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Mountain Pass Theorem and Nonuniformly Elliptic Equations

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Abstract. In this paper we improve Mountain Pass Theorem and Saddle Point Theorem. Our results only require that the functionals belong to $C^1_w(E)$ instead of $C^1(E)$, where $C^1_w(E)$ is the set of functionals that are weakly continuously differentiable on the Banach space E. An application is the existence of infinitely many generalized solutions to a nonuniformly nonlinear elliptic equation of the form $-div(a(x, \nabla u)) = f(x, u)$ in Ω with $u \in W_0^{1,p}(\Omega)$. Here a satisfies $|a(x, \xi)| \leqslant c_0 h(x)(1 + |\xi|^{p-1})$ for any ξ in $\mathbb{R}^n$, a.e. $x \in \Omega$, where $h \in L^{\frac{p}{p-1}}(\Omega)$.

1. Introduction

In this paper we use the following concept $C^1_w(E)$.

Definition 1.1. *Let I be a functional from a real Banach space E into $\mathbb{R}$. We say that I is weakly continuously differentiable on E if and only if the following conditions are satisfied:*

(i) *I is continuous on E.*

(ii) *For any $u \in E$ there exists a linear map $DI(u)$ from E into $\mathbb{R}$ such that*

$$\lim_{t \to 0} \frac{I(u + tv) - I(u)}{t} = DI(u)(v) \quad \forall v \in E.$$

(ii) *For any $v \in E$, the map $u \mapsto DI(u)(v)$ is continuous on E.*

Denote by $C^1_w(E)$ the set of weakly continuously differentiable functionals on E.

It is clear that $C^1(E) \subset C^1_w(E)$, where $C^1(E) \equiv C^1(E, \mathbb{R})$ denotes the set of continuously Fréchet differentiable functionals on E.

The Mountain-Pass Theorem, Saddle Point Theorem and the Z_2 version of Mountain Pass Theorem were proved in [9] for functionals of class $C^1(E)$. In the present paper, we extend these results to functionals of class $C^1_w(E)$. Now we recall some definitions. Let I be in $C^1_w(E)$, we put

$$||DI(u)|| = \sup\{|DI(u)(h)| : h \in E \text{ and} \|h\| = 1\}$$

for any $u \in E$, where $||DI(u)||$ may be ∞.

We say I satisfies the Palais-Smale condition if any sequence (u_m) in E for which $I(u_m)$ is bounded and $\lim_{m\to\infty} \|DI(u_m)\| = 0$ possesses a convergent subsequence.

In [3], the Mountain-pass Theorem in [9, p.7] was generalized as follows.

Theorem 1.2. *Let E be a real Banach space, I belong to $C^1_w(E)$, and I satisfy the Palais-Smale condition. Assume that $I(0) = 0$ and there exist a positive real number r and $z_0 \in E$ such that $\|z_0\| > r$, $I(z_0) \leqslant 0$ and $\alpha \equiv \inf\ \{I(u) : u \in E, \|u\| = r\}\ > 0$.*

Put $G = \{\varphi \in C([0,1], E) : \varphi(0) = 0, \varphi(1) = z_0\}$. Assume that $G \neq \emptyset$. Set $\beta = \inf\{\max I(\varphi([0,1]))\ :\ \varphi \in G\}$.

Then $\beta \geq \alpha$ and β is a critical value of I.

Our main results are the following theorems, which generalize the Z_2 version of Mountain Pass Theorem and Saddle Point Theorem in [9, p. 24, Theorem 4.6 and p. 55, Theorem 9.12] for functionals of class $C^1_w(E)$.

Theorem 1.3. *Let E be an infinite dimensional Banach space, B_r be the open ball in E of radius r centered at $0, \partial B_r$ be its boundary and I be in $C^1_w(E)$ such that I satisfies the Palais-Smale condition and $I(0) = 0$. Suppose $E = V \oplus X$, where V is a finite dimensional linear subspace of E. Moreover, assume that I is even and satisfies the following conditions*

(i) *There are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho \cap X} \geq \alpha$.*

(ii) *For each finite dimensional linear subspace $\hat{E}$ in E, there is a positive number $R = R(\hat{E})$ such that $I \leqslant 0$ on $\hat{E} \setminus B_{R(\hat{E})}$.*

Then I possesses an unbounded sequence of critical values.

Theorem 1.4. *Let E be a real Banach space such that $E = V \oplus X$, where $V \neq \{0\}$ and is a finite dimensional linear subspace of E. Suppose I belongs to $C^1_w(E)$ and satisfies the Palais-Smale condition. Assume the following conditions hold*

(i) *There exist a bounded neighborhood D of 0 in V and a constant α such that $I|_{\partial D} \leqslant \alpha$.*

(ii) *There is a constant $\beta > \alpha$ such that $I|_X \geq \beta$.*

Then I has a critical value $c \geq \beta$. Moreover, c can be characterized as

$$c = \inf_{h\in\Gamma} \max_{u\in\overline{D}} I(h(u)),$$

where $\Gamma = \{h \in C(\overline{D}, E) \ : \ h = id \ on \ \partial D\}$.

In Sec. 2 we prove our theorems. In the last section we apply these results to study the existence of nontrivial solutions of the following Dirichlet elliptic problem on a bounded domain $\Omega \subset \mathbb{R}^n$:

$$(P) \qquad \begin{cases} -div(a(x, \nabla u(x))) = f(x, u(x)) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

where $|a(x,\xi)| \leqslant c_0 h(x)(1 + |\xi|^{p-1})$ for any ξ in $\mathbb{R}^n$, a.e. $x \in \Omega$.

