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# On a System of Semilinear Elliptic Equations on an Unbounded Domain

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**Abstract.** In this paper we study the existence of weak solutions of the Dirichlet problem for a system of semilinear elliptic equations on an unbounded domain in $\mathbb{R}^n$. The proof is based on a fixed point theorem in Banach spaces.

## 1. Introduction

In the present paper we consider the following Dirichlet problem:

$$
\begin{aligned}
-\Delta u + q(x)u &= \alpha u + \beta v + f_1(u,v) \quad \text{in} \quad \Omega \qquad (1.1)\\
-\Delta v + q(x)v &= \Delta u + \gamma v + f_2(u,v)\\
u|_{\partial\Omega} &= 0, \quad v|_{\partial\Omega} = 0\\
u(x) \to 0, &\quad v(x) \to 0 \quad \text{as} \quad |x| \to +\infty \qquad (1.2)
\end{aligned}
$$

where $\Omega$ is a unbounded domain with smooth boundary $\partial\Omega$ in a $\mathbb{R}^n$, $\alpha, \beta, \delta, \gamma$ are given real numbers, $\beta > 0, \delta > 0; q(x)$ is a function defined in $\Omega$, $f_1(u,v), f_2(u,v)$ are nonlinear functions for $u, v$ such that

$$
\begin{gathered}
q(x) \in C^0(\mathbb{R}), \quad \text{and} \quad \exists q_0 > 0, q(x) \geq q_0 \quad , \forall x \in \Omega \qquad (1.3)\\
q(x) \to +\infty \quad \text{as} \quad |x| \to +\infty
\end{gathered}
$$

$f_i(u,v)$ are Lipschitz continuous in $\mathbb{R}^n$ with constants $k_i (i = 1, 2)$

$$|f_i(u,v) - f_i(\bar{u},\bar{v})| \leqslant k_i(|u - \bar{u}| + |v - \bar{v}|), \quad \forall (u,v), (\bar{u},\bar{v}) \in \mathbb{R}^2. \qquad (1.4)$$

The aim of this paper is to study the existence of weak solution of the problem (1.1)-(1.2) under hypothesis (1.3), (1.4) and suitable conditions for the parameters $\alpha, \beta, \delta, \gamma$.

We firstly notice that the problem Dirichlet for the system (1.1) in a bounded smooth domain have been studied by Zuluaga in [6].

Throughout the paper, $(.,.)$ and $\|.\|$ denotes the usual scalar product and the norm in $L^2(\Omega)$; $H^1(\Omega), \overset{\circ}{H}{}^1(\Omega)$ are the usual Sobolev's spaces.

## 2. Preliminaries and Notations

We define in $C_0^\infty(\Omega)$ the norm (as in [1])

$$\|u\|_{q,\Omega} = \Big(\int\limits_\Omega |Du|^2 + qu^2 dx\Big)^{\frac{1}{2}}, \quad \forall u \in C_0^\infty(\Omega) \tag{2.1}$$

and the scalar product

$$a_q(u,v) = (u,v)_q = \int\limits_\Omega (DuDv + qu.v)dx \tag{2.2}$$

$$\text{where} \qquad Du = \Big(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \cdots, \frac{\partial u}{\partial x_n}\Big), \quad \forall u,v \in C_0^\infty(\Omega).$$

Then we introduce the space $V_q^0(\Omega)$ defined as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|.\|_{q,\Omega}$. Furthermore, the space $V_q^0(\Omega)$ can be considered as a Sobolev-Slobodeski's space with weight.

**Proposition 2.1.** (see [1]) *$V_q^0(\Omega)$ is a Hilbert space which is dense in $L^2(\Omega)$, and the embedding of $V_q^0(\Omega)$ into $L^2(\Omega)$ is continuous and compact.*

We define by the Lax-Milgram lemma a unique operator $H_q$ in $L^2(\Omega)$ such that

$$(H_q u, v) = a_q(u,v), \quad \forall u \in D(H_q), \forall v \in V_q^0(\Omega)$$

where $D(H_q) = \{u \in V_q^0(\Omega) : H_q u = (-\Delta + q)u \in L^2(\Omega)\}$.

It is obvious that the operator

$$H_q : D(H_q) \subset L^2(\Omega) \to L^2(\Omega)$$

is a linear operator with range $R(H_q) \subset L^2(\Omega)$.

