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On the Parametric Affine Variational Inequality Approach to Linear Fractional Vector Optimization Problems

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Abstract. Yen and Phuong (2000) have shown that the efficient solution set of a linear fractional vector optimization problem can be regarded as the image of the solution map of a specific parametric monotone affine variational inequality. This paper establishes some facts about the domain, the image and the continuity of this solution map (called the basic multifunction), provided that the linear fractional vector optimization problem under consideration satisfies an additional assumption. The results can lead to some upper estimates for the number of components in the solution sets of linear fractional vector optimization problems.

1. Introduction

The problem of minimizing or maximizing several linear fractional objective functions on a polyhedral convex set is called a *linear fractional vector optimization problem* (LFVO problem for short). LFVO problems have a significant role both in the management science and in the theory of vector optimization.

Linear fractional (ratio) criteria are frequently encountered in finance. The reader is referred to [15, p. 337] for concrete examples of linear fractional criteria in Corporate Planning and Bank Balance Sheet Management. Fractional objectives also occur in other areas of management (for example, in transportation management, education management, and medicine management).

In the theory of vector optimization, LFVO problems are important examples of the so-called strictly quasiconvex vector minimization problems (or, the same, strictly quasiconcave vector maximization problems) which have attracted much attention from researchers during the last two decades (see [1, 8, 16], and the references therein). Meanwhile, the class of the LFVO problems encompasses the class of the linear vector optimization problems.

Topological properties of the solution sets of LFVO problems were studied in [1–3, 6, 8, 15, 17]. It is well known that the efficient solution set and the weakly efficient solution set of a LFVO problem with a bounded feasible set are connected [1, 3, 17]. Recently, it has been shown that the efficient solution set and the weakly efficient solution sets of LFVO problems may be not contractible even if they are path connected [7]. The example given by Choo and Atkins [3] demonstrates that the efficient solution set and the weakly efficient solution set of a LFVO problem may be disconnected if the feasible set is unbounded. In [6], the authors have proved that for any integer m there exist LFVO problems with m objective criteria whose efficient solution set and weakly efficient solution set have exactly m components.

Algorithms for solving LFVO problems and/or the related post-optimization problems have been proposed in [2, 12, 13].

Using the first-order necessary and sufficient optimality conditions for LFVO problems, which were established by Malivert [12], Yen and Phuong [17] have shown that the efficient solution set of any LFVO problem can be represented as the image of the solution map of a specific parametric monotone affine variational inequality. A similar representation is also valid for the weakly efficient solution set. This *parametric affine variational inequality approach* to LFVO problems has proved to be useful for studying topological properties of the solution sets and solution stability of LFVO problems.

The aim of this paper is to develop furthermore the parametric affine variational inequality approach to LFVO problems given in [17]. Using the solution existence theorem for affine variational inequalities due to Gowda and Pang [4], we will obtain an upper estimate for the number of components in the domain of the *basic multifunction* in the formulae for computing the solution sets of a certain type of LFVO problems. We will discuss the two conjectures related to the upper continuity and the image of the basic multifunction, which were stated in our preprint paper [5]. Further investigations in this direction can lead to establishing tight upper estimates for the number of components in the solution sets of LFVO problems.

The rest of the paper is organized as follows. Sec. 2 presents some preliminaries. Sec. 3 gives an estimate for the number of components in the domain of the basic multifunction. In Sec. 4 we construct a counterexample for the two conjectures proposed in [5]. It demonstrates the striking facts that the basic multifunction might not be upper semicontinuous on a component of its domain, and the image of a line segment though the basic multifunction might be disconnected. Some concluding remarks and proposals for further investigations are given in Sec. 5.

We now recall some standard notions and notation which will be used later on. Let X, Y be some subsets of Euclidean spaces. A multifunction $G : X \to 2^Y$ is *upper semicontinuous* (usc) at $x \in X$ if for every open set $V \subset Y$ satisfying $G(x) \subset V$ there exists a neighborhood U of x, such that $G(x') \subset V$ for all