If h belongs to $L^\infty(\Omega)$, the problem has been studied in [2, 4, 6, 8, 10] and the references therein. Here we study the case in which h belongs to $L^{\frac{p}{p-1}}(\Omega)$. The equation now may be non-uniformly elliptic.

A prototypes of (P) is the following equations, which could not be handled by [2, 4, 6, 8, 10].

$$\begin{cases} -div(h(x)|\nabla u|^{p-2}\nabla u) = f(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

$$\begin{cases} -div(h(x)(1 + |\nabla u|^2)^{\frac{p-2}{2}}\nabla u) = f(x, u(x)) & \text{in } \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

where $p \geq 2$, $h \in L^{\frac{p}{p-1}}(\Omega)$.

The variational form of the problem (P) is $DJ(u) = 0$, where

$$J(u) = \int_\Omega A(x, \nabla u)dx - \int_\Omega F(x, u)\, dx.$$

For instance, the functional J for the problem (1.2) is defined by

$$J(u) = \int_\Omega \frac{1}{p} h(x)[(1 + |\nabla u|^2)^{\frac{p}{2}} - 1]dx - \int_\Omega F(x, u)\, dx.$$

If $h \equiv 1$, then J belongs to $C^1(W^{1,p}(\Omega))$ and satisfies conditions in classical Mountain Pass Theorem. This situation has been studied in [8].

In this paper, we consider $h \in L^{\frac{p}{p-1}}(\Omega)$. In this case, the value $J(u)$ may be infinity for some $u \in W^{1,p}(\Omega)$, that is, the functional may not be defined on throughout $W^{1,p}(\Omega)$. In order to overcome this difficulty, we choose a subspace Y of $W^{1,p}(\Omega)$ and an appropriate norm $\|.\|_Y$ such that Y is a Banach space and J is defined on Y. The space Y that satisfies this property is defined by $Y = \{\ u \in W_0^{1,p}(\Omega) : \int_\Omega h(x)|\nabla u|^p dx < +\infty\ \}$ with $\|u\|_Y = \left(\int_\Omega h(x)|\nabla u|^p\, dx\right)^{\frac{1}{p}}$ for any $u \in Y$. However, J may not be of class $C^1(Y)$, and hence we can not apply classical Mountain Pass Theorem to J. We see that J is weakly continuously differentiable (see Definition 1.1) and satisfies the conditions of generalized Mountain Pass Theorem, therefore we can apply Theorems 1.2 and 1.3 to such a functional J.

2. Mountain Pass Theorems

In this section, we prove Theorems (1.2)-(1.4), which are generalizations of Mountain Pass Theorem and Saddle Point Theorem. The main tool for proving these theorems is Theorem 2.2, which is a generalized deformation theorem. Hence, Theorem 2.2 is most important in this section. The following lemma is necessary for proving Theorem 2.2.

Lemma 2.1. *Let E be a real Banach space, and $I \in C^1_w(E)$. Assume that there exist an open set E_1, closed sets E_2, E_3 such that $E_3 \subset E_1$, $E_2 \cap E_3 = \emptyset$ and $E_1 \cup E_2 = E$. Suppose there exists a positive real number b such that $\|DI(u)\| \geq b$ for any $u \in E_1$.*

Then, there exists a vector field W from E into E such that

(i) *$\|W(y)\| \leqslant 1$ for any y in E and $\|W(z)\| = 0$ for any z in E_2.*

(ii) *$DI(u)W(u) \geq \dfrac{b}{2}$ for all $u \in E_3$ and $DI(u)W(u) \geq 0$ for all $u \in E$.*

(iii) *W is locally Lipschitz continuous on E.*

Moreover, if E_1, E_2, and E_3 are symmetric with respect to the origin and I is even on E. Then there exists a vector field W such that (i), (ii), (iii) hold and

(iv) *W is odd on E.*

Proof. For each $u \in E$, we can find a vector $w(u) \in E$ such that $\|w(u)\| = 1$ and $DI(u)w(u) \geq \dfrac{2}{3}\|DI(u)\|$. If $u \in E_1$, we have $DI(u)w(u) > \dfrac{b}{2}$. Hence there exists an open neighborhood N_u of u in E_1 such that $DI(v)w(u) > \dfrac{b}{2}$ for all $v \in N_u$ since $v \mapsto DI(v)w(u)$ is continuous on E. Because $\{N_u\ : u \in E_1\}$ is an open covering of E_1, it possesses a locally finite refinement which will be denoted by $\{N_{u_j}\}_{j\in J}$. Let $\rho_j(x)$ denote the distance from $x \in E_1$ to the complement of N_{u_j} for any j in J. Then ρ_j is Lipschitz continuous on E_1 and $\rho_j(x) = 0$ if $x \notin N_{u_j}$. Set $\beta_j(x) = \Big(\sum_{k\in J} \rho_k(x)\Big)^{-1} \rho_j(x)$ for any $x \in E_1$.

Since each x belongs to only finitely many sets N_{u_k}, $\sum_{k\in J} \rho_k(x)$ is only a finite sum. Set $W_0(x) \equiv \sum_{j\in J} \beta_j(x)w(u_j)$ for any $x \in E_1$. Then W_0 is locally Lipschitz continuous on E_1 and $W_0(x) > \dfrac{b}{2}$ for any $x \in E_1$.