Since $q(x)$ is positive, the operator $H_q$ is positive in the sense that:

$$(H_q u, u)_{L^2(\Omega)} \geq 0, \quad \forall u \in D(H_q)$$

and selfadjoint

$$(H_q u, v)_{L^2(\Omega)} = (u, H_q v)_{L^2(\Omega)}, \quad \forall u, v \in D(H_q).$$

Its inverse $H_q^{-1}$ is defined on $R(H_q) \cap L^2(\Omega)$ with range $D(H_q)$, considered as an operator into $L^2(\Omega)$. By Proposition 2.1 it follows that $H_q^{-1}$ is a compact operator in $L^2(\Omega)$. Hence the spectrum of $H_q$ consists of a countable sequence of eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$, each with finite multiplicity and the first eigenvalue $\lambda_1$ is isolated and simple:

$$0 < \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_k \cdots, \lambda_k \to +\infty \quad \text{as} \quad k \to +\infty.$$

Every eigenfunction $\varphi_k(x)$ associated with $\lambda_k \quad (k = 1, 2, \cdots)$ is continuous and bounded on $\Omega$ and there exist positive constants $\alpha$ and $\beta$ such that

$$|\varphi_k(x)| \leqslant \alpha e^{-\beta|x|} \quad \text{for } |x| \text{ large enough.}$$

Moreover eigenfunction $\varphi_1(x) > 0$ in $\Omega$ (see [1]).

**Proposition 2.2.** (Maximum principle. see [1]) *Assume that $q(x)$ satisfies the hypothesis* (1.3)*, and $\lambda < \lambda_1$. Then for any $g(x)$ in $L^2(\Omega)$, there exists a unique solution $u(x)$ of the following problem:*

$$H_q u - \lambda u = g(x) \text{ in } \Omega$$
$$u|_{\partial\Omega} = 0, \quad u(x) \to 0 \text{ as } |x| \to +\infty.$$

*Furthermore if $g(x) \geq 0$ , $g(x) \not\equiv 0$ in $\Omega$ then $u(x) > 0$ in $\Omega$.*

By Proposition 2.2 it follows that with $\lambda < \lambda_1$, the operator $H_q - \lambda$ is invertible, $D(H_q - \lambda) = D(H_q) \subset V_q^0(\Omega)$, and its inverse $(H_q - \lambda)^{-1} : L^2(\Omega) \to D(H_q) \subset L^2(\Omega)$ is considered as an operator into $L^2(\Omega)$, it follows from Proposition 2.1 that $(H_q - \lambda)^{-1}$ is a compact operator.

Observe further that

$$(H_q - \lambda)^{-1}\varphi_k(x) = \frac{1}{\lambda_k - \lambda}\varphi_k(x), \quad k = 1, 2, ... \tag{2.3}$$

**Definition.** *A pair $(u, v) \in V_q^0(\Omega) \times V_q^0(\Omega)$ is called a weak solution of the problem* (1.1)*,* (1.2) *if:*

$$\begin{aligned} a_q(u, \varphi) &= \alpha(u, \varphi) + \beta(v, \varphi) + (f_1(u, v), \varphi) \\ a_q(v, \varphi) &= \delta(u, \varphi) + \gamma(v, \varphi) + (f_2(u, v), \varphi), \quad \forall\varphi \in C_0^\infty(\Omega). \end{aligned} \tag{2.4}$$

*It is seen that if $u, v \in C^2(\Omega)$ then the weak solution $(u, v)$ is a classical solution of the problem.*

## 3. Existence of Weak Solutions for the Dirichlet Problem

3.1. Suppose that

$$\gamma < \min(q_0, \lambda_1),$$

where $\lambda_1$ is the first eigenvalue of the operator $H_q$.

Let $u_0$ be fixed in $V_q^0(\Omega)$. We consider the Dirichlet problem

$$(H_q - \gamma)v = \delta u_0 + f_2(u_0, v) \text{ in } \Omega \qquad (3.1)$$
$$v|_{\partial\Omega} = 0, \quad v(x) \to 0 \text{ as } |x| \to +\infty.$$

