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A Characterization of Morrey Type Besov and Triebel-Lizorkin Spaces*

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Abstract. In this paper the author gives a maximal function characterization of the Morrey-type Besov and Triebel-Lizorkin spaces, $MB^{s,\beta}_{p,q}(\mathbb{R}^n)$ and $MF^{s,\beta}_{p,q}(\mathbb{R}^n)$, which are the generalizations of the well-known Morrey-type spaces and the inhomogeneous Besov and Triebel-Lizorkin spaces.

1. Introduction

In recent years, the Morrey-type space continues to attract the attention of many authors. Many problems of partial differential equation based on Morrey space and Morrey type Besov space have been considered in [1 - 6, 11, 16]. Many results obtained parallel with the theory of standard Besov and Triebel-Lizorkin spaces and new applications have also been given. Actually, in [7] Mazzuato established some decompositions of Morrey type Besov spaces (in [7], they were called Besov-Morrey spaces) in terms of smooth wavelets, molecules concentrated on dyadic cubes, and atoms supported on dyadic cubes. In [10], Tang Lin and the author obtained some properties including lift properties and a Fourier multiplier theorem on Morrey type Besov and Triebel-Lizorkin spaces, and a discrete characterization of these spaces. Moreover, in [10] the authors also considered the boundedness of a class pseudo-differential operators on these spaces.

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For readers interesting in standard Besov and Triebel-Lizorkin spaces and their applications, we recommend them Triebel's books [12 - 15].

Motivated by [8], our purpose is to give a maximal function inequality on Morrey-type Besov and Triebel-Lizorkin spaces, which is a characterization of Morrey-type Besov and Triebel-Lizorkin spaces. Before stating it, we recall some notations and the definition of Morrey-type Besov and Triebel-Lizorkin spaces (see, e.g., [10]).

Let $\mathbb{R}^n$ be the n-dimensional real Euclidean space. Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$. Let $\mathcal{S}'(\mathbb{R}^n)$ be the set of all the tempered distribution on $\mathbb{R}^n$. If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, then $\widehat{\varphi}$ denotes the Fourier transform of φ, and $\varphi^\vee$ denotes the inverse Fourier transform of φ.

Definition 1. *If $0 < q \leqslant p < \infty$ and $f \in L^q_{Loc}(\mathbb{R}^n)$, we say $f \in M^p_q(\mathbb{R}^n)$ provided that, for any ball $B_{R,x}$ centered at x with radius R,*

$$\|f\|_{M^p_q} =: \sup_{x\in\mathbb{R}^n, R>0} R^{n(1/p-1/q)}\Big(\int\limits_{B_{R,x}} |f(y)|^q dy\Big)^{1/q} < \infty.$$

Morrey spaces can be seen as a complement to L^p spaces. In fact, $M^p_q \equiv L^p$ and $L^p \subset M^p_q$.

For $j \in \mathbb{N}$ we put $\varphi_j(x) = 2^{nj}\varphi(2^j x),\ x \in \mathbb{R}^n$. Let functions $A, \theta \in \mathcal{S}(\mathbb{R}^n)$ satisfy the following conditions:

$$|\widehat{A}(\xi)| > 0 \quad \text{on} \quad \{|\xi| < 2\}, \quad \text{supp}\ \widehat{A} \subset \{|\xi| < 4\},$$
$$|\widehat{\theta}(\xi)| > 0 \quad \text{on} \quad \{1/2 < |\xi| < 2\}, \quad \text{supp}\ \widehat{\theta} \subset \{1/4 < |\xi| < 4\}.$$

Now the Morrey type Besov and Triebel-Lizorkin spaces can be defined as follows.

Definition 2. *Let $-\infty < s < \infty,\ 0 < q \leqslant p < \infty,\ 0 < \beta \leqslant \infty$, and $A,\ \theta$ be as above, then we define*
(i) *The Morrey type Besov spaces as*

$$MB^{s,\beta}_{p,q}(\mathbb{R}^n) = \big\{f \in S'(\mathbb{R}^n) :\ \|f\|_{MB^{s,\beta}_{p,q}} = \|A*f\|_{M^p_q} + \big\|\{2^{sj}\theta_j*f\}_1^\infty\big\|_{\ell_\beta(M^p_q)} < \infty\big\}.$$