Put $\alpha(x) = \dfrac{\|x - E_2\|}{\|x - E_2\| + \|x - E_3\|}$ for any $x \in E$. Then $\alpha = 0$ on E_2, $\alpha = 1$ on E_3, $0 \leqslant \alpha \leqslant 1$ on E, and α is Lipschitz continuous on E.

Set $W_1(x) = \alpha(x)W_0(x)$ for any $x \in E_1$ and $W_1(x) = 0$ for any $x \in E \setminus E_1$. It is clear that W_1 has the following properties:

(a) $\|W_1(y)\| \leqslant 1$ for any y in E, and $\|W_1(z)\| = 0$ for any z in E_2.

(b) $DI(u)W_1(u) \geq \frac{b}{2}$ for all $u \in E_3$ and $DI(u)W_1(u) \geq 0$ for all $u \in E$.

(c) W_1 is locally Lipschitz continuous on E.

If we choose $W = W_1$, properties (a)-(c) give (i)-(iii).

To prove (iv), we assume that E_1, E_2 and E_3 are symmetric with respect to the origin and I is even on E. Then we choose $W(u) = \frac{1}{2}(W_1(u) - W_1(-u))$ for all u in E. Property (i) comes from property (a) of W_1. We now use property (b) of W_1 to prove (ii). If u is in E_3, then $(-u)$ is also in E_3, so that

$$\begin{aligned} DI(u)W(u) &= \frac{1}{2}DI(u)\left(W_1(u) - W_1(-u)\right) = \frac{1}{2}\left(DI(u)W_1(u) - DI(u)W_1(-u)\right) \\ &= \frac{1}{2}\left(DI(u)W_1(u) + DI(-u)W_1(-u)\right) \geq \frac{1}{2}(\frac{b}{2} + \frac{b}{2}) = \frac{b}{2}. \end{aligned}$$

Moreover, $DI(u)W(u) = \frac{1}{2}\left(DI(u)W_1(u) + DI(-u)W_1(-u)\right) \geq 0$ for all $u \in E$. Hence (ii) holds. It is clear that (iii)-(iv) are satisfied. The proof is complete.

■

Let I be a real function on E, c be a real number and δ be a positive real number. We define

$$\begin{aligned} A_c &= \{u \in E \ : \ I(u) \leqslant c\}, \\ K_c &= \{u \in E \ : \ I(u) = c \quad \text{and} \quad DI(u) = 0\}, \\ N_\delta &= \begin{cases} \emptyset & \text{if} \quad K_c = \emptyset, \\ \{u \in E \ : \ \|u - K_c\| < \delta\} & \text{if} \quad K_c \neq \emptyset. \end{cases} \end{aligned}$$

We shall generalize Theorem A.4 in [9, p. 82] for functionals of class $C^1_w(E)$ as follows.

Theorem 2.2 (Deformation Theorem) *Let E be a real Banach space, and $I \in C^1_w(E)$. Suppose I satisfies the Palais-Smale condition. Let $c \in \mathbb{R}$, $\overline{\varepsilon} > 0$ be given and let $\mathcal{O}$ be any neighborhood of K_c. Then there exist a number $\varepsilon \in (0, \overline{\varepsilon})$ and $\eta \in C([0,\infty) \times E, E)$ such that*

(i) $\eta(0,u) = u \quad \forall\ u \in E$.

(ii) $\eta(t,u) = u \quad \forall\ t \in [0,\infty)$, $u \in E \setminus I^{-1}[c - \overline{\varepsilon}, c + \overline{\varepsilon}]$.

(iii) $\eta(t,.)$ *is a homeomorphism of E onto E for each* $t \in [0,\infty)$.

(iv) $\|\eta(t,u) - u\| \leqslant t$ *for all* $t \in [0,\infty)$, $u \in E$.

(v) *For any* $u \in E$, $I(\eta(t,u))$ *is non-increasing in* t.

(vi) $\eta(1, A_{c+\varepsilon} \setminus \mathcal{O}) \subset A_{c-\varepsilon}$.

(vii) *If* $K_c = \emptyset$, $\eta(1, A_{c+\varepsilon}) \subset A_{c-\varepsilon}$.

(viii) *If I is even on E*, $\eta(t,.)$ *is odd on E*.

Proof. Since I satisfies the Palais-Smale condition, K_c is empty or compact. Thus we can choose δ suitably small such that $N_\delta \subset \mathcal{O}$.

We claim there are positive constants $b, \hat{\varepsilon}$ such that

$$\|DI(u)\| \geq b \quad \forall\ u \in A_{c+\hat{\varepsilon}} \setminus (A_{c-\hat{\varepsilon}} \cup N_{\frac{\delta}{8}}). \tag{2.1}$$

Assume by contradiction that there are a sequence $\{u_m\}$ in $A_{c+\hat{\varepsilon}_m} \setminus (A_{c-\hat{\varepsilon}_m} \cup N_{\frac{\delta}{8}})$ and two sequences of positive real numbers $\{b_m\}$ and $\{\hat{\varepsilon}_m\}$ such that $\|DI(u_m)\| < b_m$ and $\lim\limits_{m\to\infty} b_m = \lim\limits_{m\to\infty} \hat{\varepsilon}_m = 0$. We see that $I(u_m) \to c$ and $\|DI(u_m)\| \to 0$. By the Palais-Smale condition, there is a subsequence of $\{u_m\}$ converging to some u in K_c. Moreover, $u \in E \setminus N_{\frac{\delta}{8}}$ since u_m belongs to a closed set $E \setminus N_{\frac{\delta}{8}}$ for any m. Therefore $u \in K_c \setminus N_{\frac{\delta}{8}}$, where $K_c \subset N_{\frac{\delta}{8}}$. This is a contradiction. Hence, there are positive constants b and $\hat{\varepsilon}$ as in (2.1). Choose

$$\varepsilon = \frac{1}{2}\min\{\hat{\varepsilon}, \overline{\varepsilon}, \frac{b\delta}{32}, \frac{b}{4}\}. \tag{2.2}$$

