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A Remark on the Dirichlet Problem*

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Abstract. Given a positive measure μ on a strongly pseudoconvex domain in $\mathbf{C}^n$. We study the Dirichlet problem $(dd^c u)^n = \mu$ in a new class of plurisubharmonic function. This class includes the classes $\mathcal{E}_p$ $(p \geq 1)$ introduced by Cegrell in [5].

1. Introduction.

Let Ω be a bounded domain in $\mathbf{C}^n$. By $PSH(\Omega)$ we denote the set of plurisubharmonic (psh) functions on Ω. By the fundamental work of Bedford and Taylor [1, 2], the complex Monge-Ampere operator $(dd^c)^n$ is well defined over the class $PSH(\Omega) \cap L^\infty_{loc}(\Omega)$ of locally bounded psh functions on Ω, more precisely, if $u \in PSH(\Omega) \cap L^\infty_{loc}(\Omega)$ is a positive Borel measure. Furthermore, this operator is continuous with respect to increasing and decreasing sequences. Later, Demailly has extended the domain of definition of the operator $(dd^c u)^n$ to the class of psh functions which are locally bounded near $\partial\Omega$. Recently in [5, 6], Cegrell introduced the largest class of upper bounded psh functions on a bounded hyperconvex domain Ω such that the operator $(dd^c u)^n$ can be defined on it. In these papers, he also studied the Dirichlet problems for the classes $\mathcal{F}_p$ (see Sec. 2 for details). The aim of our work is to investigate the Dirichlet problem for a new class of psh function. This class consist, in particular, the sum of a function in the class $\mathcal{E}_p$ and a function in $\mathcal{B}^a_{loc}$ (see Sec. 2 for the definitions of these classes).

Now we are able to formulate the main result of our work

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Main theorem.

(i) *Let Ω be a bounded strongly pseudoconvex domain in $\mathbf{C}^n$ and let μ be a positive measure on Ω, $h \in C(\partial\Omega)$ such that there exists $v \in \widehat{\mathcal{E}_p + \mathcal{B}^a_{loc}}$ (resp. $\widehat{\mathcal{F}_p + \mathcal{B}^a_{loc}}$) with $(dd^c v)^n \geq \mu$*
Then there exists $u \in \widehat{\mathcal{E}_p + \mathcal{B}^a_{loc}}$ (resp. $\widehat{\mathcal{F}_p + \mathcal{B}^a_{loc}}$) such that $(dd^c u)^n = \mu$ and $\lim\limits_{z\to\xi} u(z) = h(\xi)$, $\forall \xi \in \partial\Omega$.

(ii) *There exists $f \in L^1(\Omega)$ such that there exists no function $u \in \widehat{\mathcal{E}_p + \mathcal{B}^a_{loc}}$ which satisfying $f d\lambda \leq (dd^c u)^n$.*

For the definitions of $\widehat{\mathcal{E}_p + \mathcal{B}^a_{loc}}$ and $\widehat{\mathcal{F}_p + \mathcal{B}^a_{loc}}$ see Sec. 2.

Note that the main theorem for the subclass $\mathcal{B}$ of $\mathcal{B}^a_{loc}$ consisting of psh functions which are bounded near $\partial\Omega$ was proved by Xing in [13] and for the classes $\mathcal{E}_p$ and $\mathcal{F}_p$, $p \geq 1$ by Cegrell in [5].

The key element in the proof of our main theorem is a comparison principle (Theorem 3.1), which is an extension of Lemma 4.4, Theorem 4.5 in [5].

2. Preliminaries

In this section we recall some elements and results of pluripotential theory that will be used through out the paper. All this can be found in [2, 3, 5, 6, 11...].

2.0. Unless otherwise specified, Ω will be a bounded hyperconvex domain in $\mathbf{C}^n$ meaning that there exists a negative exhaustive psh function for Ω .

2.1. Let Ω be a bounded domain in $\mathbf{C}^n$. The C_n-capacity in the sense of Bedford and Taylor on Ω is the set function given by

$$C_n(E) = C_n(E,\Omega) = \sup\Big\{ \int\limits_E (dd^c u)^n \ : \ u \in PSH(\Omega),\ -1 \leq u \leq 0 \Big\}$$

for every Borel set E in Ω.

