatorname{Re}\Big(\frac{z(Q^{\alpha}_{\beta}f(z))'}{Q^{\alpha}_{\beta}f(z)}-1\Big) \leqslant 1-\sigma$$

which is equivalent to (10).
Conversely, if $f \in \mathcal{T}Q(\alpha,\beta,\sigma)$ and z is real, then Definition 3 yields

$$\frac{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}na_nz^{n-1}}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}a_nz^{n-1}}-\sigma \geq \frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}(n-1)a_nz^{n-1}}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}a_nz^{n-1}}.$$

Let $z \to 1^-$ along the real axis, then we get

$$\frac{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}na_n}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}a_n}-\frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}(n-1)a_n}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}a_n} \geq \sigma$$

or

$$1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}(2n-1)a_n \geq \sigma\left(1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}a_n\right)$$

which gives the required result. ∎

Our assertion in Theorem 1 is sharp, for functions of the form

$$F_n(z)=Q_\beta^\alpha f_n(z)=z-\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\frac{\Gamma(\alpha+\beta+n)}{\Gamma(\beta+n)}\frac{1-\sigma}{2n-1-\sigma}\,z^n, \quad n \geq 2 \tag{13}$$

which belong to the class $\mathcal{T}Q(\alpha,\beta,\sigma)$.

Corollary 1. *If $f \in \mathcal{T}Q(\alpha,\beta,\sigma)$ then*

$$a_n \leqslant \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\frac{\Gamma(\alpha+\beta+n)}{\Gamma(\beta+n)}\frac{1-\sigma}{2n-1-\sigma}\,, \quad n \geq 2. \tag{14}$$

Proof. Since $f \in \mathcal{T}Q(\alpha,\beta,\sigma)$, Theorem 1 gives

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}(2n-1-\sigma)a_n \leqslant 1-\sigma.$$

Next, note that

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}(2n-1-\sigma)a_n \leqslant \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}(2n-1-\sigma)a_n.$$

Therefore

$$a_n \leqslant \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\frac{\Gamma(\alpha+\beta+n)}{\Gamma(\beta+n)}\frac{1-\sigma}{2n-1-\sigma}\,, \quad n \geq 2.$$

∎

Corollary 2. *If $f \in \mathcal{T}Q(\alpha,\beta,\sigma)$ and $|z| = r < 1$, then*

(i) $|Q^{\alpha}_{\beta} f(z)| \leqslant r + \dfrac{1-\sigma}{3-\sigma}\, r^2$

(ii) $|Q^{\alpha}_{\beta} f(z)| \geq r - \dfrac{1-\sigma}{3-\sigma}\, r^2$.

The results are sharp.

Proof. First, it is obvious that

$$\begin{aligned}&\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\frac{\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)}(3-\sigma)\sum_{n=2}^{\infty} a_n\\ &\leqslant \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}(2n-1-\sigma)a_n\end{aligned}$$

and as $f \in \mathcal{T}Q(\alpha,\beta,\sigma)$, using the inequality in Theorem 1 yields

$$\sum_{n=2}^{\infty} a_n \leqslant \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\beta+2)}\frac{1-\sigma}{3-\sigma}. \tag{15}$$

From (8) with $|z| = r(r < 1)$, we have

$$\begin{aligned}|Q^{\alpha}_{\beta} f(z)| &\leqslant r + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}a_n r^n\\ &\leqslant r + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\frac{\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)}\sum_{n=2}^{\infty}a_n r^2\end{aligned}$$

and

$$\begin{aligned}|Q^{\alpha}_{\beta} f(z)| &\geq r - \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}a_n r^n\\ &\geq r - \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\frac{\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)}\sum_{n=2}^{\infty}a_n r^2.\end{aligned}$$

Finally, using (15) in the above inequalities gives us both results (i) and (ii). We note that (i) and (ii) are sharp for the following function

$$Q^{\alpha}_{\beta} f_2(z) = z - \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\beta+2)}\frac{1-\sigma}{3-\sigma}\, z^2$$

at $z = \pm ir, \pm r$. ∎

Definition 5. *Let $\mathcal{T}Q(\alpha,\beta,\sigma,\gamma)$ be the class of functions $f \in \mathcal{T}$ satisfying the condition*

$$Re\left(\frac{z(Q^{\alpha}_{\beta} f(z))'}{Q^{\alpha}_{\beta} f(z)}\right) \geq \sigma\left|\frac{z(Q^{\alpha}_{\beta} f(z))'}{Q^{\alpha}_{\beta} f(z)} - 1\right| + \gamma,\ \ \alpha > 0,\ \beta > -1,\ 0 \leqslant \sigma \leqslant 1,\ 0 \leqslant \gamma \leqslant 1 \tag{16}$$

where $Q^{\alpha}_{\beta} f$ is defined as in (3).