Put

$$E_1 = \begin{cases} \left\{u \in E \ : \ c - \hat{\varepsilon} < I(u) < c + \hat{\varepsilon} \text{ and } \|u - K_c\| > \dfrac{\delta}{8}\right\} & \text{if } K_c \neq \emptyset, \\ \{u \in E \ : \ c - \hat{\varepsilon} < I(u) < c + \hat{\varepsilon}\} & \text{if } K_c = \emptyset, \end{cases}$$

$$E_2 = \begin{cases} \left\{u \in E : I(u) \leqslant c - \dfrac{4\varepsilon}{3} \text{ or } I(u) \geq c + \dfrac{4\varepsilon}{3} \text{ or } \|u - K_c\| \leqslant \dfrac{3\delta}{16}\right\} & \text{if } K_c \neq \emptyset, \\ \left\{u \in E : I(u) \leqslant c - \dfrac{4\varepsilon}{3} \text{ or } I(u) \geq c + \dfrac{4\varepsilon}{3}\right\} & \text{if } K_c = \emptyset, \end{cases}$$

$$E_3 = \begin{cases} \left\{u \in E \ : \ c - \varepsilon \leqslant I(u) \leqslant c + \varepsilon \ \text{ and } \|u - K_c\| \geq \dfrac{\delta}{4}\right\} & \text{if } K_c \neq \emptyset, \\ \{u \in E \ : \ c - \varepsilon \leqslant I(u) \leqslant c + \varepsilon\} & \text{if } K_c = \emptyset. \end{cases}$$

It is clear that E_1, E_2 and E_3 satisfy the conditions in Lemma 2.1 and there exists a vector field W on E as in Lemma 2.1.

Let us consider the following Cauchy problem

$$\begin{cases} \dfrac{d\eta}{dt} = -W(\eta), \\ \eta(0, u) = u. \end{cases} \tag{2.3}$$

Since W is locally Lipschitz continuous throughout E and $\|W(.)\| \leqslant 1$ on E, there exists a global solution η in $C^1([0,\infty) \times E, E)$ to the problem (2.3). The initial condition of (2.3) gives (i). Since $W(.) = 0$ on E_2 and $I^{-1}(\mathbb{R}\setminus[c-\overline{\varepsilon}, c+\overline{\varepsilon}]) \subset E_2$, the property (ii) is satisfied. The semigroup property for solutions of the problem (2.3) gives (iii).

By (2.3) we have $\|\eta(t,u) - \eta(0,u)\| = \|\int_0^t W(\eta(s,u))ds\| \leqslant \int_0^t 1ds = t$ for every $t \geq 0$. It implies (iv).

From (2.3) and (ii) of Lemma 2.1, we infer that

$$\frac{dI(\eta(t,u))}{dt} = DI(\eta(t,u))(-W(\eta(t,u))) = -DI(\eta(t,u))W(\eta(t,u)) \leqslant 0 \ \ \forall\, t \in (0,\infty),$$

which yields (v).

Since $N_\delta \subset \mathcal{O}$, we now prove $\eta(1, A_{c+\varepsilon} \setminus N_\delta) \subset A_{c-\varepsilon}$ instead of (vi). If $u \in A_{c-\varepsilon}$, then $I(\eta(1,u)) \leqslant c - \varepsilon$ by (v), so that $\eta(1,u) \in A_{c-\varepsilon}$. Hence we need only prove that $\eta(1, A_{c+\varepsilon} \setminus (A_{c-\varepsilon} \cup N_\delta)) \subset A_{c-\varepsilon}$.

Let u be in $A_{c+\varepsilon} \setminus (A_{c-\varepsilon} \cup N_\delta)$. For $t \geq 0$, put $T(t) = \{\eta(s,u) : 0 \leqslant s \leqslant t\}$. By (v),

$$I(\eta(t,u)) \leqslant I(\eta(0,u)) = I(u) \leqslant c + \varepsilon \quad \forall\, t \geq 0,$$

which implies that $T(1)$ belongs to $A_{c+\varepsilon}$.

Assume by contradiction that

$$T(1) \cap A_{c-\varepsilon} = \emptyset. \tag{2.4}$$

Since $T(0) = \{u\}$ is a subset of the closed set E_3, there exists t_0 such that

$$t_0 = \max\{t \in [0,1] : T(t) \subset E_3\}.$$

It is clear that $E_3 \subset A_{c+\hat{\varepsilon}} \setminus \left(A_{c-\hat{\varepsilon}} \cup N_{\frac{\delta}{8}}\right)$. By (2.1), we obtain

$$\begin{aligned} I(\eta(0,u)) - I(\eta(t_0,u)) &= \int_{t_0}^{0} \frac{dI(\eta(s,u))}{ds} ds \\ &= \int_0^{t_0} DI(\eta(s,u))W(\eta(s,u))ds \geq \int_0^{t_0} \frac{b}{2} ds = \frac{b}{2} t_0. \end{aligned}$$

On the other hand, because $\eta(0,u), \eta(t_0,u) \in T(1) \subset A_{c+\varepsilon} \setminus A_{c+\varepsilon}$, we get

$$I(\eta(0,u)) - I(\eta(t,u)) < 2\varepsilon.$$

Hence $2\varepsilon > \dfrac{b}{2} t_0$, and we have

$$t_0 < \frac{4\varepsilon}{b} < \frac{\delta}{8}. \tag{2.5}$$

By (iv), we have $\|\eta(t,u) - \eta(0,u)\| \leqslant t \leqslant t_0 < \dfrac{\delta}{8}$ for every $t \in [0,t_0]$, where $\eta(0,u) \in E \setminus N_\delta$. So that, $T(t_0) \in E \setminus N_{\frac{7}{8}\delta}$.