First, we remark that since $\gamma < \min(q_0, \lambda_1)$, $q(x) - \gamma > 0$ in $\Omega$. Then $H_q - \gamma$ is a positive, selfadjoint operator in $L^2(\Omega)$. Furthermore, the operator $(H_q - \gamma)$ is invertible and

$$(H_q - \gamma)^{-1} : L^2(\Omega) \to D(H_q) \subset L^2(\Omega)$$

is continuous compact in $L^2(\Omega)$. Hence the spectrum of $H_q - \gamma$ consists of a countable sequence of eigenvalues $\{\hat{\lambda}_k\}_{k=1}^{\infty}$ where $\hat{\lambda}_k = \lambda_k - \gamma$:

$$0 < \hat{\lambda}_1 < \hat{\lambda}_2 \leqslant \cdots \leqslant \hat{\lambda}_k \leqslant \cdots$$

Besides, we have

$$\|(H_q - \gamma)^{-1}\|_{L^2(\Omega)} \leqslant \frac{1}{\lambda_1 - \gamma}.$$

Under hypothesis (1.4), for $v$ fixed in $V_q^0(\Omega)$, $f_2(u_0, v) \in L^2(\Omega)$. Then the problem

$$(H_q - \gamma)w = \delta u_0 + f_2(u_0, v) \text{ in } \Omega \qquad (3.2)$$
$$w|_{\partial\Omega} = 0, \quad w(x) \to 0 \text{ as } |x| \to +\infty$$

has a unique solution $w = w(u_0, v)$ in $D(H_q)$ defined by

$$w = (H_q - \gamma)^{-1}[\delta u_0 + f_2(u_0, v)].$$

Thus, for any $u_0$ fixed in $V_q^0(\Omega)$, there exists an operator $A = A(u_0)$ mapping $V_q^0(\Omega)$ into $D(H_q) \subset V_q^0(\Omega)$, such that

$$Av = A(u_0)v = w = (H_q - \gamma)^{-1}[\delta u_0 + f_2(u_0, v)]. \qquad (3.3)$$

**Proposition 3.1.** *For all $v, \bar{v} \in V_q^0(\Omega)$ we have the following estimate:*

$$\|Av - A\bar{v}\| \leqslant \frac{k_2}{\lambda_1 - \gamma}\|v - \bar{v}\| \qquad (3.4)$$

*where $\|.\|$ is the norm in $L^2(\Omega)$.*

*Proof.* For $v, \bar{v} \in V_q^0(\Omega)$ we have

$$\|Av - A\bar{v}\| = \|(H_q - \gamma)^{-1}[f_2(u_0, v) - f_2(u_0, \bar{v})]\|$$
$$\leqslant \frac{1}{\lambda_1 - \gamma}\|f_2(u_0, v) - f_2(u_0, \bar{v})\|.$$

By hypothesis (1.4) it follows that

$$\|f_2(u_0, v) - f_2(u_0, \bar{v})\| \leqslant k_2\|v - \bar{v}\|.$$

From this we obtain the estimate (3.4). ∎

**Theorem 3.2.** *Suppose that*

$$\gamma < \min(q_0, \lambda_1), \quad \frac{k_2}{\lambda_1 - \gamma} < 1. \tag{3.5}$$

*Then for every $u_0$ fixed in $V_q^0(\Omega)$ there exists a weak solution $v = v(u_0)$ of the Dirichlet problem* (3.1).

*Proof.* Form (3.3), (3.4) and (3.5) it follows that the operator

$$A = A(u_0) : L^2(\Omega) \supset V_q^0(\Omega) \to D(H_q) \subset L^2(\Omega)$$

such that for $v \in V_q^0(\Omega)$,

$$Av = (H_q - \gamma)^{-1}[\delta u_0 + f_2(u_0, v)]$$

is a contraction operator in $L^2(\Omega)$.

Let $v_0 \in V_q^0(\Omega)$. We denote by

$$v_1 = Av_0, v_k = Av_{k-1} \quad k = 1, 2, ...$$

Then we obtain a sequence $\{v_k\}_{k=1}^{\infty}$ in $D(H_q)$. Since $A = A(u_0)$ is a contraction operator in $L^2(\Omega)$, $\{v_k\}_{k=1}^{\infty}$ is a fundamental sequence in $L^2(\Omega)$.

Therefore there exists a limit $\lim\limits_{k \to +\infty} v_k = v$ in $L^2(\Omega)$, or in other words:

$$\lim_{k \to +\infty} \|v_k - v\| = 0. \tag{3.6}$$

Moreover $v$ is fixed point of the operator $A : v = Av$ in $L^2(\Omega)$.