We write $\mathcal{T}Q(\alpha,\beta,1,\gamma) = \mathcal{T}Q(\alpha,\beta,\gamma)$.

Theorem 2. *Let the functions $f \in \mathcal{T}$. A function $f \in \mathcal{T}Q(\alpha,\beta,\sigma,\gamma)$ for some $\alpha > 0, \beta > -1, 0 \leqslant \sigma \leqslant 1$ and $0 \leqslant \gamma \leqslant 1$ if and only if*

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}[n(1+\sigma)-(\sigma+\gamma)]a_n \leqslant 1-\gamma. \tag{17}$$

Proof. In view of Definition 5, it suffices to show that

$$\sigma\left|\frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)}-1\right| \leqslant \mathrm{Re}\left\{\frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)}\right\}-\gamma. \tag{18}$$

The condition (18) is equivalent to

$$\sigma\left|\frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)}-1\right| - \mathrm{Re}\left(\frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)}-1\right) \leqslant 1-\gamma.$$

Then

$$\sigma\left|\frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)}-1\right| - \mathrm{Re}\left(\frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)}-1\right) \leqslant (\sigma+1)\left|\frac{z(Q_\beta^\alpha f(z))'}{Q_\beta^\alpha f(z)}-1\right|$$

$$\leqslant \frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}(\sigma+1)(n-1)|a_n||z|^{n-1}}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}|a_n||z|^{n-1}}$$

$$\leqslant \frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}(\sigma+1)(n-1)a_n}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}a_n}.$$

The above expression is bounded by $1-\gamma$ if and only if (17) is satisfied.

Conversely, if $f \in \mathcal{T}Q(\alpha,\beta,\sigma,\gamma)$ and z is real, we get

$$\frac{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}na_nz^{n-1}}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}a_nz^{n-1}}$$

$$\geq \sigma\frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}(n-1)a_nz^{n-1}}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}a_nz^{n-1}}+\gamma.$$

Let $z \to 1^-$ along the real axis, which gives

$$\frac{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}na_n}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}a_n}$$

$$\geq \sigma\frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}(n-1)a_n}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}a_n}+\gamma,$$

that is equivalent to

$$1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}(n+\sigma n-\sigma)a_n$$

$$\geq \gamma\left(1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}a_n\right)$$

and gives the required result. Then, the proof is complete. ∎

Our assertion in Theorem 2 is sharp for functions of the form

$$F_n(z) = Q_\beta^\alpha f_n(z) = z - \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\beta+n)} \frac{1-\gamma}{n(1+\sigma)-(\sigma+\gamma)}\, z^n, \quad n \geq 2 \tag{19}$$

which belong to the class $\mathcal{T}Q(\alpha,\beta,\sigma,\gamma)$.

Corollary 3. *If $f \in \mathcal{T}Q(\alpha,\beta,\sigma,\gamma)$ then*

$$a_n \leqslant \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\beta+n)} \frac{1-\gamma}{n(1+\sigma)-(\sigma+\gamma)}\,, \quad n \geq 2. \tag{20}$$

Proof. Since $f \in \mathcal{T}Q(\alpha,\beta,\sigma,\gamma)$, Theorem 2 yields

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} [n(1+\sigma)-(\sigma+\gamma)] a_n \leqslant 1-\gamma.$$

Next, note that

$$\begin{aligned} &\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} [n(1+\sigma)-(\sigma+\gamma)] a_n \\ &\leqslant \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} [n(1+\sigma)-(\sigma+\gamma)] a_n. \end{aligned}$$

Therefore

$$a_n \leqslant \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\beta+n)} \frac{1-\gamma}{n(1+\sigma)-(\sigma+\gamma)}\,, \quad n \geq 2.$$ ∎