Assume by contradiction that $t_0 < 1$. Then, there exists $t_1 \in (t_0, 1]$ such that $T(t_1) \subset E \setminus N_{\frac{\delta}{4}}$. Therefore, $T(t_1) \subset A_{c+\varepsilon} \setminus \left(A_{c-\varepsilon} \cup N_{\frac{\delta}{4}}\right) \subset E_3$. This contradicts the definition of t_0. Hence, $t_0 = 1$. By (2.5), we have $\varepsilon > \dfrac{b}{4}$, which contradicts (2.2). This together with (2.4) implies that $T(1) \cap A_{c-\varepsilon} \neq \emptyset$.

Hence, there exists $t_3 \in [0,1]$ such that $I(\eta(t_3,u)) \leqslant c - \varepsilon$. By (v), $I(\eta(1,u)) \leqslant c - \varepsilon$. It implies that $\eta(1,u) \in A_{c-\varepsilon}$.

Thus, $\eta(1, A_{c+\varepsilon} \setminus (A_{c-\varepsilon} \cup N_\delta)) \subset A_{c-\varepsilon}$. We deduce that $\eta(1, A_{c+\varepsilon} \setminus N_\delta) \subset A_{c-\varepsilon}$. Hence, (vi) and (vii) hold.

It remains only to prove (viii).

If I is even on E, then E_1, E_2, E_3 are symmetric sets with respect to the origin. Therefore, W is odd by (iv) of Lemma 2.1. Hence $\eta(t,.)$ is odd on E and (viii) follows. The proof is complete. ∎

2.3. Proof of Theorem 1.2

Theorem 1.2 is an application of Theorem 2.1 in [3, p. 433] with $F = E$ and $f = I$.

2.4. Proof of Theorem 1.3

Theorem 1.3 is similar to the Z_2 version of the Mountain Pass Theorem in [9, p. 55, Theorem 9.12], but the functional I in Theorem 1.3 belongs to $C^1_w(E)$ instead of $C^1(E)$ as in [9].

The proof of the Z_2 version of the Mountain Pass Theorem in [9, p. 55, Theorem 9.12] bases on Theorem 8.1 in [9, p. 55], which relies on the Deformation theorem in [9, p. 81,Theorem A.4]. Using Theorem 2.2 of the present paper instead of the cited Theorem A.4, and arguing as in the proofs of the cited Theorems 8.1 and 9.12, we get the desired result.

2.5. Proof of Theorem 1.4

Arguing as in the proof of Theorem 1.3 above, we have Theorem 1.4 .

3. Application

We first introduce some hypotheses.

Let p be in $(1, +\infty)$ and Ω be a bounded domain in $\mathbb{R}^n$ having C^2 boundary $\partial\Omega$. Let A be a measurable function on $\Omega \times R^n$ such that $A(x, 0) = 0$ and $a(x,\xi) \equiv \dfrac{\partial A(x,\xi)}{\partial \xi}$ is a Carathéodory function on $\Omega \times \mathbb{R}^n$. Assume that there are positive real numbers c_0, k_0, k_1 and a nonnegative measurable function h on Ω such that $h \in L^{\frac{p}{p-1}}(\Omega)$, and $h(x) \geq 1$ for a.e. x in Ω. Suppose the following conditions hold:

$(A1)$ $\qquad |a(x,\xi)| \leqslant c_0 h(x)(1 + |\xi|^{p-1}) \qquad \forall \xi \in \mathbb{R}^n$, a.e. $x \in \Omega$.

$(A2)$ A is p-uniformly convex, that is,

$$A(x, t\xi + (1-t)\eta) + k_1 h(x)|\xi - \eta|^p \leqslant tA(x,\xi) + (1-t)A(x,\eta),$$
$$\forall (\xi, \eta, t) \in \mathbb{R}^n \times \mathbb{R}^n \times [0,1], \quad \text{a.e. } x \in \Omega.$$

$(A3)$ A is p-subhomogeneous:

$$0 \leqslant a(x,\xi)\cdot\xi \leqslant pA(x,\xi) \quad \forall \xi \in \mathbb{R}^n, \text{ a.e. } x \in \Omega.$$

$(A4)$ $\qquad A(x,\xi) \geq k_0 h(x)|\xi|^p \qquad \forall \xi \in R^n$, a.e. $x \in \Omega$.

Let f be a real Carathéodory function on $\Omega \times \mathbb{R}$ having the following properties

$(F1)$ $\qquad |f(x,s)| \leqslant c_1(1 + |s|^{q-1}) \qquad \forall s \in \mathbb{R}$, a.e. $x \in \Omega$,

where c_1 is a positive real number, $q \in (p, +\infty)$ if $p \geq n$, and $q \in (p, p^*)$ with $p^* = np/(n-p)$ if $p < n$.