On the other hand for all $k, l \in \mathbb{N}^*$ we have

$$a_q(v_k - v_l, \varphi) = \big(H_q(v_k - v_l), \varphi\big) = (v_k - v_l, H_q\varphi), \quad \forall \varphi \in C_0^{\infty}(\Omega).$$

By applying the Schwarz's estimate we get

$$|a_q(v_k - v_l, \varphi)| \leqslant \|v_k - v_l\|.\|H_q\varphi\|, \quad \forall \varphi \in C_0^{\infty}(\Omega).$$

Letting $k, l \to +\infty$, since $\lim\limits_{k,l \to +\infty} \|v_k - v_l\| = 0$, from the last inequality we obtain that

$$\lim_{k,l \to +\infty} a_q(v_k - v_l, \varphi) = 0, \quad \forall \varphi \in C_0^{\infty}(\Omega).$$

Thus $\{v_k\}_{k=1}^{\infty}$ is a weakly convergent sequence in the Hilbert space $V_q^0(\Omega)$.

Then there exists $\bar{v} \in V_q^0(\Omega)$ such that

$$\lim_{k \to +\infty} a_q(v_k, \varphi) = a_q(\bar{v}, \varphi), \quad \varphi \in C_0^{\infty}(\Omega). \tag{3.7}$$

Since the embedding of $V_q^0(\Omega)$ into $L^2(\Omega)$ is continuous and compact then the sequence $\{v_k\}_{k=1}^{\infty}$ weakly converges to $\bar{v}$ in $L^2(\Omega)$. From this it follows that $v = \bar{v}$.

Besides, under hypothesis (1.4) we have the estimate:

$$\|f_2(u_0, v_k) - f_2(u_0, v)\| \leqslant k_2\|v_k - v\|.$$

By using (3.6), letting $k \to +\infty$ we obtain

$$\lim_{k\to+\infty} f_2(u_0, v_k) = f_2(u_0, v) \text{ in } L^2(\Omega). \tag{3.8}$$

In the sequel we will prove that $v$ defined by (3.6) is a weak solution of the problem (3.1).

For any $\varphi \in C_0^\infty(\Omega)$,

$$\begin{aligned} a_q(v_k, \varphi) &= (H_q v_k, \varphi) = \big((H_q - \gamma)v_k, \varphi\big) + \gamma(v_k, \varphi) \\ &= \big(v_k, (H_q - \gamma)\varphi\big) + \gamma(v_k, \varphi) \\ &= \big(Av_{k-1}, (H_q - \gamma)\varphi\big) + \gamma(v_k, \varphi) \\ &= \big((H_q - \gamma)^{-1}[\delta u_0 + f_2(u_0, v_{k-1})], (H_q - \gamma)\varphi\big) + \gamma(v_k, \varphi) \\ &= \big(\delta u_0 + f_2(u_0, v_{k-1}), \varphi\big) + \gamma(v_k, \varphi) \\ &= \delta(u_0, \varphi) + \big(f_2(u_0, v_{k-1}), \varphi\big) + \gamma(v_k, \varphi). \end{aligned}$$

Letting $k \to +\infty$ under (3.6), (3.7) and (3.8) we get

$$a_q(v, \varphi) = \delta(u_0, \varphi) + \gamma(v, \varphi) + \big(f_2(u_0, v), \varphi\big), \quad \forall \varphi \in C_0^\infty(\Omega).$$

Thus, $v$ is a weak solution of the Dirichlet problem (3.1). The proof of the Theorem 3.2 is complete. ■

3.2. Under hypothesis (3.5) according to Theorem 3.2 for any $u \in V_q^0(\Omega)$ there exists a weak solution $v = v(u)$ of the Dirichlet problem (3.1).

Let us denote $B$ as an operator mapping from $V_q^0(\Omega)$ into $D(H_q) \subset V_q^0(\Omega)$ such that for every $u \in V_q^0(\Omega)$

$$Bu = v = (H_q - \gamma)^{-1}[\delta u + f_2(u, Bu)]. \tag{3.9}$$

**Proposition 3.3.** *For every $u, \bar{u} \in V_q^0(\Omega)$ we have the following estimate:*

$$\|Bu - B\bar{u}\| \leqslant \frac{\delta + k_2}{\lambda_1 - \gamma - k_2}\|u - \bar{u}\|. \tag{3.10}$$