Corollary 4. *If $f \in \mathcal{T}Q(\alpha,\beta,\sigma,\gamma)$ and $|z| = r < 1$, then*

(i) $|Q_\beta^\alpha f(z)| \leqslant r + \dfrac{1-\gamma}{2+\sigma-\gamma}\, r^2,$

(ii) $|Q_\beta^\alpha f(z)| \geq r - \dfrac{1-\gamma}{2+\sigma-\gamma}\, r^2.$

Proof. First, it is obvious that

$$\begin{aligned} &\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \frac{\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)} (2+\sigma-\gamma) \sum_{n=2}^{\infty} a_n \\ &\leqslant \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} [n(1+\sigma)-(\sigma+\gamma)] a_n, \end{aligned}$$

and as $f \in \mathcal{T}Q(\alpha,\beta,\sigma,\gamma)$, using the inequality in Theorem 2 yields

$$\sum_{n=2}^{\infty} a_n \leqslant \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\beta+2)} \frac{1-\gamma}{2+\sigma-\gamma}. \tag{21}$$

From (8) with $|z| = r(r<1)$, we have

$$
\begin{aligned}
|Q_\beta^\alpha f(z)| &\leqslant r + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} a_n r^n \\
&\leqslant r + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \frac{\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)} \sum_{n=2}^{\infty} a_n r^2
\end{aligned}
$$

and

$$
\begin{aligned}
|Q_\beta^\alpha f(z)| &\geq r - \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} a_n r^n \\
&\geq r - \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \frac{\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)} \sum_{n=2}^{\infty} a_n r^2.
\end{aligned}
$$

Finally, using (21) in the above inequalities gives us both results (i) and (ii). ∎

We note that (i) and (ii) are sharp for the following function

$$
Q_\beta^\alpha f_2(z) = z - \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\beta+2)} \frac{1-\gamma}{2+\sigma-\gamma} z^2
$$

at $z = \pm ir, \pm r$.

Remark 1. By taking $\beta = \alpha$ and $\alpha = 1$ in (8), analogous results for $\mathcal{T}J(\alpha,\sigma)$ are also obtained.

3. Properties of $\mathcal{UCV}Q_T(\alpha,\beta,\sigma)$

Now, let us draw our attention to the following new subclasses of $\mathcal{UCV}(\alpha)$ and find their coefficient criterion.

Definition 6. *Let $\mathcal{UCV}Q_T(\alpha,\beta,\sigma)$, $\alpha > 0, \beta > -1$ and $0 \leqslant \sigma \leqslant 1$ be the class of functions $f \in \mathcal{T}$ which satisfy the condition*

$$
\left| \frac{z(Q_\beta^\alpha f(z))''}{(Q_\beta^\alpha f(z))'} \right| \leqslant \operatorname{Re}\left(1 + \frac{z(Q_\beta^\alpha f(z))''}{(Q_\beta^\alpha f(z))'} - \sigma\right), \quad z \in D \tag{22}
$$

where $Q_\beta^\alpha f$ is defined as in (3).

Definition 7. *Let $\mathcal{UCV}J_T(\alpha,\sigma)$, $\alpha > -1$ and $0 \leqslant \sigma \leqslant 1$ be the class of functions $f \in \mathcal{T}$ which satisfy the condition*

$$
\left| \frac{z(J_\alpha f(z))''}{(J_\alpha f(z))'} \right| \leqslant \operatorname{Re}\left(1 + \frac{z(J_\alpha f(z))''}{(J_\alpha f(z))'} - \sigma\right), \quad z \in D, \tag{23}
$$

where $J_\alpha f$ is defined as in (4).

Next, we give some results for $\mathcal{UCV}Q_T(\alpha,\beta,\sigma)$ as the following.

Theorem 3. *Let the functions $f \in \mathcal{T}$. Then*

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}(2n-1-\sigma)na_n \leqslant 1-\sigma \tag{24}$$

for some $\alpha > 0, \beta > -1$ *and* $0 \leqslant \sigma \leqslant 1$ *if and only if* $f \in \mathcal{UCVQ}_T(\alpha,\beta,\sigma)$.