2.2. According to Xing (see [13]), a sequence of positive measures $\{\mu_j\}$ on Ω is called uniformly absolutely continuous with respect to C_n in a subset E of Ω if

$$\forall \epsilon > 0,\ \exists \delta > 0 \ : \ F \subset E,\ C_n(F) < \delta \Rightarrow \mu_j(F) < \epsilon, \quad \forall j \geq 1$$

We write $\mu_j \ll C_n$ in E uniformly for $j \geq 1$.

2.3. By $\mathcal{B}^a_{loc} = \mathcal{B}^a_{loc}(\Omega)$ we denote the set of upper bounded psh functions u which are locally bounded near $\partial\Omega$ such that $(dd^c u)^n \ll C_n$ in every $E \subset\subset \Omega$.

2.4. The following classes of psh functions were introduced by Cegrell in [5] and [6]

$$\mathcal{E}_0 = \mathcal{E}_0(\Omega) = \Big\{ \varphi \in PSH(\Omega) \cap L^\infty(\Omega) : \lim_{z\to\partial\Omega} \varphi(z) = 0,\ \int\limits_\Omega (dd^c\varphi)^n < +\infty \Big\},$$

$$\mathcal{E}_p = \mathcal{E}_p(\Omega) = \Big\{ \varphi \in PSH(\Omega) : \exists \mathcal{E}_0 \ni \varphi_j \searrow \varphi,\ \sup_{j\geq 1} \int\limits_\Omega (-\varphi_j)^p (dd^c\varphi_j)^n < +\infty \Big\},$$

$$\mathcal{F}_p = \mathcal{F}_p(\Omega) = \Big\{\varphi \in PSH(\Omega) : \exists \mathcal{E}_0 \ni \varphi_j \searrow \varphi,\ \sup_{j\geq 1} \int_\Omega (-\varphi_j)^p (dd^c\varphi_j)^n, < +\infty, \sup_{j\geq 1} \int_\Omega (dd^c\varphi_j)^n < +\infty\Big\}$$

$\mathcal{E} = \mathcal{E}(\Omega) = \Big\{\varphi \in PSH(\Omega) : \forall z_0 \in \Omega\ \exists$ a neighborhood $\omega \ni z_0, \mathcal{E}_0 \ni \varphi_j \searrow \varphi$ on $\omega,\ \sup_{j\geq 1} \int_\Omega (dd^c\varphi_j)^n < +\infty\}$.

The following inclusions are obvious $\mathcal{E}_0 \subset \mathcal{F}_p \subset \mathcal{E}_p \subset \mathcal{E}$. It is also known that these inclusion are strict (see [5, 6]).

The interesting theorem below was proved by Cegrell (see [6])

Theorem 2.5. *The class $\mathcal{E}$ has the following properties*

1. *$\mathcal{E}$ is a convex cone.*
2. *If $u \in \mathcal{E},\ v \in PSH^-(\Omega) = \{\varphi \in PSH(\Omega) : \varphi \leq 0\}$, then* $\max(u, v) \in \mathcal{E}$.
3. *If $u \in \mathcal{E},\ PSH(\Omega) \cap L^\infty_{loc}(\Omega) \ni u_j \searrow u$, then $(dd^c u_j)^n$ is weakly convergent.* *Conversely if $\mathcal{K} \subset PSH^-(\Omega)$ satisfies* 2 *and* 3, *then $\mathcal{K} \subset \mathcal{E}$*

Since $\mathcal{B}^-_{loc} = \mathcal{B}^a_{loc} \cap PSH^-(\Omega)$ satisfies 2 and 3 we have by [8] $\mathcal{B}^-_{loc} \subset \mathcal{E}$.

2.6. Cegrell also studied the following Dirichlet problem: Given μ a positive measure on Ω, find $u \in \mathcal{F}_p$ such that $(dd^c u)^n = \mu$. He gave a necessary and sufficient condition for this problem to have a solution (Theorem 5.2 in [5]).

2.7. We define

$$\widehat{\mathcal{E}_p + \mathcal{B}^a_{loc}} = \big\{u \in PSH(\Omega) : \exists\ \varphi \in \mathcal{E}_p,\ f \in \mathcal{B}^a_{loc} : \varphi + f \leq u \leq \sup_\Omega u < +\infty\big\},$$
$$\widehat{\mathcal{F}_p + \mathcal{B}^a_{loc}} = \big\{u \in PSH(\Omega) : \exists\ \varphi \in \mathcal{F}_p,\ f \in \mathcal{B}^a_{loc} :\ \varphi + f \leq u \leq \sup_\Omega u < +\infty\big\}.$$

It follows that if $\varphi + f \leq u < \sup_\Omega u < +\infty,\ \varphi \in \mathcal{E},\ f \in \mathcal{B}^a_{loc}$ then

$$u - c = \max(u - c, \varphi + f - c) \in \mathcal{E}, \ \text{ because }\ \varphi + (f - c) \in \mathcal{E},$$

where $c = \max(\sup_\Omega f,\ \sup_\Omega u)$.