*Proof.* For $u, \bar{u} \in V_q^0(\Omega)$ we have

$$\begin{aligned} \|Bu - B\bar{u}\| &= \|(H_q - \gamma)^{-1}[\delta(u - \bar{u}) + f_2(u, Bu) - f_2(\bar{u}, B\bar{u})]\| \\ &\leqslant \frac{1}{\lambda_1 - \gamma}\big(\delta\|u - \bar{u}\| + k_2\|u - \bar{u}\| + k_2\|Bu - B\bar{u}\|\big) \\ &\leqslant \frac{\delta + k_2}{\lambda_1 - \gamma}\|u - \bar{u}\| + \frac{k_2}{\lambda_1 - \gamma}\|Bu - B\bar{u}\|. \end{aligned}$$

Under (3.5), $\lambda_1 - \gamma - k_2 > 0$, it follows that

$$\left(1 - \frac{k_2}{\lambda_1 - \gamma}\right)\|Bu - B\bar{u}\| \leqslant \frac{\delta + k_2}{\lambda_1 - \gamma}\|u - \bar{u}\|.$$

From that we obtain the estimate (3.10). ■

3.3. Assume that

$$\alpha < \min(q_0, \lambda_1)$$

where $\lambda_1$ is the first eigenvalue of the operator $H_q$.

For any $u \in V_q^0(\Omega)$, $Bu \in D(H_q) \subset V_q^0(\Omega)$, where $B$ is the operator defined by (3.9). Under hypothesis (1.4) $f_1(u, Bu) \in L^2(\Omega)$ then $\beta Bu + f_1(u, Bu) \in L^2(\Omega)$

Therefore for every $u \in V_q^0(\Omega)$ the variational problem:

$$\begin{gathered}(H_q - \alpha)U = \beta Bu + f_1(u, Bu) \text{ in } \Omega \\ U|_{\partial\Omega} = 0 \quad , \quad U(x) \to 0 \text{ as } |x| \to +\infty.\end{gathered} \tag{3.11}$$

has a unique solution

$$U = (H_q - \alpha)^{-1}[\beta Bu + f_1(u, Bu)] \quad \text{in } D(H_q).$$

Thus, there exists an operator

$$T : V_q^0(\Omega) \to D(H_q) \subset V_q^0(\Omega)$$

such that for every $u \in V_q^0(\Omega)$

$$U = Tu = (H_q - \alpha)^{-1}[\beta Bu + f_1(u, Bu)] \tag{3.12}$$

is a solution of the problem (3.11). Using a similar approach as for Proposition 3.3 we get the following proposition.

**Proposition 3.4.** *For all $u, \bar{u} \in V_q^0(\Omega)$ we have the estimate*

$$\|Tu - T\bar{u}\| \leqslant h\|u - \bar{u}\| \tag{3.13}$$

*where*

$$h = \frac{(\beta + k_1)(\delta + k_2) + k_1(\lambda_1 - \gamma - k_2)}{(\lambda_1 - \alpha)(\lambda_1 - \gamma - k_2)}.$$

Remark that $T$ considered as an operator into $L^2(\Omega)$, is a contraction operator if:

$$h = \frac{(\beta + k_1)(\delta + k_2) + k_1(\lambda_1 - \gamma - k_2)}{(\lambda_1 - \alpha)(\lambda_1 - \gamma - k_2)} < 1.$$

It is clear that this inequality is satisfied if and only if

$$\lambda_1 - \alpha - k_1 > 0 \quad \text{and} \quad \frac{(\beta + k_1)(\delta + k_2)}{(\lambda_1 - \alpha - k_1)(\lambda_1 - \gamma - k_2)} < 1. \tag{3.14}$$

**Theorem 3.5.** *Suppose that the conditions* (3.5), (3.14) *are satisfied. Then there exists a weak solution $u$ in $V_q^0(\Omega)$ of the following variational problem:*

$$\begin{gathered}(H_q - \alpha)u = \beta Bu + f_1(u, Bu) \\ u|_{\partial\Omega} = 0, u(x) \to 0 \text{ as } |x| \to +\infty.\end{gathered} \tag{3.15}$$

*Proof.* Under conditions (3.14), the operator $T$ defined by (3.12) is a contraction operator in $L^2(\Omega)$.