$x' \in U$. If G is upper semicontinuous at every $x \in X$, then it is said that G is an upper semicontinuous multifunction. A subset Z of an Euclidean space is said to be *connected* if one cannot find any pair (Z_1, Z_2) of disjoint nonempty open subsets Z_1, Z_2 of Z in the induced topology such that $Z = Z_1 \cup Z_2$. One says that Z is *path connected* if for any $a, b \in Z$ there exists a continuous mapping $\gamma : [0,1] \to Z$ such that $\gamma(0) = a$, $\gamma(1) = b$. If for any given points $a, b \in Z$ there exists a sequence of line segments $[z_i, z_{i+1}] \subset Z$ $(i = 0, \dots, k-1)$ such that $z_0 = a$ and $z_k = b$, then Z is said to be *connected by line segments*. If Z is disconnected, then we denote by $\chi(Z)$ *the (cardinal) number of components* of Z. By definition, a subset $M \subset Z$ is said to be a *component* of Z if M is connected and it is not a proper subset of any connected subset of Z. The cone generated by Z and the convex hull of Z are denoted by $\operatorname{cone} Z$ and $\operatorname{co} Z$, respectively. For any $w, w' \in \mathbb{R}^m$, the inequality $w \leqslant w'$ (resp., $w < w'$) means $w_i \leqslant w'_i$ (resp., $w_i < w'_i$) for all $i = 1, \dots, m$. If $M \subset \mathbb{R}^n$ is a convex set, then $\dim M$ denotes the *dimension* of M, i.e., the dimension of the affine hull of M. If M is a cone, then we say that M is *pointed* if $M \cap (-M) = \{0\}$.

2. Preliminaries

Let $f_i : \mathbb{R}^n \to \mathbb{R} \quad (i = 1, 2, \dots, m)$ be m linear fractional functions, that is

$$f_i(x) = \frac{a_i^T x + \alpha_i}{b_i^T x + \beta_i}$$

for some $a_i \in \mathbb{R}^n, b_i \in \mathbb{R}^n, \alpha_i \in \mathbb{R}$, and $\beta_i \in \mathbb{R}$. (Here and in the sequel, ^{T} denotes the matrix transposition.) Let

$$\Lambda = \{\lambda \in \mathbb{R}^m_+ : \sum_{i=1}^m \lambda_i = 1\},$$

where $\mathbb{R}^m_+ = \{\lambda = (x_1, \dots, \lambda_m) \in \mathbb{R}^m : \lambda_i \geqslant 0 \quad \text{for all } i\}$. Then

$$\operatorname{ri} \Lambda = \{\lambda \in \mathbb{R}^m_+ : \sum_{i=1}^m \lambda_i = 1, \ \ \lambda_i > 0 \ \text{ for all } i\}$$

is the relative interior of Λ.

Consider the *linear fractional vector optimization problem*

$$\text{(P)} \qquad \begin{cases} \text{Minimize } \ f(x) = (f_1(x), \dots, f_m(x)) \text{ subject to} \\ x \in D := \{x \in \mathbb{R}^n \ : \ Cx \leqslant d\}, \end{cases}$$

where C is an $(r \times n)$-matrix, d is an r-dimensional column vector. Throughout this paper, it is assumed that $b_i^T x + \beta_i \neq 0$ for every i and for every $x \in D$.

Definition 2.1. *A vector $x \in D$ is said to be an efficient solution of* (P) *if there exists no $y \in D$ such that $f(y) \leqslant f(x)$ and $f(y) \neq f(x)$. If $x \in D$ and there does not exist $y \in D$ such that $f(y) < f(x)$, then x is called a weakly efficient solution of* (P).

The set of the efficient solutions (resp., the weakly efficient solutions) of (P) is denoted by $E(\mathrm{P})$ (resp., $E^w(\mathrm{P})$).

Theorem 2.1 (see [12]). *For any $x \in D$, the following assertions hold:*

(i) $x \in E(P)$ *if and only if there exists* $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathrm{ri}\,\Lambda$ *such that*

$$\Big\langle \sum_{i=1}^{m} \lambda_i \big[\big(b_i^T x + \beta_i\big)a_i - \big(a_i^T x + \alpha_i\big)b_i\big], y - x \Big\rangle \geqslant 0 \quad \forall y \in D, \tag{2.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^n$.

(ii) $x \in E^w(\mathrm{P})$ *if and only if there exists* $\lambda = (\lambda_1, \ldots, \lambda_m) \in \Lambda$ *such that* (2.1) *holds.*

(iii) *Condition* (2.1) *is satisfied if and only if there exists* $\mu = (\mu_1, \ldots, \mu_r)$, $\mu_j \geqslant 0$ *for all* $j = 1, \ldots, r$, *such that*

$$\sum_{i=1}^{m} \lambda_i \left[\big(b_i^T x + \beta_i\big)a_i - \big(a_i^T x + \alpha_i\big)b_i\right] + \sum_{j \in I(x)} \mu_j C_j^T = 0, \tag{2.2}$$

where C_j denotes the j-th row of the matrix C and $I(x) = \{j : C_j x = d_j\}$.

A detailed proof of Theorem 2.1 can be found also in [10].