(ii) *The Morrey type Triebel-Lizorkin spaces as*

$$MF^{s,\beta}_{p,q}(\mathbb{R}^n) = \big\{f \in S'(\mathbb{R}^n) :\ \|f\|_{MF^{s,\beta}_{p,q}} = \|A*f\|_{M^p_q} + \big\|\big\{2^{sj}\theta_j*f\big\}_1^\infty\big\|_{M^p_q(\ell_\beta)} < \infty\big\}.$$

Obviously, for $s \in \mathbb{R},\ 0 < p = q < \infty$, and $0 \leqslant \beta \leqslant \infty$, then $MB^{s,\beta}_{q,q} = B^s_{q,\beta}$ and $MF^{s,\beta}_{q,q} = F^s_{q,\beta}$, standard Besov and Triebel-Lizorkin spaces respectively; see [22].

To make these space meaningful, the key point is to show that Definition 2 is independent of the choice of functions A and θ. Actually, by the method of Triebel's book [12] we had proved this in a modified definition in [10]. In this paper, we will consider this by using maximal function again. The following Theorem 1 is stronger than what we obtained in [10].

Let $\Psi, \psi \in \mathcal{S}(\mathbb{R}^n), \epsilon > 0$, an integer $S \geq -1$ be such that

$$|\widehat{\Psi}(\xi)| > 0 \quad \text{on} \quad \{|\xi| < 2\epsilon\}, \tag{1}$$
$$|\widehat{\psi}(\xi)| > 0 \quad \text{on} \quad \{\epsilon/2 < |\xi| < 2\epsilon\},$$

and

$$D^\tau \widehat{\psi}(0) = 0 \quad \text{for all} \quad |\tau| \leqslant S. \tag{2}$$

Here (1) are Tauberian conditions, while (2) expresses moment conditions on ψ.

For any $a > 0$, $f \in \mathcal{S}'(\mathbb{R}^n)$, and $x \in \mathbb{R}^n$, we introduce maximal functions,

$$\Psi_a^* f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\Psi * f(y)|}{(1 + |x - y|)^a}, \tag{3}$$

and

$$\psi_{j,a}^* f(x) = \sup_{y \in \mathbb{R}^n} \frac{|\psi_j * f(y)|}{(1 + 2^j |x - y|)^a}. \tag{3'}$$

In what follows, by writing $A_1 \lesssim A_2$ we mean that $A_1 \leqslant C A_2$, C is a positive constant independent of $f \in \mathcal{S}'(\mathbb{R}^n)$.

Theorem 1.

(i) *Let $s < S + 1$, $0 < \beta \leqslant \infty$, $0 < q, p \leqslant \infty, a > n/q$. Then for all $f \in \mathcal{S}'(\mathbb{R}^n)$*

$$\|\Psi_a^* f\|_{M_q^p} + \|\{2^{sj} \psi_{j,a}^* f\}_1^\infty\|_{\ell_\beta(M_q^p)} \lesssim \|f\|_{M_q^p B_\beta^s}$$
$$\lesssim \|\Psi * f\|_{M_q^p} + \|\{2^{js} \psi_j * f\}_1^\infty\|_{\ell_\beta(M_q^p)}. \tag{4}$$

(ii) *Let $s < S + 1$, $0 < \beta \leqslant \infty$, $0 < q, p < \infty, a > n/\min(q, \beta)$. Then for all $f \in \mathcal{S}'$*

$$\|\Psi_a^* f\|_{M_q^p} + \|\{2^{sj} \psi_{j,a}^* f\}_1^\infty\|_{M_q^p(\ell_\beta)} \lesssim \|f\|_{M_q^p F_\beta^s}$$
$$\lesssim \|\Psi * f\|_{M_q^p} + \|\{2^{js} \psi_j * f\}_1^\infty\|_{M_q^p(\ell_\beta)}. \tag{5}$$

The remainder of the paper is to give the proof of Theorem 1. To do this, we need some lemmas, which will be given in Sec. 2. The complete proof will be given in Sec. 3. Finally, we point that letter C will denote various positive constants. The constants may in general depend on all fixed parameters, and sometimes we show this dependence explicitly by writing, e.g., C_N. In the sequel, for convenience we omit the range of integration when it is $\mathbb{R}^n$.