Let $u_0 \in V_q^0(\Omega)$. We denote

$$u_1 = Tu_0, \quad u_k = Tu_{k-1}, \quad k = 1, 2, ...$$

Then we obtain a sequence $\{u_k\}_{k=1}^{\infty}$ in $D(H_q)$. Since $T$ is a contraction operator in $L^2(\Omega)$, $\{u_k\}_{k=1}^{\infty}$ is a fundamental sequence in $L^2(\Omega)$. Therefore there is a limit: $\lim\limits_{k\to+\infty} u_k = u$ in $L^2(\Omega)$, or in other words:

$$\lim_{k\to+\infty} \|u_k - u\| = 0. \tag{3.16}$$

Moreover $u$ is a fixed point of the operator $T : u = Tu$ in $L^2(\Omega)$.

By using a similar approach as for the proof of Theorem 3.2 it follows that the sequence $\{u_k\}_{k=1}^{\infty}$ is weakly convergent in $V_q^0(\Omega)$ and there exists $\bar{u} \in V_q^0(\Omega)$ such that

$$\lim_{k\to+\infty} a_q(u_k, \varphi) = a_q(\bar{u}, \varphi), \quad \forall \varphi \in C_0^{\infty}(\Omega). \tag{3.17}$$

Since the embedding of $V_q^0(\Omega)$ into $L^2(\Omega)$ is continuous and compact then the sequence $\{u_k\}_{k=1}^{\infty}$ weakly converges to $\bar{v}$ in $L^2(\Omega)$. From this it follows that $v = \bar{v}$. Besides, under hypothesis (1.4) and inequality (3.10) we have

$$\|f_1(u_k, Bu_k) - f_1(u, Bu)\| \leqslant k_1\big(\|u_k - u\| + \|Bu_k - Bu\|\big)$$

and

$$\|Bu_k - Bu\| \leqslant \frac{\delta + k_2}{\lambda_1 - \gamma - k_2}\|u_k - u\|.$$

Letting $k \to +\infty$ from (3.16) it follows that

$$\lim_{k\to+\infty} Bu_k = Bu \text{ in } L^2(\Omega) \tag{3.18}$$

$$\lim_{k\to+\infty} f_1(u_k, Bu_k) = f_1(u, Bu) \text{ in } L^2(\Omega).$$

Furthermore for any $\varphi(x) \in C_0^{\infty}(\Omega)$

$$\begin{aligned} a_q(u_k, \varphi) &= (H_q u_k, \varphi) = (u_k, H_q\varphi) = \big(u_k, (H_q - \alpha)\varphi\big) + \alpha(u_k, \varphi) \\ &= \big((H_q - \alpha)^{-1}\big(\beta Bu_{k-1} + f_1(u_{k-1}, Bu_{k-1})\big), (H_q - \alpha)\varphi\big) + \alpha(u_k.\varphi) \\ &= \big(\beta Bu_{k-1} + f_1(u_{k-1}, Bu_{k-1}), \varphi\big) + \alpha(u_k, \varphi) \\ &= \beta(Bu_{k-1}, \varphi) + \big(f_1(u_{k-1}, Bu_{k-1}), \varphi\big) + \alpha(u_k, \varphi). \end{aligned}$$

Letting $k \to +\infty$ under (3.17), (3.18) we get

$$a_q(u, \varphi) = \beta(Bu, \varphi) + \big(f_1(u, Bu), \varphi\big) + \alpha(u, \varphi), \quad \forall \varphi \in C_0^{\infty}(\Omega).$$

Thus, $u$ is a weak solution of the problem (3.15). ■

**Theorem 3.6.** *Suppose that the conditions* (3.5), (3.14) *are satisfied. Then there exists a weak solution* $(u_0, v_0) \in V_q^0(\Omega) \times V_q^0(\Omega)$ *of the Dirichlet problem* (1.1)*,* (1.2).

*Proof.* Under hypothesis (3.5), from Theorem 3.2 there exists an operator

$$B : V_q^0(\Omega) \to D(H_q) \subset V_q^0(\Omega)$$

such that for every $u \in V_q^0(\Omega)$,

$$Bu = (H_q - \gamma)^{-1}[\delta u + f_2(u, Bu)].$$

On the other hand by Theorem 3.5 under hypothesis (3.14) the variational problem (3.15) has a weak solution $u_0 \in V_q^0(\Omega)$.

We denote $v_0 = Bu_0$. Then $(u_0, v_0)$ is a weak solution of the problem (1.1), (1.2). ■

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