$(F2)$ There are a constant $\theta > p$ and a positive real number s_0 such that

$$0 < \theta F(x,s) \leqslant f(x,s)s \quad \forall s \in \mathbb{R} \setminus (-s_0, s_0), \text{ a.e. } x \in \Omega,$$

where $F(x,s) = \int\limits_0^s f(x,t)dt$.

$(F3)$ There are $\mu \in (0, k_0 p\lambda_1)$ and a positive real number δ such that

$$\frac{f(x,s)}{|s|^{p-2}s} \leqslant \mu \quad \forall s \in (-\delta, \delta) \setminus \{0\}, \quad \text{a.e. } x \in \Omega,$$

where $\lambda_1 = \inf \{\int\limits_\Omega h(x)|\nabla u|^p\, dx \Big(\int\limits_\Omega |u|^p\, dx\Big)^{-1} : u \in W_0^{1,p}(\Omega)\backslash\{0\}\}$.

The following theorem is our main result in this section.

Theorem 3.1. *Under the conditions* $(A1)-(A4)$ *and* $(F1)-(F3)$*, let us consider the following Dirichlet problem*

$$(P) \qquad \begin{cases} -div(a(x,\nabla u(x))) = f(x,u(x)) & \textit{in } \Omega,\\ u = 0 & \textit{on } \partial\Omega. \end{cases}$$

(i) *Then the problem* (P) *has at least one nontrivial generalized solution in* $W_0^{1,p}(\Omega)$.

(ii) *Moreover, suppose* A *and* F *are even with respect to the second variable:*

$$A(x,-\xi) = A(x,\xi) \quad \forall \xi \in \mathbb{R}^n, \quad \text{a.e. } x \in \Omega,$$
$$F(x,-s) = F(x,s) \quad \forall s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega.$$

Then the problem (P) *has infinitely many nontrivial generalized solutions in* $W_0^{1,p}(\Omega)$.

Remark. By Poincaré inequality and $h(x) \geq 1$, there exists a positive constant λ' such that

$$\lambda' \leqslant \Big(\int\limits_\Omega |\nabla u|^p\, dx\Big)\Big(\int\limits_\Omega |u|^p\, dx\Big)^{-1} \leqslant \Big(\int\limits_\Omega h(x)|\nabla u|^p\, dx\Big)\Big(\int\limits_\Omega |u|^p\, dx\Big)^{-1}.$$

Hence $\lambda_1 > 0$ and $\int\limits_\Omega |u|^p\, dx \leqslant \dfrac{1}{\lambda_1}\displaystyle\int_\Omega h(x)|\nabla u|^p\, dx \quad \forall u \in W_0^{1,p}(\Omega)$.

We denote by J the functional defined by

$$\begin{aligned} J(u) &= \int\limits_\Omega A(x,\nabla u)dx - \int_\Omega F(x,u)\, dx \\ &= P(u) - T(u), \quad \forall u \in W_0^{1,p}(\Omega), \end{aligned}$$

where $P(u) = \int\limits_\Omega A(x,\nabla u)dx$ and $T(u) = \int\limits_\Omega F(x,u)dx$.

The variational form of the problem (P) is $DJ(u) = 0$. We shall apply generalized Mountain Pass Theorem to prove the existence of critical points of the functional J. We first choose a real Banach space Y such that J is defined and weakly continuously differentiable on Y. This space Y is defined as in the following lemma.

Lemma 3.2. *Suppose* $Y = \{\, u \in W_0^{1,p}(\Omega) \;:\; \int\limits_\Omega h(x)|\nabla u|^p\, dx < +\infty \,\}$ *and put*

$$\|u\|_Y = \left(\int_\Omega h(x)|\nabla u|^p \, dx \right)^{\frac{1}{p}} \quad \textit{for all} \;\; u \in Y.$$

Then the following properties hold:

(i) $\left(\int |\nabla u|^p dx \right)^{\frac{1}{p}} \leqslant \|u\|_Y$ *for any* $u \in Y$ *, where* $\left(\int |\nabla u|^p dx \right)^{\frac{1}{p}}$ *is the usual norm of* u *in the Sobolev space* $W_0^{1,p}(\Omega)$.

(ii) $C_c^\infty(\Omega)$ *is a subset of* Y.

(iii) $(Y, \|.\|_Y)$ *is an infinite dimensional Banach space.*

(iv) $Y = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega h(x)|\nabla u|^p \, dx + \int_\Omega f(x,u) dx < +\infty \right\}$.

Proof.
(i) Since $h(x) \geq 1$ for a.e $x \in \Omega$, we deduce (i).
(ii) Suppose $u \in C_c^\infty(\Omega)$. Since h is bounded on support(u), u is in Y.
(iii) It is clear that Y is a normed space and has infinite dimension. Now we prove that the space Y is complete. Let $\{u_m\}$ be a Cauchy sequence in Y. Then $\lim\limits_{m\to\infty} \liminf_{j\to\infty} \int_\Omega h|\nabla u_j - \nabla u_m|^p \, dx = 0$ and $\{\|u_m\|_Y\}_m$ is bounded above. Moreover, by (i) the sequence $\{u_m\}$ is also a Cauchy sequence in the usual Sobolev space $W_0^{1,p}(\Omega)$. So that, the sequence $\{u_m\}$ converges to some u in $W_0^{1,p}(\Omega)$. Therefore $\{\nabla u_m(x)\}$ converges to $\nabla u(x)$ for a.e. x in Ω. Applying Fatou's lemma we get

$$\int_\Omega h(x)|\nabla u|^p \, dx \leqslant \liminf_{m\to\infty} \int_\Omega h(x)|\nabla u_m|^p \, dx = \liminf_{m\to\infty} \|u_m\|_Y^p < \infty.$$