The problem of finding $x \in D$ satisfying (2.1) can be rewritten in the form of a *parametric affine variational inequality* problem as follows

$$(\mathrm{VI})_\lambda \quad \text{Find } \ x \in D \ \text{ such that } \ \langle M(\lambda)x + q(\lambda), y - x \rangle \geqslant 0 \ \text{ for all } y \in D.$$

We put $M(\lambda) = (M_{kj}(\lambda))$,

$$M_{kj}(\lambda) = \sum_{i=1}^{m} \lambda_i \big(b_{i,j}a_{i,k} - a_{i,j}b_{i,k}\big), \quad 1 \leqslant k \leqslant n, \ \ 1 \leqslant j \leqslant n,$$

and

$$q(\lambda) = \big(q_k(\lambda)\big), \quad q_k(\lambda) = \sum_{i=1}^{m} \lambda_i(\beta_i a_{i,k} - \alpha_i b_{i,k}), \quad 1 \leqslant k \leqslant n,$$

where $a_{i,k}$ and $b_{i,k}$ are the k-th components of a_i and b_i, respectively.

As it has been noted in [17], since $\big(M(\lambda)\big)^T = -M(\lambda)$, $\langle M(\lambda)v, v \rangle = 0$ for every $v \in \mathbb{R}^n$. Hence $M(\lambda)$ is a positive semidefinite matrix. (Recall that an $(n \times n)$-matrix M is said to be positive semidefinite if $\langle Mv, v \rangle \geqslant 0$ for every $v \in \mathbb{R}^n$.) Denote by $F(\lambda)$ the solution set of $(\mathrm{VI})_\lambda$. By the Minty lemma (see [9]), $F(\lambda)$ is a closed convex set (possibly empty). By Theorem 2.1 we have

$$E(\mathrm{P}) = \bigcup_{\lambda \in \mathrm{ri}\,\Lambda} F(\lambda) \tag{2.3}$$

and

$$E^w(\mathrm{P}) = \bigcup_{\lambda \in \Lambda} F(\lambda). \tag{2.4}$$

Definition 2.2. *The multifunction* $F : \Lambda \to 2^{\mathbb{R}^n}$, $\lambda \to F(\lambda)$, *is said to be the basic multifunction associated to the problem* (P).

Using (2.3), (2.4), and the following lemma, one can show that $E(\mathrm{P})$ and $E^w(\mathrm{P})$ are connected if D is bounded (see [17]).

Lemma 2.1 (see [16]). *Suppose that* $X \subset \mathbb{R}^k$ *is a connected set, and* Y *is a subset of* $\mathbb{R}^s$. *If a multifunction* $G : X \to 2^Y$ *is upper semicontinuous at every* $x \in X$ *and, for every* $x \in X$, *the set* $G(x)$ *is nonempty and connected, then the set* $G(X) := \bigcup_{x\in X} G(x)$ *is connected.*

Choo and Atkins [3] showed that if D is bounded, then $E^w(\mathrm{P})$ is connected by line segments. Up to now it is still not clear whether $E(\mathrm{P})$ is also connected by line segments if D is bounded. If D is unbounded, then $E(\mathrm{P})$ and $E^w(\mathrm{P})$ may be disconnected.

Example 2.1 (see [3]). Consider problem (P) with

$$D = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geqslant 2,\ 0 \leqslant x_2 \leqslant 4\},$$
$$f_1(x) = -x_1/(x_1 + x_2 - 1), \quad f_2(x) = -x_1/(x_1 - x_2 + 3).$$

Then $E(\mathrm{P}) = E^w(\mathrm{P}) = \{(x_1, 0) : x_1 \geqslant 2\} \cup \{(x_1, 4) : x_1 \geqslant 2\}$.

It is of interest to know whether the estimates

$$\chi(E(\mathrm{P})) \leqslant m, \quad \chi(E^w(\mathrm{P})) \leqslant m \tag{2.5}$$

hold true, or not. In our opinion, the parametric affine variational inequality approach can help to study these estimates.

The following solution existence theorem for monotone affine variational inequality problems will be needed in the sequel.

Theorem 2.2. (see [4, p. 432] and [10, p. 103]). *Let* M *be an* $(n \times n)$*-matrix,* $q \in \mathbb{R}^n$ *a given vector, and* $D \subset \mathbb{R}^n$ *a nonempty polyhedral convex set. Suppose that* M *is positive semidefinite. Then the affine variational inequality problem*

$$\text{Find } x \in D \text{ such that } \langle Mx + q, y - x\rangle \geqslant 0 \text{ for all } y \in D$$

has a solution if and only if there exists $x \in D$ *such that*

$$\langle Mx + q, v\rangle \geqslant 0 \quad \forall v \in 0^+D,$$

where $0^+D = \{v \in \mathbb{R}^n : x + tv \in D$ *for all* $x \in D$ *and* $t \in \mathbb{R}_+\}$ *is the recession cone of* D.