Thus we can define $(dd^c u)^n$ for $u \in \widehat{\mathcal{E}_p + B^a_{loc}}$.

2.8. The aim of this work is to study a Dirichlet problem similar to the one considered by Cegrell but for the classes $\widehat{\mathcal{E}_p + \mathcal{B}^a_{loc}}$ and $\widehat{\mathcal{F}_p + \mathcal{B}^a_{loc}}$. Namely, given a positive measure μ on Ω and $h \in C(\partial\Omega)$, find $u \in \widehat{\mathcal{E}_p + \mathcal{B}^a_{loc}}$ (resp. $\widehat{\mathcal{F}_p + \mathcal{B}^a_{loc}}$) such that $(dd^c u)^n = \mu$ and $\overline{\lim}_{z\to\xi} u(z) = h(\xi)\ \forall \xi \in \partial\Omega$.

2.9. Let μ be a positive measure on Ω and $h \in C(\partial\Omega)$. Following Cegrell, we define

$$B(\mu,h)=\{v\in PSH(\Omega)\cap L^\infty_{loc}(\Omega)\ :\ (dd^cv)^n\geq\mu,\ \overline{\lim_{z\to\xi}}\,v(z)\leq h(\xi)\},$$
$$U(\mu,h)(z)=\sup\{v(z)\ :\ v\in B(\mu,h)\},\ z\in\Omega.$$

Observe that $B(\mu,h)\neq\emptyset$ implies that μ vanishes on pluripolar sets. The function $U(\mu,h)$ plays a crucial role in solving the Dirichlet problem.

3. The Comparison Principle for $\widehat{\mathcal{E}_p+\mathcal{B}^a_{loc}}$

In order to prove the main theorem, in this section we prove the following comparison principle

Theorem 3.1. *Let u,v be functions in $\widehat{\mathcal{E}_p+\mathcal{B}^a_{loc}}$ satisfying*

$$\underline{\lim}_{z\to\partial\Omega}[u(z)-v(z)]\geq 0.$$

Then

$$\int_{\{u<v\}}(dd^cv)^n\leq\int_{\{u<v\}}(dd^cu)^n.$$

We need the following result

Lemma 3.2. *Let $PSH(\Omega)\cap L^\infty(\Omega)\ni u_j\searrow u$. Assume that*

$$\lim_{s\to+\infty}s^nC_n(\{u<-s\}\cap D)=0,\quad\forall D\subset\subset\Omega.$$

Then $(dd^cu_j)^n\ll C_n$ in every $D\subset\subset\Omega$ uniformly for $j\geq 1$.

Proof. Given $D\subset\subset\Omega$. Without loss of generality we may assume that D is hyperconvex and $u_j\leq 0$ on D. By [6] for each $j\geq 1$ there exists $u_j^k\in PSH(D)\cap C(\bar D)$ such that $u_j^k\searrow u_j$ on D and $u_j^k=0$ on ∂D. As in [9] for every $k,j\geq 1$ and $s>0$ put

$$D_{kj}(s)=\{u_j^k<-s\}\cap D,\ D_j(s)=\{u_j<-s\}\cap D,\ D(s)=\{u<-s\}\cap D,$$
$$a_{kj}(s)=C_n(D_{kj}(s)),\ a_j(s)=C_n(D_j(s)),\ a(s)=C_n(D(s)),$$
$$b_{kj}(s)=\int_{D_{kj}(s)}(dd^cu_j^k)^n,\ b_j(s)=\int_{D_j(s)}(dd^cu_j)^n.$$

For $0<s<t$ we have $\max(u_j^k,-t)=u_j^k$ on $\{u_j^k>-t\}$ an open neighborhood of $\partial D_{kj}(s)$. It follows that

$$a_{kj}(s)\geq t^{-n}\int_{D_{kj}(s)}(dd^c\max(u_j^k,-t))^n=t^{-n}\int_{D_{kj}(s)}(dd^cu_j^k)^n=t^{-n}b_{kj}(s).$$