Proof. First, we consider

$$\begin{aligned}&\left|\frac{z(Q^{\alpha}_{\beta}f(z))''}{(Q^{\alpha}_{\beta}f(z))'}\right| - \mathrm{Re}\left(\frac{z(Q^{\alpha}_{\beta}f(z))''}{(Q^{\alpha}_{\beta}f(z))'}\right) \leqslant 2\left|\frac{z(Q^{\alpha}_{\beta}f(z))''}{(Q^{\alpha}_{\beta}f(z))'}\right|\\
&\leqslant \frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}2n(n-1)|a_n||z|^{n-1}}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}n|a_n||z|^{n-1}}\\
&\leqslant \frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}2n(n-1)a_n}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}na_n},\end{aligned}$$

where $1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}n|a_n| > 0$ by getting use of Lemma 1.

The above expression is bounded by $1-\sigma$ if and only if (24) is satisfied. Consequently, we can write

$$\left|\frac{z(Q^{\alpha}_{\beta}f(z))''}{(Q^{\alpha}_{\beta}f(z))'}\right| \leqslant \mathrm{Re}\,\frac{z(Q^{\alpha}_{\beta}f(z))''}{(Q^{\alpha}_{\beta}f(z))'} + 1-\sigma.$$

which is equivalent to (22).

Conversely, if $f \in \mathcal{UCVQ}_T(\alpha,\beta,\sigma)$ and z is real, then Definition 6 yields

$$\begin{aligned}&1-\frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}n(n-1)a_nz^{n-1}}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}na_nz^{n-1}}-\sigma\\
&\geq \frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}n(n-1)a_nz^{n-1}}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}na_nz^{n-1}}.\end{aligned}$$

Let $z \to 1^-$ along the real axis, then we get

$$(1-\sigma) \geq \frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}2n(n-1)a_n}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}na_n}$$

or

$$\begin{aligned}&(1-\sigma)\Big(1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}na_n\Big)\\
&\geq \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}2n(n-1)a_n,\end{aligned}$$

which gives the required result. ■

Our assertion in Theorem 3 is sharp for functions of the form

$$F_n(z) = Q^\alpha_\beta f_n(z) = z - \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\beta+n)} \frac{1-\sigma}{n(2n-1-\sigma)}\, z^n, \quad n \geq 2 \tag{25}$$

which belong to the class $\mathcal{UCV}Q_T(\alpha,\beta,\sigma)$.

Corollary 5. *If* $f \in \mathcal{UCV}Q_T(\alpha,\beta,\sigma)$ *then*

$$a_n \leqslant \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\beta+n)} \frac{1-\sigma}{n(2n-1-\sigma)}\ , \quad n \geq 2. \tag{26}$$

Proof. Since $f \in \mathcal{UCV}Q_T(\alpha,\beta,\sigma)$, Theorem 3 gives

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} n(2n-1-\sigma) a_n \leqslant 1-\sigma.$$

Next, note that

$$\begin{aligned} &\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} n(2n-1-\sigma) a_n \\ &\leqslant \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} n(2n-1-\sigma) a_n. \end{aligned}$$

Therefore

$$a_n \leqslant \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\beta+n)} \frac{1-\sigma}{n(2n-1-\sigma)}\ , \quad n \geq 2.$$

Corollary 6. *If* $f \in \mathcal{UCV}Q_T(\alpha,\beta,\sigma)$ *and* $|z| = r < 1$, *then*

(i) $|Q^\alpha_\beta f(z)| \leqslant r + \dfrac{1-\sigma}{2(3-\sigma)}\, r^2$,

(ii) $|Q^\alpha_\beta f(z)| \geq r - \dfrac{1-\sigma}{2(3-\sigma)}\, r^2$.

Proof. First, it is obvious that

$$\begin{aligned} &\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \frac{\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)} 2(3-\sigma) \sum_{n=2}^{\infty} a_n \\ &\leqslant \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} n(2n-1-\sigma) a_n, \end{aligned}$$

and as $f \in \mathcal{UCV}Q_T(\alpha,\beta,\sigma)$, using the inequality in Theorem 3 yields

$$\sum_{n=2}^{\infty}\ a_n \leqslant \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\beta+2)} \frac{1-\sigma}{2(3-\sigma)}. \tag{27}$$

From (8) with $|z| = r (r < 1)$, we have

$$\begin{aligned} |Q^\alpha_\beta f(z)| &\leqslant r + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} a_n r^n \\ &\leqslant r + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \frac{\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)} \sum_{n=2}^{\infty} a_n r^2 \end{aligned}$$

and

$$|Q_\beta^\alpha f(z)| \geq r - \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} a_n r^n$$

$$\geq r - \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \frac{\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)} \sum_{n=2}^{\infty} a_n r^2.$$

Finally, using (27) in the above inequalities gives us both results (i) and (ii). ■

We note that (i) and (ii) are sharp for the following function

$$Q_\beta^\alpha f_2(z) = z - \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\beta+2)} \frac{1-\sigma}{2(3-\sigma)} z^2$$

at $z = \pm ir, \pm r$.