Note that for the case $D = \{x \in \mathbb{R}^n : Cx \leqslant d\}$, we have $0^+D = \{v \in \mathbb{R}^n : Cv \leqslant 0\}$ (see [14, p. 62]).

3. Domain of the Basic Multifunction

In this section we establish some facts about *the domain*

$$\mathrm{dom}F := \{\lambda \in \Lambda : F(\lambda) \neq \emptyset\}$$

of the basic multifunction $F : \Lambda \to 2^{\mathbb{R}^n}$, which plays a key role in the formulae (2.3) and (2.4). If F is usc on Λ, then combining these facts with Lemma 2.1 we get some upper estimates for the number of components in the solution sets of LFVO problems.

The main result of this section can be stated as follows.

Theorem 3.1. *For problem* (P), *the following assertions are valid:*

(i) *If there exists* $\overline{v} \in \mathbb{R}^n \setminus \{0\}$ *such that* $0^+D = \mathrm{cone}\{\overline{v}\}$, *then* $\mathrm{dom}F$ *is a compact subset of* Λ, $\chi(\mathrm{dom}F) \leqslant m$. *Moreover, each point in* $\mathrm{dom}F$ *can be joined with at least one vertex of* Λ *by a line segment which is contained in* $\mathrm{dom}F$.

(ii) *If for each* $i \in \{1, \ldots, m\}$ *either* $b_i^T x + \beta_i \equiv 1$ *(i.e.,* f_i *is an affine function) or* $a_i^T x + \alpha_i \equiv 1$ *(i.e.,* $1/f_i$ *is an affine function), then* $\mathrm{dom}F$ *is a polyhedral convex set.*

The assumption stated in (i) is equivalent to saying that the cone $0^+D = \{v \in \mathbb{R}^n : Cv \leqslant 0\}$ is pointed and $\dim 0^+D = 1$. There are many examples of sets D satisfying this rather strict assumption. For the set D in Example 2.1, we have $0^+D = \mathrm{cone}\{\overline{v}\}$, where $\overline{v} = (0, 1)$. If

$$D = \Big\{x \in \mathbb{R}^2 : -1 \leqslant x_2 - x_1 \leqslant 1\Big\},$$

then $0^+D = \mathrm{cone}\{\overline{v}\}$, where $\overline{v} = (1, 1)$. If

$$D = \Big\{x \in \mathbb{R}^3 : x_1 + x_2 - 2x_3 \leqslant 1, x_1 - 2x_2 + x_3 \leqslant 1, -2x_1 + x_2 + x_3 \leqslant 1, x_1 + x_2 + x_3 \geqslant 1\Big\},$$

then $0^+D = \mathrm{cone}\{\overline{v}\}$, where $\overline{v} = (1, 1, 1)$.

For proving Theorem 3.1, we first establish two lemmas. Let $\Omega = \Lambda \setminus \mathrm{dom}F$. The next lemma shows that Ω is a convex set if the recession cone 0^+D has a simple structure.

Lemma 3.1. *If there exists* $\overline{v} \in \mathbb{R}^n \setminus \{0\}$ *such that* $0^+D = \mathrm{cone}\{\overline{v}\}$, *then* Ω *is a convex set, which is open in the induced topology of* Λ.

Proof. Applying Theorem 2.2 to the problem $(\mathrm{VI})_\lambda$, where $\lambda \in \Lambda$, we deduce that $F(\lambda) = \emptyset$ if and only if

$$\forall x \in D \;\; \exists v \in 0^+D \quad \text{such that} \quad \langle M(\lambda)x + q(\lambda), v \rangle < 0. \tag{3.1}$$

Since $0^+D = \mathrm{cone}\{\overline{v}\}$, (3.1) is equivalent to the following property:

$$\forall x \in D \quad \text{it holds} \quad \langle M(\lambda)x + q(\lambda), \overline{v} \rangle < 0. \tag{3.2}$$

From the formulae of $M(\lambda)$ and $q(\lambda)$ given in the preceding section it follows that
$$M(t\lambda^1 + (1-t)\lambda^2) = tM(\lambda^1) + (1-t)M(\lambda^2)$$
and
$$q(t\lambda^1 + (1-t)\lambda^2) = tq(\lambda^1) + (1-t)q(\lambda^2)$$
for all $t \in [0,1]$ and $\lambda^1, \lambda^2 \in \Lambda$. Combining this with the fact that $\lambda \in \Omega$ if and only if (3.2) is valid, we conclude that Ω is a convex set.