Letting $t \searrow s$ we get

$$s^n a_{kj}(s) \geq b_{kj}(s) \text{ for } k, j \geq 1 \text{ and } s > 0. \tag{1}$$

Given $\epsilon > 0$. By the hypothesis there exists $s_0 > 0$ such that

$$s_0^n a(s_0) < \epsilon. \tag{2}$$

Let $E \subset D$ with $C_n(E) < \frac{\epsilon}{s_0^n}$. Take an open neighborhood G of E such that $C_n(G) < \frac{\epsilon}{s_0^n}$. Since $(dd^c u_j^k)^n \to (dd^c u_j)^n$ weakly as $k \to \infty$ we have

$$\begin{aligned}\int\limits_E (dd^c u_j)^n \leq \int\limits_G (dd^c u_j)^n &\leq \varliminf_{k\to\infty} \int\limits_G (dd^c u_j^k)^n \\ &\leq \varliminf_{k\to\infty} \Big[\int\limits_{D_{kj}(s_0)} (dd^c u_j^k)^n + \int\limits_{G\setminus D_{kj}(s_0)} (dd^c u_j^k)^n \Big] \\ &\leq \varliminf_{k\to\infty} [s_0^n a_{kj}(s_0) + s_0^n C_n(G)] \leq s_0^n a(s_0) + \epsilon < 2\epsilon\end{aligned}$$

for $j \geq 1$. Hence $(dd^c u_j)^n \ll C_n$ in D uniformly for $j \geq 1$.

Proof of Theorem 3.1. We may assume that $u, v \leq 0$ and $\varliminf_{z\to\partial\Omega} [u(z) - v(z)] > \delta > 0$. By hypothesis $u, v \in \widehat{\mathcal{E}_p + \mathcal{B}^a_{loc}}$ it is easy to find $\varphi \in \mathcal{E}_p,\ g \in \mathcal{B}^-_{loc}$ such that $\varphi + g \leq \min(u, v)$. Let $\varphi_j \searrow \varphi$ be a sequence decreasing to φ as in the definition of $\mathcal{E}_p$. For each $j \geq 1$ put

$$g_j = \max(g, -j),\ u_j = \max(u, \varphi_j + g_j),\ v_j = \max(v, \varphi_j + g_j).$$

It follows that g_j, u_j, v_j are bounded and $g_j \searrow g,\ u_j \searrow u,\ v_j \searrow v$. By the comparison principle for bounded psh functions we have

$$\int\limits_{\{u_j<v_k\}} (dd^c v_k)^n \leq \int\limits_{\{u_j<v_k\}} (dd^c u_j)^n$$

for $k \geq j \geq 1$.

On the other hand, since

$$s^n C_n(\{u < -s\} \cap D) \leq s^n C_n(\{\varphi < -\frac{s}{2}\} \cap D) + s^n C_n\Big(\Big\{g < -\frac{s}{2}\Big\} \cap D\Big) \to 0$$

as $s \to +\infty$ (see [5, 9])

By Lemma 3.2 $(dd^c u_j)^n + (dd^c v_j)^n \ll C_n$ in every $D \subset\subset \Omega$ uniformly for $j \geq 1$. Thus by the quasicontinuity of psh functions as Theorem 2.2.6 in [4] we obtain

$$\int\limits_{\{u<v\}} (dd^c v)^n \leq \int\limits_{\{u\leq v\}} (dd^c u)^n.$$

By replacing u by $u + \delta,\ \delta > 0$ and then let $\delta \searrow 0$, we have

$$\int\limits_{\{u<v\}} (dd^c v)^n \leq \int\limits_{\{u<v\}} (dd^c u)^n.$$

This is the desired conclusion.

From Theorem 3.1, as Corollary 2.2.8 in [4], we get the following dominant principle.

Corollary 3.3. *Assume that u and v are as in Theorem* 3.1 *and* $(dd^c u)^n \leq (dd^c v)^n$. *Then* $u \geq v$.