Definition 8. *Let $\mathcal{UCV}Q_T(\alpha, \beta, \sigma, \gamma)$ be the class of functions $f \in \mathcal{T}$ satisfying the condition*

$$Re\left(1 + \frac{z(Q_\beta^\alpha f(z))''}{(Q_\beta^\alpha f(z))'}\right) \geq \sigma \left|\frac{z(Q_\beta^\alpha f(z))''}{(Q_\beta^\alpha f(z))'}\right| + \gamma, \;\; \alpha > 0, \beta > -1, \; 0 \leqslant \sigma \leqslant 1, \; 0 \leqslant \gamma \leqslant 1. \tag{28}$$

where $Q_\beta^\alpha f$ is defined as in (3).

We write $\mathcal{UCV}Q_T(\alpha, \beta, 1, \gamma) = \mathcal{UCV}Q_T(\alpha, \beta, \gamma)$.

Theorem 4. *Let the functions $f \in \mathcal{T}$. A function $f \in \mathcal{UCV}Q_T(\alpha, \beta, \sigma, \gamma)$ for some $\alpha > 0, \beta > -1, 0 \leqslant \sigma \leqslant 1$ and $0 \leqslant \gamma \leqslant 1$ if and only if*

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} n[n(1+\sigma) - (\sigma+\gamma)] a_n \leqslant 1 - \gamma. \tag{29}$$

Proof. In view of Definition 8, it suffices to show that

$$\sigma \left|\frac{z(Q_\beta^\alpha f(z))''}{(Q_\beta^\alpha f(z))'}\right| \leqslant \mathrm{Re}\left(\frac{z(Q_\beta^\alpha f(z))''}{(Q_\beta^\alpha f(z))'} + 1\right) - \gamma. \tag{30}$$

Then, we have

$$\sigma \left|\frac{z(Q_\beta^\alpha f(z))''}{(Q_\beta^\alpha f(z))'}\right| - \mathrm{Re}\left(\frac{z(Q_\beta^\alpha f(z))''}{(Q_\beta^\alpha f(z))'}\right) \leqslant (\sigma+1) \left|\frac{z(Q_\beta^\alpha f(z))''}{(Q_\beta^\alpha f(z))'}\right|$$

$$\leqslant \frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} (\sigma+1) n(n-1) |a_n| |z|^{n-1}}{1 - \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} n |a_n| |z|^{n-1}}$$

$$\leqslant \frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} (\sigma+1) n(n-1) a_n}{1 - \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} n a_n}.$$

The above expression is bounded by $1-\gamma$ if and only if (29) is satisfied.

Conversely, if $f \in \mathcal{UCV}Q_T(\alpha,\beta,\sigma,\gamma)$ and z is real, we get

$$1-\frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}n(n-1)a_nz^{n-1}}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}na_nz^{n-1}}$$
$$\geq \sigma\frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}n(n-1)a_nz^{n-1}}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)n}a_nz^{n-1}}+\gamma.$$

Let $z \to 1^-$ along the real axis, which gives

$$(1-\gamma)\geq(1+\sigma)\frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}n(n-1)a_n}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}na_n},$$

and equivalent to

$$(1-\gamma)\left(1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}na_n\right)$$
$$\geq(1+\sigma)\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}n(n-1)a_n.$$

Thus, the proof is complete. ∎

Our assertion in Theorem 4 is sharp for functions of the form

$$F_n(z)=Q_\beta^\alpha f_n(z)=z-\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\frac{\Gamma(\alpha+\beta+n)}{\Gamma(\beta+n)}\frac{1-\gamma}{n[n(1+\sigma)-(\sigma+\gamma)]}\,z^n,\quad n\geq 2 \tag{31}$$

which belong to the class $\mathcal{UCV}Q_T(\alpha,\beta,\sigma,\gamma)$.