We now show that Ω is open in the induced topology of Λ. As D is a polyhedral convex set, by [14, Theorem 19.1] there exist $k \in N$ and $z^1, \ldots, z^k \in D$ such that

$$D = \Big\{ x = \sum_{i=1}^{k} \eta_i z^i + \rho \overline{v} : \eta_i \geqslant 0 \ \text{ for } \ i = 1, \ldots, k, \ \ \sum_{i=1}^{k} \eta_i = 1, \ \ \rho \geqslant 0 \Big\}.$$

Then, from (3.2) and the property $\langle M(\lambda)\overline{v}, \overline{v} \rangle = 0$ it follows that
$$\Omega = \big\{ \lambda \in \Lambda : \langle M(\lambda) z^i + q(\lambda), \overline{v} \rangle < 0 \ \ \forall i = 1, \ldots, k \big\}.$$
This formula and the continuity of the functions
$$\lambda \mapsto \langle M(\lambda) z^i + q(\lambda), \overline{v} \rangle \qquad (i = 1, \ldots, k)$$
imply that Ω is an open subset of Λ in the induced topology. ∎

In connection with Lemma 3.1, we would like to raise the following open question:

Question 1. Without any additional assumption on the recession cone 0^+D, is it true that Ω is a convex set, which is open in the induced topology of Λ?

Lemma 3.2. *If $\Omega \subset \Lambda$ is a convex set, then $\chi(\Lambda \setminus \Omega) \leqslant m$. Moreover, each point in $\Lambda \setminus \Omega$ can be joined with at least one of the vertices*
$$e^i = (0, \ldots, \underbrace{1}_{i-th}, 0, \ldots, 0) \quad (i = 1, \ldots, m)$$
of Λ by a line segment, which is contained in $\Lambda \setminus \Omega$.

Proof. (This is a refined version of the proof given in [5]). It suffices to prove that each point in $\Lambda \setminus \Omega$ can be joined with at least one of the vertices of Λ by a line segment contained in $\Lambda \setminus \Omega$, because the inequality $\chi(\Lambda \setminus \Omega) \leqslant m$ is a direct consequence of this property.

Given any point $\lambda \in \Lambda \setminus \Omega$, we consider the line segments
$$[\lambda, e^i] = \{ t\lambda + (1-t)e^i : t \in [0,1] \} \qquad (i = 1, \ldots, m).$$
To obtain a contradiction, suppose that $[\lambda, e^i] \cap \Omega \neq \emptyset$ for all $i = 1, \ldots, m$. Then for each i we can find a point $\lambda^i \in [\lambda, e^i] \cap \Omega$. Of course, $\lambda^i \neq \lambda$. Hence $\lambda^i = t_i \lambda + (1-t_i)e^i$ for some $t_i \in [0,1)$. From this we deduce that

$$e^i = \frac{1}{1-t_i}\lambda^i - \frac{t_i}{1-t_i}\lambda. \tag{3.3}$$

As $\lambda \in \mathrm{co}\{e^1, e^2, \ldots, e^m\}$, there exist $\mu_i \geqslant 0$, $\sum_{i=1}^m \mu_i = 1$, such that $\lambda = \sum_{i=1}^m \mu_i e^i$. Combining this with (3.3) we obtain

$$\lambda = \Big(1 + \sum_{i=1}^m \frac{\mu_i t_i}{1-t_i}\Big)^{-1} \sum_{i=1}^m \frac{\mu_i}{1-t_i}\lambda^i. \tag{3.4}$$

Since $\mu_i/(1-t_i) \geqslant 0$ for all i and

$$\sum_{i=1}^m \frac{\mu_i}{1-t_i} = \sum_{i=1}^m \Big(\mu_i + \frac{\mu_i t_i}{1-t_i}\Big) = 1 + \sum_{i=1}^m \frac{\mu_i t_i}{1-t_i},$$

(3.4) shows that $\lambda \in \mathrm{co}\{\lambda^1, \lambda^2, \ldots, \lambda^m\}$. By the convexity of Ω, from this we conclude that $\lambda \in \Omega$, a contradiction. The proof is complete. ∎

It is likely that under the assumption of Lemma 3.2 the property "$\chi(\mathrm{ri}\,\Lambda \setminus \Omega) \leqslant m$ *and each component of* $\mathrm{ri}\,\Lambda \setminus \Omega$ *is connected by line segments*" is valid. But we still do not have any proof for this fact.