2. Some Lemmas

Lemma 1. (see [8]) *Let* $\mu, \nu \in \mathcal{S}(\mathbb{R}^n)$, $M \geq -1$ *integer,*

$$D^\tau \widehat{\mu}(0) = 0 \quad \text{for all} \quad |\tau| \leqslant M.$$

Then for any $N > 0$ *there is a constant* C_N *such that*

$$\sup_{z\in\mathbb{R}^n} |\mu_t * \nu(z)|(1+|z|)^N \leqslant C_N t^{M+1}.$$

The following Lemma 2 is easy to obtain. For its proof one can also see [8].

Lemma 2. *Let* $0 < \beta \leqslant \infty, \delta > 0$. *For any sequence* $\{g_j\}_0^\infty$ *of nonnegative measurable functions on* $\mathbb{R}^n$, *put*

$$G_j(x) = \sum_{k=0}^{\infty} 2^{-|k-j|^\delta} g_k(x), \quad x \in \mathbb{R}^n.$$

Then

$$\|\{G_j(x)\}_0^\infty\|_{\ell_\beta} \leqslant C\|\{g_j(x)\}_0^\infty\|_{\ell_\beta} \tag{6}$$

holds, where C *is a constant only dependent on* β, δ.

Lemma 3. *Let* $0 < p,\ q, \beta \leqslant \infty$, $\delta > 0$. *For any sequence* $\{g_j\}_0^\infty$ *of nonnegative measurable functions on* $\mathbb{R}^n$, *set*

$$G_j(x) = \sum_{k=0}^{\infty} 2^{-|k-j|^\delta} g_k(x), \quad x \in \mathbb{R}^n.$$

Then

$$\|\{G_j\}_0^\infty\|_{M_q^p(\ell_\beta)} \leqslant C_1\|\{g_j\}_0^\infty\|_{M_q^p(\ell_\beta)}, \tag{7}$$

and

$$\|\{G_j\}_0^\infty\|_{\ell_\beta(M_q^p)} \leqslant C_2\|\{g_j\}_0^\infty\|_{\ell_\beta(M_q^p)} \tag{8}$$

hold with some constants $C_1 = C_1(\beta,\delta)$ *and* $C_2 = C_2(p,q,\beta,\delta)$.

Proof. By Lemma 2, (7) follows immediately from (6). Now we prove (8) by considering two cases.

Case 1. $q \geq 1$. Since $\|\cdot\|_{M_q^p}$ is a norm, by Minkowski's inequality, we have

$$\|G_j\|_{M_q^p} \leqslant \sum_{k=0}^{\infty} 2^{-|k-j|\delta}\|g_k\|_{M_q^p}.$$

Hence (8) follows from Lemma 2.

Case 2. $q \leqslant 1$. By Definition 1

$$\begin{aligned}\|G_j\|^q_{M^p_q} &= \sup_{x\in\mathbb{R}^n, R>0} R^{nq(1/p-1/q)} \int_{B_{R,x}} |G_j(y)|^q\,dy \\ &\leqslant \sup_{x\in\mathbb{R}^n, R>0} R^{nq(1/p-1/q)} \sum_{k=0}^{\infty} 2^{-q|k-j|\delta} \int_{B_{R,x}} |g_k(y)|^q\,dy \\ &\leqslant \sum_{k=0}^{\infty} 2^{-q|k-j|\delta} \sup_{x\in\mathbb{R}^n, R>0} R^{nq(1/p-1/q)} \int_{B_{R,x}} |g_k(y)|^q\,dy \\ &= \sum_{k=0}^{\infty} 2^{-|k-j|q\delta} \|g_k\|^q_{M^p_q}.\end{aligned}$$

By Lemma 2 with β, and δ replaced by β/q and $q\delta$ respectively, we have

$$\big\|\{\|G_j\|^q_{M^p_q}\}\big\|_{\ell_{\beta/q}} \leqslant C\big\|\{\|g_j\|^q_{M^p_q}\}\big\|_{\ell_{\beta/q}}.$$

Raising the above inequality to power $1/q$, we obtain (8).

This completes the proof of Lemma 3. ■

Lemma 4. (see [10]) *Let $1 < \beta < \infty$ and $1 < q \leqslant p < \infty$. If $\{f_j\}_{j=0}^{\infty}$ is a sequence of local integral functions on $\mathbb{R}^n$, then*

$$\|(\sum_{j=0}^{\infty} |\mathcal{M} f_j|^{\beta})^{\frac{1}{\beta}}\|_{M^p_q} \leqslant C\|\Big(\sum_{j=0}^{\infty} |f_j|^{\beta}\Big)^{\frac{1}{\beta}}\|_{M^p_q},$$

where the constant C is independent of $\{f_j\}_{j=0}^{\infty}$ and $\mathcal{M}$ denotes standard Hardy-Littlewood maximal operator.