Corollary 7. *If* $f\in\mathcal{UCV}Q_T(\alpha,\beta,\sigma,\gamma)$ *then*

$$a_n\leqslant\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\frac{\Gamma(\alpha+\beta+n)}{\Gamma(\beta+n)}\frac{1-\gamma}{n[n(1+\sigma)-(\sigma+\gamma)]}\;,\quad n\geq 2. \tag{32}$$

Proof. Since $f\in\mathcal{UCV}Q_T(\alpha,\beta,\sigma,\gamma)$, Theorem 4 gives

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}n[n(1+\sigma)-(\sigma+\gamma)]a_n\leqslant 1-\gamma.$$

Next, note that

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}n[n(1+\sigma)-(\sigma+\gamma)]a_n$$
$$\leqslant\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}n[n(1+\sigma)-(\sigma+\gamma)]a_n.$$

Therefore

$$a_n \leqslant \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+n)}{\Gamma(\beta+n)} \frac{1-\gamma}{n[n(1+\sigma)-(\sigma+\gamma)]}, \quad n \geq 2.$$

Corollary 8. *If $f \in \mathcal{UCV}Q_T(\alpha,\beta,\sigma,\gamma)$ and $|z| = r < 1$, then*
(i) $|Q^\alpha_\beta f(z)| \leqslant r + \frac{1-\gamma}{2(2+\sigma-\gamma)}\, r^2$
(ii) $|Q^\alpha_\beta f(z)| \geq r - \frac{1-\gamma}{2(2+\sigma-\gamma)}\, r^2$.
The results are sharp.

Proof. First, it is obvious that

$$\begin{aligned}&\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \frac{\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)} 2(2+\sigma-\gamma)a_n \\ &\leqslant \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} n[n(1+\sigma)-(\sigma+\gamma)]a_n,\end{aligned}$$

and as $f \in \mathcal{UCV}Q_T(\alpha,\beta,\sigma,\gamma)$, using the inequality in Theorem 4 yields

$$\sum_{n=2}^{\infty} a_n \leqslant \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\beta+2)} \frac{1-\gamma}{2(2+\sigma-\gamma)}. \tag{33}$$

From (8) with $|z| = r(r < 1)$, we have

$$\begin{aligned}|Q^\alpha_\beta f(z)| &\leqslant r + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} a_n r^n \\ &\leqslant r + \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \frac{\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)} \sum_{n=2}^{\infty} a_n r^2\end{aligned}$$

and

$$\begin{aligned}|Q^\alpha_\beta f(z)| &\geq r - \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \sum_{n=2}^{\infty} \frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)} a_n r^n \\ &\geq r - \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)} \frac{\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)} \sum_{n=2}^{\infty} a_n r^2.\end{aligned}$$

Finally, using (33) in the above inequalities gives us both results (i) and (ii).

We note that (i) and (ii) are sharp for the following function

$$Q^\alpha_\beta f_2(z) = z - \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\beta+2)} \frac{1-\gamma}{2(2+\sigma-\gamma)}\, z^2$$

at $z = \pm ir, \pm r$. ∎

Definition 9. *Let*

$$\begin{aligned}&\mathcal{C}Q_T(\alpha,\beta,\sigma) \\ &= \left\{ f \in \mathcal{T};\left| \frac{z(Q^\alpha_\beta f(z))''}{(Q^\alpha_\beta f(z))'} + 1 - \sigma \right| \leqslant \mathrm{Re}\left\{ \frac{z(Q^\alpha_\beta f(z))''}{(Q^\alpha_\beta f(z))'} + 1 + \sigma \right\},\ \alpha > 0,\ \beta > -1,\ \sigma > 0 \right\}\end{aligned}$$

Theorem 5. *Let $f \in \mathcal{T}$. Then $f \in \mathcal{C}Q_T(\alpha,\beta,\sigma)$ if and only if*

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}n(n-1+\sigma)a_n \leqslant \sigma. \tag{34}$$