Proof of Theorem 3.1. Since assertion (i) is immediate from Lemmas 3.1 and 3.2, we have to show only that (ii) is valid. If for each $i \in \{1, \ldots, m\}$ either $b_i^T x + \beta_i \equiv 1$ or $a_i^T x + \alpha_i \equiv 1$, then from the formulae

$$M(\lambda) = (M_{kj}(\lambda)), \quad M_{kj}(\lambda) = \sum_{i=1}^m \lambda_i \left(b_{i,j}a_{i,k} - a_{i,j}b_{i,k}\right)$$

for all $1 \leqslant k \leqslant n,\ 1 \leqslant j \leqslant n$ it follows that, for every $\lambda \in \Lambda$, $M(\lambda)$ collapses to the zero matrix. Hence, by Theorem 2.2, an element $\lambda \in \Lambda$ belongs to $\mathrm{dom}F$ if and only if

$$\langle q(\lambda), v\rangle \geqslant 0 \quad \forall v \in 0^+D.$$

Since $q(\lambda)$ is a linear function and 0^+D is a polyhedral convex cone, this implies that $\mathrm{dom}F$ is a polyhedral convex set. ∎

Theorem 3.1 shows that $\chi(\mathrm{dom}\,F) \leqslant 1$ provided that every objective function is either an affine function or the reverse of an affine function. Note that connectedness of the efficient set of a vector optimization problem with linear objective functions and a polyhedral convex feasible set, which is called a *linear vector optimization problem*, is a classical result (see [11]).

Example 3.1. Let us consider once again the problem given in Example 2.1 and observe that the assumption of Lemma 3.1 is satisfied for this problem. Indeed, since $0^+D = \{(\alpha, 0) : \alpha \in R_+\}$, one can choose $\overline{v} = (1, 0)$. An elementary investigation on the parametric affine variational inequality $(\mathrm{VI})_\lambda$ shows that

$$\Omega = (\overline{\lambda}, \hat{\lambda}) := \{t\overline{\lambda} + (1-t)\hat{\lambda} \ : \ 0 < t < 1\},$$

where $\overline{\lambda} = \left(\frac{1}{4}, \frac{3}{4}\right)$ and $\hat{\lambda} = \left(\frac{3}{4}, \frac{1}{4}\right)$. Therefore,

$$\begin{aligned} \operatorname{dom} F = \Lambda \setminus \Omega &= \operatorname{co}\{(0,1),(1,0)\} \setminus \Omega \\ &= \operatorname{co}\{(0,1),\overline{\lambda}\} \cup \operatorname{co}\{\hat{\lambda},(1,0)\}. \end{aligned}$$

We see that $\operatorname{dom} F$ has two components.

In connection with the first assertion of Theorem 3.1, the following open question seems to be interesting.

Question 2. Is it true that the conclusion of the first part of Theorem 3.1 is still valid without the additional assumption on the recession cone 0^+D?.

In order to derive information about the numbers $\chi(E^w(\mathrm{P}))$ and $\chi(E(\mathrm{P}))$ from the information about the number $\chi(\mathrm{dom}F)$, one has to investigate furthermore the behavior of the basic multifunction. The following two conjectures were stated in our preprint paper [5].

Conjecture 1. The basic multifunction $F : \Lambda \to 2^{R^n}$ is upper semicontinuous on Λ.

Conjecture 2. If $\lambda^1, \lambda^2 \in \Lambda$ are such that $[\lambda^1, \lambda^2] \subset \mathrm{dom}F$, then the set $F([\lambda^1, \lambda^2])$ is connected by line segments.

Note that both the conjectures are valid for the problem considered in Example 3.1. If Conjecture 1 is true, then from Theorem 3.1 and Lemma 2.1 it follows that "If there exists $\overline{v} \in R^n \setminus \{0\}$ such that $0^+D = \mathrm{cone}\{\overline{v}\}$, then $\chi(E^w(P)) \leqslant m$". If Conjecture 2 is true, by Theorem 3.1 one can assert that "If $0^+D = \mathrm{cone}\{\overline{v}\}$ for some $\overline{v} \in R^n \setminus \{0\}$, then $\chi(E^w(P)) \leqslant m$. Moreover, each component of $E^w(P)$ is connected by line segments".

Unfortunately, the counterexample given in the next section shows that both the conjectures are not true. We believe that the counterexample not only solves the conjectures, but it is also very useful for understanding the behavior of the basic multifunction.

4. Image of a Line Segment through the Basic Multifunction

To analyze the behavior of the basic multifunction $\lambda \mapsto F(\lambda)$, we consider problem (P) with the following data:

$$D = \left\{x \in R^2 : x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_1 + x_2 \geqslant 1\right\},$$
$$f_1(x) = \frac{x_1 + 1}{2x_1 + x_2}, \quad f_2(x) = \frac{-x_1 - 2}{x_1 + x_2}.$$