Lemma 5. (see [8]) *Let $0 < r \leqslant 1$, and let $\{b_j\}_0^{\infty}$, $\{d_j\}_0^{\infty}$ be two sequences taking values in $(0, +\infty]$ and $(0, +\infty)$ respectively. Assume that for some $N_0 > 0$*

$$d_j = O(2^{jN_0}), \quad j \to \infty,$$

and that for any $N > 0$, and $j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, there exists a constant C_N independent of j such that

$$d_j \leqslant C_N \sum_{k=j}^{\infty} 2^{(j-k)N} b_k d_k^{1-r}.$$

Then for any $N > 0$ and $j \in \mathbb{N}_0$,

$$d_j^r \leqslant C_N \sum_{k=j}^{\infty} 2^{(j-k)Nr} b_k$$

hold with the same constants C_N as above.

3. Proof of Theorem 1

The idea of the proof is from Rychkov[8]. In fact, we will use the method in [8] with Lemma 3 and Lemma 4. To do the end, we give the proof in three steps.

Step 1. Take any pair of functions $\Phi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ so that for an $\varepsilon' > 0$

$$\begin{aligned} &|\widehat{\Phi}(\xi)| > 0 \quad \text{on} \quad \{|\xi| < 2\varepsilon'\}, \\ &|\widehat{\varphi}(\xi)| > 0 \quad \text{on} \quad \{\varepsilon'/2 < |\xi| < 2\varepsilon'\}, \end{aligned} \tag{9}$$

and define $\Phi^*_a f,\ \varphi^*_{j,a} f$ as (3) and (3').

For any $a > 0, s < S+1, 0 < p,\ q,\ \beta \leqslant \infty$, we will prove that for all $f \in \mathcal{S}'(\mathbb{R}^n)$ the following estimates hold.

$$\|\Psi^*_a f\|_{M^p_q} + \|\{2^{sj}\psi^*_{j,a} f\}_1^\infty\|_{\ell_\beta(M^p_q)} \lesssim \|\Phi^*_a f\|_{M^p_q} + \|\{2^{js}\varphi^*_{j,a} f\}_1^\infty\|_{\ell_\beta(M^p_q)}. \tag{10}$$

$$\|\Psi^*_a f\|_{M^p_q} + \|\{2^{sj}\psi^*_{j,a} f\}_1^\infty\|_{M^p_q(\ell_\beta)} \lesssim \|\Phi^*_a f\|_{M^p_q} + \|\{2^{js}\varphi^*_{j,a} f\}_1^\infty\|_{M^p_q(\ell_\beta)}. \tag{11}$$

Actually, it follows from (9) that there exist two functions $\Lambda, \lambda \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\begin{aligned} &\operatorname{supp} \widehat{\Lambda} \subset \{|\xi| < 2\varepsilon'\}, \\ &\operatorname{supp} \widehat{\lambda} \subset \{\varepsilon'/2 < |\xi| < 2\varepsilon'\}, \end{aligned}$$

and

$$\widehat{\Lambda}(\xi)\widehat{\Phi}(\xi) + \sum_{j=1}^\infty \widehat{\lambda}(2^{-j}\xi)\widehat{\varphi}(2^{-j}\xi) \equiv 1, \text{ for all } \xi \in \mathbb{R}^n.$$

Then, for all $f \in \mathcal{S}'(\mathbb{R}^n)$, we have the identity,

$$f = \Lambda * \Phi * f + \sum_{k=1}^\infty \lambda_k * \psi_k * f.$$

Thus we can write

$$\psi_j * f = \psi_j * \Lambda * \Phi * f + \sum_{k=1}^\infty \psi_j * \lambda_k * \psi_k * f.$$