4. Proof of the Main Theorem

(i) We can assume $v \leq 0$. Since $(dd^c v)^n$ vanishes on every pluripolar set in Ω, by Theorem 6.3 in [5] we can find $\psi \in \mathcal{E}_0$ and $0 \leq f \in L^1_{loc}((dd^c\psi)^n)$ such that $\mu = f(dd^c\psi)^n$. Put $\mu_k = \min(f,k)(dd^c\psi)^n$. Then $\mu_k \leq (dd^c k^{\frac{1}{n}}\psi)^n$. By Theorem 2 in [13] there exists $\omega_k \in \mathcal{E}_0$ such that $(dd^c\omega_k)^n = \mu_k$. The comparison principle implies that $0 \geq \omega_k \searrow \omega \geq v$. Hence $\omega \in \widehat{\mathcal{E}_p + \mathcal{B}^a_{loc}}$ and $(dd^c\omega)^n = \mu$. We show that $\overline{\lim\limits_{z\to\xi}}\,\omega(z) = 0$ for $\xi \in \partial\Omega$. Assume the contrary, then $\overline{\lim\limits_{z\to\xi_0}}\,\omega(z) < -\epsilon$ for some $\xi_0 \in \partial\Omega,\ \epsilon > 0$. Take $\delta > 0$ such that $\omega(z) < -\epsilon$ for $z \in B(\xi_0,\delta)\cap\Omega$. Let $\tau \in C(\partial\Omega)$ such that $\tau|_{B(\xi_0,\frac{\delta}{2})\cap\partial\Omega} = \epsilon$, $\mathrm{supp}\tau \subset B(\xi_0,\delta)\cap\partial\Omega$. By [2] there exists $\phi \in PSH(\Omega)\cap C(\bar{\Omega})$ such that $(dd^c\phi)^n = 0$ and $\phi|_{\partial\Omega} = \tau$. Since $\underline{\lim\limits_{z\to\xi}}[\omega_k(z) - (\omega(z)+\phi(z))] \geq 0$ for $\xi \in \partial\Omega$ and $(dd^c\omega_k)^n = \mu_k \leq \mu = (dd^c\omega)^n \leq (dd^c(\omega+\phi))^n$, we have $\omega_k \geq \omega + \phi$ on Ω for $k \geq 1$. Thus $\omega \geq \omega + \phi$ on Ω. Hence $\phi \leq 0$ on $\Omega\backslash\{\omega = -\infty\}$. Since ϕ is plurisubharmonic, $\phi \leq 0$ on Ω. This is impossible, because $\phi(\xi) = \tau(\xi) = \epsilon$ for $\xi \in B(\xi_0,\frac{\delta}{2})\cap\partial\Omega$. Hence $\overline{\lim\limits_{z\to\xi}}\,\omega(z) = 0$ for $\xi \in \partial\Omega$. From the relations

$$\begin{cases} U((dd^c(\omega_k + U(0,h)))^n, h) = \omega_k + U(0,h), \\ (dd^c(\omega + U(0,h)))^n \geq \mu_k, \end{cases}$$

and from Theorem 8.1 in [5] it follows that

$$\begin{cases} (dd^c U(\mu_k,h))^n = \mu_k, \\ U(0,h) \geq U(\mu_k,h) \geq \omega_k + U(0,h). \end{cases}$$

Theorem 3.1 implies that $U(\mu_k,h) \searrow u \in \widehat{\mathcal{E}_p + \mathcal{B}^a_{loc}}$ with $(dd^c u)^n = \mu$ and $U(0,h) \geq u \geq \omega + U(0,h)$. Thus for $\xi \in \partial\Omega$ we have

$$\begin{aligned} h(\xi) = &\ \overline{\lim_{z\to\xi}}\,U(0,h) \geq \overline{\lim_{z\to\xi}}\,u(z) \geq \overline{\lim_{z\to\xi}}[\omega(z) + U(0,h)(z)] \\ = &\ \overline{\lim_{z\to\xi}}\,\omega(z) + \lim_{z\to\xi} U(0,h)(z) = h(\xi). \end{aligned}$$

Consequently $u \in \widehat{\mathcal{E}_p + \mathcal{B}^a_{loc}}$ such that $(dd^c u)^n = \mu$ and $\overline{\lim\limits_{z\to\xi}}\,u(z) = h(\xi)\ \forall \xi \in \partial\Omega$.