Then, in the notation of Sec. 2, we have

$$C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ -1 & -1 \end{pmatrix}, \quad d = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Claim 1. *The following formula is valid:*

$$F(\lambda) = \begin{cases} \{0\} \times [2, +\infty) \quad \text{if } \lambda = \left(\frac{2}{3}, \frac{1}{3}\right) \\ \left\{\left(\frac{2-3\lambda_1}{2\lambda_1 - 1}, 2\right)\right\} \quad \text{if } \lambda = (\lambda_1, 1-\lambda_1),\ \frac{1}{2} < \lambda_1 < \frac{2}{3} \\ [1, +\infty) \times \{0\} \quad \text{if } \lambda = \left(\frac{1}{2}, \frac{1}{2}\right) \\ \{(1,0)\} \quad \text{if } \lambda = (\lambda_1, 1-\lambda_1),\ 0 \leqslant \lambda_1 < \frac{1}{2} \\ \emptyset \quad \text{if } \lambda = (\lambda_1, 1-\lambda_1),\ \frac{2}{3} < \lambda_1 \leqslant 1. \end{cases} \tag{4.1}$$

Proof. Let $x \in D$. By Theorem 2.1, $x \in E^w(\mathrm{P})$ if and only if there exist $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 0$, $\lambda_1 + \lambda_2 = 1$, $\mu_1 \geqslant 0$, $\mu_2 \geqslant 0$, $\mu_3 \geqslant 0$, such that

$$\lambda_1 \left[(2x_1 + x_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} - (x_1 + 1) \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right] + \lambda_2 \left[(x_1 + x_2) \begin{pmatrix} -1 \\ 0 \end{pmatrix} - (-x_1 - 2) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] + \sum_{j \in I(x)} \mu_j C_j^T = 0$$

or, equivalently,

$$\begin{pmatrix} (\lambda_1 - \lambda_2)(x_2 - 2) \\ -\lambda_1(x_1 + 1) + \lambda_2(x_1 + 2) \end{pmatrix} + \sum_{j \in I(x)} \mu_j C_j^T = 0. \tag{4.2}$$

Case 1. $I(x) = \emptyset$. Then we have $x_1 > 0$, $x_2 > 0$, $x_1 + x_2 > 1$. In this case, (4.2) is equivalent to the system of two conditions: $x_2 = 2$, $\lambda_2 = \lambda_1[1 - 1/(x_1 + 2)]$. Taking into account the equality $\lambda_1 + \lambda_2 = 1$, we obtain $\lambda_1 = (x_1 + 2)/(2x_1 + 3)$, $\lambda_2 = 1 - \lambda_1$. Since $x_1 \in (0, +\infty)$, it holds $\frac{1}{2} < \lambda_1 < \frac{2}{3}$. We can express $x = (x_1, x_2)$ via $\lambda = (\lambda_1, \lambda_2)$ as follows

$$x_1 = \frac{2 - 3\lambda_1}{2\lambda_1 - 1}, \quad x_2 = 2.$$

Case 2. $I(x) = \{1\}$. In this case we have $x_1 = 0$, $x_2 > 1$. The equation (4.2) can be rewritten as the following

$$\begin{pmatrix} (\lambda_1 - \lambda_2)(x_2 - 2) \\ -\lambda_1 + 2\lambda_2 \end{pmatrix} + \mu_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} = 0.$$

Combining this with the equality $\lambda_1 + \lambda_2 = 1$, we obtain $x_2 \geqslant 2$, $\lambda_1 = 2/3$, $\lambda_2 = 1/3$, $\mu_1 = \frac{1}{3}\lambda_1(x_2 - 2)$.

Case 3. $I(x) = \{2\}$. In this case we have $x_2 = 0$, $x_1 > 1$. We rewrite (4.2) equivalently as follows

$$\begin{pmatrix} 2(\lambda_2 - \lambda_1) \\ -\lambda_1(x_1 + 1) + \lambda_2(x_1 + 2) \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = 0.$$

As $\lambda_1 + \lambda_2 = 1$, this implies $\lambda_1 = \lambda_2 = 1/2$, $\mu_2 = 1/2$.

Case 4. $I(x) = \{3\}$. In this case we have $x_1 > 0,\ x_2 > 0,\ x_1 + x_2 = 1$. The equation (4.2) now becomes

$$\begin{pmatrix} (\lambda_2 - \lambda_1)(x_1 + 1) \\ -\lambda_1(x_1 + 1) + \lambda_2(x_1 + 2) \end{pmatrix} + \mu_3 \begin{pmatrix} -1 \\ -1 \end{pmatrix} = 0.$$

This implies

$$\mu_3 = (\lambda_2 - \lambda_1)(x_1 + 1),\ \mu_3 = (\lambda_2 - \lambda_1)(x_1 + 1) + \lambda_2.$$

It is clear that one cannot find multipliers $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 0$, $\lambda_1 + \lambda_2 = 1$, and $\mu_3 \geqslant 0$, which satisfy these two conditions.