Therefore, by Lemma 1 we have

$$\begin{aligned} |\psi_j * \lambda_k * \varphi_k * f(y)| &\leqslant \int_{\mathbb{R}^n} |\psi_j * \lambda_k||\varphi_k * f(y-z)|\, dz \\ &\leqslant \varphi^*_{k,a} f(y) \int_{\mathbb{R}^n} |\psi_j * \lambda_k|(1 + 2^k|z|)^a\, dz \\ &\equiv \varphi^*_{k,a} f(y) I_{j,k}, \end{aligned}$$

where

$$I_{j,k} \leqslant C(\lambda, \psi) \begin{cases} 2^{(k-j)(S+1)} & \text{if}, \quad k \leqslant j, \\ 2^{(j-k)(S+1)} & \text{if}, \quad k \geq j; \end{cases}$$

see [8]. Noting that for all $x, y \in \mathbb{R}^n$,

$$\varphi^*_{k,a} f(y) \leqslant \varphi^*_{k,a} f(x)(1 + 2^k|x-y|)^a \leqslant \varphi^*_{k,a} f(x) \max(1, 2^{(k-j)a})(1 + 2^j|x-y|)^a.$$

So we have

$$\sup_{y\in\mathbb{R}^n}\frac{|\psi_j*\lambda_k*\varphi_k*f(y)|}{(1+2^j|x-y|)^a}\lesssim\varphi^*_{k,a}f(x)\times\begin{cases}2^{(k-j)(S+1)} & \text{if,}\quad k\leqslant j,\\ 2^{(j-k)(S+1)} & \text{if,}\quad k\geq j.\end{cases}$$

Note that for $k=1$, we do not use the condition $D^\tau\widehat{\lambda}(0)=0$ in the above proof of the last estimate, so by replacing respectively λ_1 and φ_1 with Λ and Φ we have a similar estimate

$$\sup_{y\in\mathbb{R}^n}\frac{|\psi_j*\Lambda*\varphi_k*f(y)|}{(1+2^j|x-y|)^a}\lesssim\Phi^*_af(x)2^{-j(S+1)}.$$

So we obtain

$$\psi^*_{j,a}f(x)\lesssim\Phi^*_af(x)2^{-j(S+1)}+\sum_{k=1}^{\infty}\varphi^*_{k,a}f(x)\times\begin{cases}2^{(k-j)(S+1)} & \text{if,}\quad k\leqslant j,\\ 2^{(j-k)(S+1)} & \text{if,}\quad k\geq j.\end{cases}$$

Hence with $\delta=\min(1,S+1-s)>0$ for all $f\in\mathcal{S}',\ x\in\mathbb{R}^n, j\in\mathbb{N}$

$$2^{js}\psi^*_{j,a}f(x)\lesssim\Phi^*_af(x)2^{-j\delta}+\sum_{k=1}^{\infty}2^{ks}\varphi^*_{k,a}f(x)2^{-|k-j|\delta}. \tag{12}$$

Again, for $j=1$ we did not use (2) to get this estimate, so we can replace ψ_1 with Ψ to obtain

$$2^{js}\Psi^*_af(x)\lesssim\Phi^*_af(x)2^{-j\delta}+\sum_{k=1}^{\infty}2^{ks}\varphi^*_{k,a}f(x)2^{-j\delta}. \tag{13}$$

The desired estimates (10), (11), follow from (12), (13) and Lemma 3.

Step 2. In this step we will show the following estimates.

In the conditions of (4), for all $f\in\mathcal{S}'(\mathbb{R})$

$$\|\Psi^*_af\|_{M^p_q}+\|\{2^{sj}\psi^*_{j,a}f\}_1^\infty\|_{\ell_\beta(M^p_q)}\lesssim\|\Psi*f\|_{M^p_q}+\|\{2^{js}\psi_j*f\}_1^\infty\|_{\ell_\beta(M^p_q)}. \tag{14}$$

And in the conditions of (5), for all $f\in\mathcal{S}'(\mathbb{R}^n)$

$$\|\Psi^*_af\|_{M^p_q}+\|\{2^{sj}\psi^*_{j,a}f\}_1^\infty\|_{M^p_q(\ell_\beta)}\lesssim\|\Psi*f\|_{M^p_q}+\|\{2^{js}\psi_j*f\}_1^\infty\|_{M^p_q(\ell_\beta)}. \tag{15}$$