Hence u is in Y. Applying again Fatou's lemma we have

$$\lim_{m\to\infty} \int_\Omega h(x)|\nabla u - \nabla u_m|^p \, dx \leqslant \lim_{m\to\infty} \left[\liminf_{j\to\infty} \int_\Omega h(x)|\nabla u_j - \nabla u_m|^p \, dx \right] = 0.$$

Hence $\{u_m\}$ converges to u in Y, so that Y is complete. Thus, Y is a Banach space. (iv) By $(F1)$, $\int_\Omega f(x,u)dx < +\infty$ for all $u \in W_0^{1,p}(\Omega)$. This give (iv).

This proof is complete. ■

Before applying generalized Mountain Pass Theorem, we need some lemmas. We list here some properties of A, F before checking properties of P, T, J.

Lemma 3.3.

(i) *A verifies the growth condition :* $|A(x,\xi)| \leqslant c_0 h(x)(|\xi| + |\xi|^p) \quad \forall \; \xi \in \mathbb{R}^n$, *a.e* $x \in \Omega$.

(ii) *There exists a constant* c_2 *such that* $|F(x,s)| \leqslant c_2(1+|s|^q) \;\; \forall s \in \mathbb{R}$, *a.e.* $x \in \Omega$.

(iii) *There exists* $\gamma \in L^\infty(\Omega)$ *such that* $\gamma(x) > 0$ *for a.e.* x *in* Ω *and* $F(x,s) \geq \gamma(x)|s|^\theta \;\; \forall s \in \mathbb{R} \setminus (-s_0, s_0)$, *a.e.* $x \in \Omega$.

Proof.
(i) By $(A1)$ we have

$$|A(x,\xi)| = \Big|\int_0^1 \frac{d}{dt}A(x,t\xi)dt\Big| = \Big|\int_0^1 a(x,t\xi)\cdot\xi dt\Big| \leqslant \int_0^1 |a(x,t\xi)|\;|\xi|\;dt$$
$$\leqslant \int_0^1 c_0 h(x)(1+|\xi|^{p-1}t^{p-1})\;|\xi|\;dt \leqslant c_0 h(x)(|\xi|+|\xi|^p)$$

(ii) This follows from $(F1)$.
(iii) It has been proved in [8, p. 1217, Property (14)].
The proof is complete. ∎

The following lemma concerns the smoothness of the functions T, P. We shall use this lemma to prove properties of J in Lemmata 3.5 and 3.6.

Lemma 3.4.
(i) *$P(tu+(1-t)z)+k_1\|u-z\|_Y^p \leqslant tP(u)+(1-t)P(z)$ for any $u,z\in Y, t\in[0,1]$.*
(ii) *T belongs to $C^1_w(Y)$ and $DT(u)(v)=\int_\Omega f(x,u)v\,dx \quad \forall u,v\in Y$.*
(iii) *If $\{u_m\}$ is a sequence weakly converging to u in $W_0^{1,p}(\Omega)$, then $T(u)=\lim_{m\to\infty} T(u_m)$ and $P(u)\leqslant \liminf_{m\to\infty} P(u_m)$.*
(iv) *P belongs to $C^1_w(Y)$ and $DP(u)(v)=\int_\Omega a(x,\nabla u)\cdot\nabla v\,dx \quad \forall u,v\in Y$.*
(v) *$P(u)-P(v)\geq DP(v)(u-v) \quad \forall u,v\in Y$.*

Proof.
(i) It comes from $(A2)$.
(ii) By Proposition 6 in [6, p. 354] and $(F1)$, the function T is continuously Fréchet differentiable on $L^q(\Omega)$ and $DT(u)(v)=\int_\Omega f(x,u)vdx \quad \forall u,v\in L^q(\Omega)$.

In view of Lemma 3.2-(i) and Sobolev Embedding Theorem (see [11, p. 97]), it implies (ii).
(iii) Let $\{u_m\}$ be a sequence weakly converging to u in $W_0^{1,p}(\Omega)$. Arguing as in (ii), we get that T is continuous on $L^q(\Omega)$. By the Rellich-Kondrachov theorem (see [5, p. 144]) we deduce $\lim_{m\to\infty} T(u_m)=T(u)$.

Applying the Rellich-Kondrachov theorem again, we see that $\{u_m\}$ converges strongly to u in $L^1(\Omega)$. By Theorem 4.5 in [5, p. 129] we have $P(u)\leqslant \liminf_{m\to\infty} P(u_m)$.
(iv) We first prove that P is continuous on Y.