(ii) Let $\{\Omega_j\}$ be an increasing exhaustion sequence of strongly pseudoconvex subdomains of Ω. For each $j \geq 1$ take a sequence of distinguished points $z_{jm} \subset \Omega_j \backslash \overline{\Omega}_{j-1}$ converging to $\xi_j \in \partial\Omega_j$ as $m \to \infty$ and a sequence $s_j \searrow 0$ such that $B(z_{jm}, s_{jm}) \subset \Omega_j \backslash \overline{\Omega}_{j-1}$ and $B(z_{jm}, s_{jm}) \cap B(z_{jt}, s_{jt}) = \emptyset$ for $m \neq t$. Let $a_{jm} > 0$ with $\sum\limits_{j,m=1}^{\infty} a_{jm} < \infty$. Put

$$f = \sum_{j,m \geq 1} \frac{a_{jm}}{d_n r_{jm}^{2n}} \chi_{B(z_{jm}, r_{jm})},$$

where $0 < r_{jm} < s_{jm}$ are chosen such that

$$\frac{1}{a_{jm}} (C_n(B(z_{jm}, r_{jm}), \Omega))^{\frac{p}{n+p}} \to 0 \text{ as } m \to \infty,$$

for $j \geq 1$ and d_n is the volume of the unit ball in $\mathbf{C}^n$.

Assume that $f d\lambda \leq (dd^c u)^n$ for some $u \in \widehat{\mathcal{E}_p + \mathcal{B}^a_{loc}}$. Take $\varphi \in \mathcal{E}_p,\ g \in \mathcal{B}^a_{loc}$ such that $\varphi + g \leq u \leq \sup\limits_{\Omega} u < +\infty$. We may assume that g and u are negative. Let $j_0 \geq 2$ and $M > 0$ such that $g > -M$ on $\Omega_{j_0} \backslash \overline{\Omega}_{j_0 - 1}$.
Put

$$\tilde{g} = \max(g, A h_{\Omega_{j_0}}) \text{ where } A = -\frac{M}{\sup\limits_{\bar{\Omega}_{j_0}} h_{\Omega_{j_0}}} > 0.$$

It follows that $\tilde{g} \in \mathcal{E}_0,\ \tilde{g} = g$ on $\Omega_{j_0} \backslash \overline{\Omega}_{j_0-1}$.

Let $\tilde{u} = \max(u, \varphi + \tilde{g})$. Since $\varphi + \tilde{g} \leq \tilde{u} \leq 0$ and $\varphi + \tilde{g} \in \mathcal{E}_p + \mathcal{E}_0 = \mathcal{E}_p$, by [5] we have $\tilde{u} \in \mathcal{E}_p$.
Moreover $\tilde{u} = u$ on $\Omega_{j_0} \backslash \overline{\Omega}_{j_0-1}$. Thus for $B_m = B(z_{j_0 m}, r_{j_0 m})$ we have

$$a_{j_0 m} = \int\limits_{B_m} f d\lambda = \int\limits_{B_m} (dd^c u)^n = \int\limits_{B_m} (dd^c \tilde{u})^n.$$

Let $\tilde{u}_k \searrow \tilde{u}$ as in the definition of $\mathcal{E}_p$. Then $(dd^c \tilde{u}_k)^n \to (dd^c \tilde{u})^n$ weakly (see [5]). Applying the Holder inequality (see [7]) we have

$$\begin{aligned}
a_{j_0 m} &= \int\limits_{B_m} (dd^c \tilde{u})^n \leq \varliminf_{k \to \infty} \int\limits_{B_m} (dd^c \tilde{u}_k)^n \\
&= \varliminf_{k \to \infty} \int\limits_{B_m} (-h_{B_m})^p (dd^c \tilde{u}_k)^n \\
&\leq \alpha_1 \varliminf_{k \to \infty} [\int\limits_{\Omega} (-h_{B_m})^p (dd^c h_{B_m})^n]^{\frac{p}{n+p}} [\int\limits_{\Omega} (-\tilde{u}_k)^p (dd^c \tilde{u}_k)^n]^{\frac{n}{n+p}} \\
&\leq \alpha_2 [\int\limits_{\Omega} (dd^c h_{B_m})^n]^{\frac{p}{n+p}} \\
&= \alpha_2 [C_n(B_m, \Omega)]^{\frac{p}{n+p}},
\end{aligned}$$

where $\alpha_2 = \alpha_1 [\sup\limits_{k \geq 1} \int\limits_{\Omega} (-\tilde{u}_k)^p (dd^c \tilde{u}_k)^n]^{\frac{n}{n+p}} < +\infty$. This is impossible, because

$$\lim_{m\to\infty} \frac{[C_n(B_m,\Omega)]^{\frac{p}{p+n}}}{a_{j_0 m}} = 0.$$

Remark. Using Theorem 7.5 in [1] we can find $u \in \mathcal{F}^a$ such that $(dd^c u)^n = f d\lambda$ where f is constructed as in (ii). Hence, there exists a function u in $\mathcal{F}^a \setminus (\widehat{\xi_p + B^a_{loc}})$.

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