Similar to (9), pick two functions $\Lambda,\lambda\ \in\mathcal{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp}\widehat{\Lambda}\subset\{|\xi|<2\varepsilon'\},\operatorname{supp}\widehat{\lambda}\subset\{\varepsilon'/2<|\xi|<2\varepsilon'\},$$

and

$$\widehat{\Lambda}(\xi)\widehat{\Phi}(\xi)+\sum_{j=1}^{\infty}\widehat{\lambda}(2^{-j}\xi)\widehat{\varphi}(2^{-j}\xi)\equiv1$$

for all $\xi\in\mathbb{R}^n$. Then, for all $f\in\mathcal{S}'(\mathbb{R}^n)$ we have the identity,

$$f = \Lambda * \Phi * f + \sum_{k=1}^{\infty} \lambda_k * \psi_k * f.$$

Thus we can write

$$\psi_j * f = \psi_j * \Lambda * \Phi * f + \sum_{k=1}^{\infty} \psi_j * \lambda_k * \psi_k * f.$$

By replacing f with $f(2^{-j}\cdot)$ for $j \in \mathbb{N}$, we obtain

$$f = \Lambda_j * \Phi_j * f + \sum_{k=j+1}^{\infty} \lambda_k * \psi_k * f.$$

Thus

$$\psi_j * f = (\Lambda_j * \Phi_j) * (\psi_j * f) + \sum_{k=j+1}^{\infty} (\psi_j * \lambda_k) * (\psi_k * f). \tag{16}$$

By Lemma 1, we know that

$$|\psi_j * \lambda_k(z)| \leqslant C_N \frac{2^{jn} 2^{(j-k)N}}{(1+2^j|z|)^a}, \quad z \in \mathbb{R}^n, \tag{17}$$

holds for $k \geq j$ with arbitrarily large $N > 0$, where C_N is a constant dependent on N. And also it is easy to see that

$$|\psi_j * \lambda_j(z)| \leqslant C \frac{2^{jn}}{(1+2^j|z|)^a}, \quad z \in \mathbb{R}^n. \tag{18}$$

By putting the last two estimates (17) and (18) into (16), we obtain that for all $f \in \mathcal{S}'(\mathbb{R}^n)$, $y \in \mathbb{R}^n$, and $j \in \mathbb{N}$,

$$|\psi_j * f(y)| \leqslant C_N \sum_{k=j}^{\infty} 2^{jn} 2^{(j-k)N} \int \frac{|\psi_k * f(z)|}{(1+2^j|y-z|)^a}\, dz. \tag{19}$$

For any $r \in (0,1]$, dividing both sides of (19) by $(1+2^j|x-y|)^a$, then in the left hand side taking the supremum over $y \in \mathbb{R}^n$, while in the right hand side making use of the following inequalities

$$(1+2^j|x-y|)(1+2^j|y-z|) \geq (1+2^j|x-y|), \tag{20}$$

$$|\psi_k * f(z)| \leqslant |\psi_k * f(z)|^r [\psi_{k,a}^* f(x)]^{1-r} (1+2^k|x-z|)^{a(1-r)},$$

and

$$\frac{(1+2^k|x-z|)^{a(1-r)}}{(1+2^j|x-z|)^a} \leqslant \frac{2^{(k-j)a}}{(1+2^k|x-z|)^{ar}},$$

we obtain that for all $f \in \mathcal{S}'(\mathbb{R}^n), x \in \mathbb{R}^n$ and $j \in \mathbb{N}$,

$$\psi_{j,a}^* f(x) \leqslant C_N \sum_{k=j}^{\infty} 2^{(j-k)N'} \int \frac{2^{kn}|\psi_k * f(z)|^r}{(1+2^k|x-z|)^{ar}}\, dz [\psi_{k,a}^* f(x)]^{1-r} \tag{21}$$

holds, where $N' = N - a + n$ can be taken arbitrarily large.

Similarly, we can prove that for all $f \in \mathcal{S}'(\mathbb{R}^n)$,

$$\psi_a^* f(x) \leqslant C_N \Big(\int \frac{|\Psi * f(z)|^r}{(1+|x-z|)^{ar}} dz [\Psi_a^* f(x)]^{1-r} + \sum_{k=1}^{\infty} 2^{-kN'} \int \frac{2^{kn}|\psi_k * f(z)|^r}{(1+2^k|x-z|)^{ar}} dz [\psi_{k,a}^* f(x)]^{1-r} \Big) \tag{22}$$