Proof. By Definition 9, it is sufficient to prove the inequality

$$\left|1+\frac{z(Q^{\alpha}_{\beta}f(z))''}{(Q^{\alpha}_{\beta}f(z))'}-\sigma\right| \leqslant \mathrm{Re}\left\{\frac{z(Q^{\alpha}_{\beta}f(z))''}{(Q^{\alpha}_{\beta}f(z))'}+1+\sigma\right\}$$

or equivalently

$$\left|1+\frac{z(Q^{\alpha}_{\beta}f(z))''}{(Q^{\alpha}_{\beta}f(z))'}-\sigma\right| - \mathrm{Re}\left\{1+\frac{z(Q^{\alpha}_{\beta}f(z))''}{(Q^{\alpha}_{\beta}f(z))'}-\sigma\right\} \leqslant 2\sigma.$$

We have

$$\begin{aligned}
&\left|1+\frac{z(Q^{\alpha}_{\beta}f(z))''}{(Q^{\alpha}_{\beta}f(z))'}-\sigma\right| - \mathrm{Re}\left\{1+\frac{z(Q^{\alpha}_{\beta}f(z))''}{(Q^{\alpha}_{\beta}f(z))'}-\sigma\right\}\\
&\leqslant 2\left|1+\frac{z(Q^{\alpha}_{\beta}f(z))''}{(Q^{\alpha}_{\beta}f(z))'}-\sigma\right|\\
&\leqslant 2\left\{\frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}n(n-1)|a_n||z|^{n-1}}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}n|a_n||z|^{n-1}}+1-\sigma\right\}\\
&\leqslant 2\left\{\frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}n(n-1)a_n}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}na_n}+1-\sigma\right\}.
\end{aligned}$$

The above expression is bounded by 2σ if and only if (34) is satisfied.

Conversely, if $f \in \mathcal{C}Q_T(\alpha,\beta,\sigma)$ and z is real, we get

$$\begin{aligned}
&1-\frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}n(n-1)a_nz^{n-1}}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}na_nz^{n-1}}\\
&\geq \frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}n(n-1)a_nz^{n-1}}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}na_nz^{n-1}}+1-2\sigma.
\end{aligned}$$

Let $z \to 1^-$ along the real axis, which gives

$$\frac{\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}n(n-1)a_n}{1-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}na_n} \leqslant \sigma$$

and gives the required result. ∎

Corollary 9. *If $f \in \mathcal{C}Q_T(\alpha,\beta,\sigma)$ and $|z| = r < 1$, then for $0 < \sigma \leqslant 1$*

(i) $|Q^{\alpha}_{\beta}f(z)| \leqslant r + \frac{\sigma}{2(1+\sigma)}\, r^2$,

(i) $|Q^{\alpha}_{\beta}f(z)| \geq r - \frac{\sigma}{2(1+\sigma)}\, r^2$.

Proof. First, it is obvious that

$$\begin{aligned}&\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\frac{\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)}2(1+\sigma)a_n\\&\leqslant\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}n(n-1+\sigma)a_n,\end{aligned}$$

and as $f\in\mathcal{C}Q_T(\alpha,\beta,\sigma)$, using the inequality in Theorem 5 yields

$$\sum_{n=2}^{\infty}a_n\leqslant\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\beta+2)}\frac{\sigma}{2(1+\sigma)}. \tag{35}$$

From (8) with $|z|=r(r<1)$, we have

$$\begin{aligned}Q_\beta^\alpha f(z)|&\leqslant r+\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}a_nr^n\\&\leqslant r+\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\frac{\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)}\sum_{n=2}^{\infty}a_nr^2\end{aligned}$$

and

$$\begin{aligned}|Q_\beta^\alpha f(z)|&\geq r-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\sum_{n=2}^{\infty}\frac{\Gamma(\beta+n)}{\Gamma(\alpha+\beta+n)}a_nr^n\\&\geq r-\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\beta+1)}\frac{\Gamma(\beta+2)}{\Gamma(\alpha+\beta+2)}\sum_{n=2}^{\infty}a_nr^2.\end{aligned}$$

Finally, using (35) in the above inequalities gives us both results (i) and (ii). ■

We note that (i) and (ii) are sharp for the following function

$$Q_\beta^\alpha f_2(z)=z-\frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\beta+2)}\frac{\sigma}{2(1+\sigma)}\ z^2$$

at $z=\pm ir,\pm r$.

Remark 2. By taking $\beta=\alpha$ and $\alpha=1$ in (8), analogous results for $\mathcal{UCV}J_T(\alpha,\sigma)$ are also obtained.

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