Case 5. $I(x) = \{1, 3\}$. Since $x_1 = 0$ and $x_2 = 1$, we can rewrite (4.2) as follows

$$\begin{pmatrix} \lambda_2 - \lambda_1 \\ 2\lambda_2 - \lambda_1 \end{pmatrix} + \mu_1 \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \mu_3 \begin{pmatrix} -1 \\ -1 \end{pmatrix} = 0.$$

It is easily seen that there exist no multipliers $\lambda_1 \geqslant 0$, $\lambda_2 \geqslant 0$, $\lambda_1 + \lambda_2 = 1$, $\mu_1 \geqslant 0$ and $\mu_3 \geqslant 0$, which satisfy this equation.
Case 6. $I(x) = \{2, 3\}$. Since $x_1 = 1$ and $x_2 = 0$, (4.2) becomes

$$\begin{pmatrix} 2(\lambda_2 - \lambda_1) \\ 3\lambda_2 - 2\lambda_1 \end{pmatrix} + \mu_2 \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \mu_3 \begin{pmatrix} -1 \\ -1 \end{pmatrix} = 0,$$

which yields

$$\mu_2 = \lambda_2, \quad \mu_3 = 2(\lambda_2 - \lambda_1), \quad \lambda_2 \geqslant \lambda_1.$$

As $\lambda_2 + \lambda_1 = 1$, we deduce that $0 \leqslant \lambda_1 \leqslant 1/2$, $\lambda_2 = 1 - \lambda_1$. (The cases $I(x) = \{1, 2\}$ and $I(x) = \{1, 2, 3\}$ are excluded, because we assume that $x \in D$).

Summarizing the above results, we obtain (4.1). ■

Claim 2. *The basic multifunction F is not upper semicontinuous on the line segment*

$$L := \mathrm{co}\left\{ \left(\frac{2}{3}, \frac{1}{3}\right), (0, 1) \right\}, \tag{4.3}$$

which coincides with the set $\mathrm{dom}\, F$.

Proof. Formula (4.1) shows that F is not upper semicontinuous at the point $(1/2, 1/2) \in L$. ■

Claim 3. *The image of the line segment L through F is disconnected.*

Proof. By (4.1) we have

$$F(L) = \Big([1, +\infty) \times \{0\}\Big) \cup \Big([0, +\infty) \times \{2\}\Big) \cup \Big(\{0\} \times [2, +\infty)\Big).$$

So $F(L)$ has two components. ■

The above example shows that the properties of the basic multifunction stated in Conjecture 1 and Conjecture 2 (see Section 3), are not available for general LFVO problems. However, if we decompose the set $L = \operatorname{dom} F$ into the union of the two disjoint subsets

$$L_1 := \left\{ t(0,1) + (1-t)\Big(\frac{1}{2}, \frac{1}{2}\Big) : 0 \leqslant t \leqslant 1 \right\},$$
$$L_2 := \left\{ t\Big(\frac{1}{2}, \frac{1}{2}\Big) + (1-t)\Big(\frac{2}{3}, \frac{1}{3}\Big) : 0 \leqslant t < 1 \right\}$$

then, by virtue of (4.1), the following assertions hold true:

(i) Each subset L_i $(i = 1, 2)$ is connected by line segments.

(ii) The restriction of F on each subset L_i $(i = 1, 2)$ is an upper semicontinuous multifunction.

(iii) The image of each subset L_i $(i = 1, 2)$ through F is connected by line segments.

Concerning the basic function F figured in (2.3) and (2.4), we say that $\lambda \in \operatorname{dom} F$ is a *regular point* if $F(\lambda)$ is bounded. If $F(\lambda)$ is unbounded, we say that λ is an *irregular point*. In order to make the above decomposition $L = L_1 \cup L_2$, we have used the irregular points

$$\lambda^1 := \Big(\frac{1}{2}, \frac{1}{2}\Big), \quad \lambda^2 := \Big(\frac{2}{3}, \frac{1}{3}\Big).$$

Observe that the sets $F(L_1)$ and $F(L_2)$ are just the two components of the weakly efficient solution set $E^w(\mathrm{P})$. Here we have $m = 2$ and $E(\mathrm{P}) = E^w(\mathrm{P})$, so the estimates in (2.5) are valid for the problem considered in this section.

5. Concluding Remarks

In this paper we investigate furthermore the parametric affine variational inequality approach to linear fractional vector optimization problems, which can help to obtain tight upper estimates for the number of components in the solution sets of these problems. Some results on the domain, the image, and the continuity of the basic multifunction have been obtained.

Although the two conjectures stated in [5] are not true, the following claims might be valid: 1) The set $\operatorname{dom} F$ can be decomposed into the union of not more than m subsets, each of them is connected by line segments; 2) The restriction of F on each of the subsets is an upper semicontinuous multifunction; 3) The image of each of the subsets through F is connected by line segments.

Thus, further efforts are needed to prove (or disprove) the estimates in (2.5). New results, even for the case $m = 2$, would be of interest.

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