We now fix any $x \in \mathbb{R}^n$ and apply Lemma 5 with

$$d_j = \psi_{j,a}^* f(x), \text{ for } j \in \mathbb{N}, \quad d_0 = \Psi_a^* f(x),$$
$$b_j = \int \frac{2^{kn}|\psi_k * f(z)|^r}{(1+2^k|x-z|)^{ar}} dz, \text{ for } j \in \mathbb{N}, \text{ and } b_0 = \int \frac{|\Psi * f(z)|^r}{(1+|x-z|)^{ar}} dz.$$

Then we have

$$[\psi_{j,a}^* f(x)]^r \leqslant C'_N \sum_{k=j}^{\infty} 2^{(j-k)Nr} \int \frac{2^{kn}|\psi_k * f(z)|^r}{(1+2^k|x-z|)^{ar}} dz, \tag{23}$$

where $C'_N = C_{N+a-n}$,

We remark that (23) also holds when $r > 1$. In fact, to see this, it suffices to take (19) with $a+n$ instead of a, apply Hölder's inequalities in k and z, and finally the inequality deduced from (20).

Since the function $\dfrac{1}{(1+|z|)^{ar}} \in L_1$, by the majorant property of the Hardy-Littlewood maximal operator $\mathcal{M}$ (see, [9], Chapter 2,(3.9)), we deduce from (23) that

$$[\psi_{j,a}^* f(x)]^r \leqslant C'_N \sum_{k=j}^{\infty} 2^{(j-k)Nr} \mathcal{M}(|\psi_k * f|^r)(x), \tag{24}$$

and a similar inequality with $\psi_{j,a}^* f(x)$ replaced by $\Psi_a^* f(x)$.

By (24) choosing $N > \max(-s, 0)$, and applying Lemma 3 with

$$g_j = 2^{jsr} \mathcal{M}(|\psi_k * f|^r), \quad j \in \mathbb{N}, \quad g_0 = \mathcal{M}(|\Psi * f|^r)$$

we obtain that for all $f \in \mathcal{S}'(\mathbb{R}^n)$

$$\|\Psi_a^* f\|_{M_q^p} + \|\{2^{sj}\psi_{j,a}^* f\}_1^\infty\|_{\ell_\beta(M_q^p)} \lesssim \|\mathcal{M}_r(\Psi * f)\|_{M_q^p} + \|\{2^{js}\mathcal{M}_r(\psi_j * f)\}_1^\infty\|_{\ell_\beta(M_q^p)}. \tag{25}$$

$$\|\Psi_a^* f\|_{M_q^p} + \|\{2^{sj}\psi_{j,a}^* f\}_1^\infty\|_{M_q^p(\ell_\beta)} \lesssim \|\mathcal{M}_r(\Psi * f)\|_{M_q^p} + \|\{2^{js}\mathcal{M}_r(\psi_j * f)\}_1^\infty\|_{M_q^p(\ell_\beta)}. \tag{26}$$

where we used the notation $\mathcal{M}_r(g) = (\mathcal{M}(|g|^r))^{1/r}$.

For (25), we choose r so that $n/a < r < \beta$. By Lemma 4, we have (14).

For (26), we choose r so that $n/a < r < \min(q, \beta)$. By Lemma 4, we have (15).

Step 3. We will check that (4), (5) follow from (10), (11), and (14), (15). For instance, we do it for (4).

The left inequality in (4) is proved by the chain of estimates

$$\text{the left side of (4)} \lesssim \|A_a^* f\|_{M_q^p} + \|\{2^{js}\theta_j * f\}\|_{\ell_\beta(M_q^p)} \lesssim \|f\|_{M_q^p B_\beta^s},$$

here we first used (10) with $\Phi = A,\ \varphi = \theta$, and then applied (15) with $\Psi = A,\ \psi = \theta$.

The right inequality in (4) is proved by another chain

$$\begin{aligned}\|f\|_{M_q^p B_\beta^s} &\lesssim \|A_a^* f\|_{M_q^p} + \|\{2^{js}\theta_j * f\}\|_{\ell(M_q^p)} \\ &\lesssim \|\Psi_a^* f\|_{M_q^p} + \|\{2^{js}\psi_{j,a}^* f\}\|_{\ell_\beta(M_q^p)} \lesssim \text{the right side of (4)},\end{aligned}$$

here the the first inequality is obvious, the second is (10) with $\Phi = \Psi,\ \varphi = \psi$, and A and θ instead of Ψ and ψ in the left hand side. Finally, the third inequality is (15).

This completes the proof. ■

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