Assume by contradiction that there exist $w\in Y$, $\varepsilon_w>0$, $\{w_m\}$ such that the sequence $\{w_m\}$ converges to w in Y and $|P(w_m)-P(w)|\geq\varepsilon_w$ for any $m\in\mathbb{N}$. Then we claim there is a subsequence $\{w_{m_j}\}_j$ of $\{w_m\}$ such that $\{P(w_{m_j})\}_j$ converges to $P(w)$. Indeed, since $\{h^{1/p}|\nabla w_m|\}_m$ converges to $h^{1/p}|\nabla w|$ in $L^p(\Omega)$, there exists a subsequence $\{h^{1/p}|\nabla w_{m_j}|\}_j$ such that $||h^{1/p}(|\nabla w_{m_{j+1}}|-|\nabla w_{m_j}|)||_{L^p}\leqslant 2^{-j}$ for any positive integer j. Put

$$g=\Big\{h^{1/p}|\nabla w_{m_1}|+\sum_1^\infty h^{1/p}|\;|\nabla w_{m_{j+1}}|-|\nabla w_{m_j}|\;|\Big\}^p.$$

We see that g is in $L^1(\Omega)$ and $|\nabla w_{m_j}|^p \leqslant h|\nabla w_{m_j}|^p \leqslant g$ a.e. in Ω for any $j \in \mathbb{N}$. According to (i) of Lemma 3.3, for any $j \in \mathbb{N}$, a.e. $x \in \Omega$ we have

$$\begin{aligned}|A(x, \nabla w_{m_j}(x))| &\leqslant c_0[h(x)|\nabla w_{m_j}(x)| + h(x)|\nabla w_{m_j}(x)|^p] \\ &\leqslant c_0[h(x)g^{1/p} + g(x)],\end{aligned}$$

where the last term on the right-hand side is integrable on Ω.

On the other hand, $\{\nabla w_{m_j}(x)\}_j$ converges to $\nabla w(x)$ for almost everywhere x in Ω since $\{h^{1/p}|\nabla w_{m_j}|\}_j$ converges to $h^{1/p}|\nabla w|$ in $L^p(\Omega)$. Hence, by the Carathéodory property of A we see that $\{A(x, \nabla w_{m_j}(x)\}$ converges to $A(x,\nabla w(x))$ for almost everywhere x in Ω. By the Lebesgue Dominated convergence theorem, we conclude that

$$\lim_{j\to\infty} \int_\Omega A(x, \nabla w_{m_j})dx = \int_\Omega A(x, \nabla w)\ dx; \text{ that is, } \lim_{j\to\infty} P(w_{m_j}) = P(w).$$

This contradiction implies that P is continuous on Y.

We now prove that $DP(u)(v) = \int\limits_\Omega a(x, \nabla u) \cdot \nabla v\ dx \quad \forall\ u,\ v\ \in\ Y.$

For any u, v in Y, any t in $(-1, 1)$ and any x in Ω, we have

$$\begin{aligned}&\Big|\frac{A(x, \nabla u(x){+}t\nabla v(x)){-}A(x, \nabla u(x))}{t}\Big| = \Big|\int_0^1 a(x, \nabla u(x){+}\tau t\nabla v(x)) \cdot \nabla v(x)\ d\tau\Big| \\ &\leqslant \int_0^1 c_0 h(x) \left(1 + |\nabla u(x) + \tau t\nabla v(x)|^{p-1}\right)\ |\nabla v(x)|\ d\tau \\ &\leqslant c_0 h(x) \left[1 + (|\nabla u(x)| + |\nabla v(x)|)^{p-1}\right] |\nabla v(x)| \\ &\leqslant c_0 h(x)|\nabla v(x)| + c_0\left[h^{\frac{1}{p}}(x)|\nabla v(x)|\right] \left(h^{\frac{1}{p}}(x)|\nabla u(x)| + h^{\frac{1}{p}}(x)|\nabla v(x)|\right)^{p-1}.\end{aligned}$$

Since $h^{\frac{p}{p-1}}$, $|\nabla v|^p$, $h|\nabla u|^p$ and $h|\nabla v|^p$ are integrable on Ω, the last term on the right-hand side is integrable on Ω. Applying the Lebesgue Dominated convergence theorem, we see that

$$DP(u)(v) = \lim_{t\to 0} \int_\Omega \frac{A(x, \nabla u + t\nabla v) - A(x, \nabla u)}{t} dx = \int_\Omega a(x, \nabla u) \cdot \nabla v\ dx.$$

It remains to prove the continuity of the map $u \mapsto DP(u)(v)$ for any v in Y.

Fix a vector v in Y. Suppose by contradiction that the map $u \mapsto DP(u)(v)$ is not continuous at some vector $w \in Y$. Then, there exist a positive ε and a sequence $\{w_m\}$ in Y such that $\|w_m - w\|_Y \to 0$ and $|DP(w_m)(v) - DP(w)(v)| > \varepsilon$ for any $m \in \mathbb{N}$.

Arguing as above, we can find a function g in $L^1(\Omega)$ such that $|\nabla w_{m_j}|^p \leqslant h|\nabla w_{m_j}|^p \leqslant g$ a.e. in Ω for any $j \in \mathbb{N}$. Therefore , for any $j \in \mathbb{N}$, a.e. $x \in \Omega$ we get

$$\begin{aligned} &|a(x, \nabla w_{m_j}(x))||\nabla v(x)| \leqslant c_o h(x)[1 + |\nabla w_{m_j}(x)|^{p-1}]|\nabla v(x)| \\ &= c_o h(x)|\nabla v(x)| + c_o \left(h(x)|\nabla w_{m_j}(x)|^p\right)^{\frac{p-1}{p}} h^{\frac{1}{p}}|\nabla v(x)| \\ &\leqslant c_o h(x).|\nabla v(x)| + c_o |g|^{\frac{p-1}{p}}(x).h^{\frac{1}{p}}(x)|\nabla v(x)|. \end{aligned}$$

Since $h, g^{\frac{p-1}{p}} \in L^{\frac{p}{p-1}}(\Omega)$ and $|\nabla v|, h^{\frac{1}{p}}|\nabla v| \in L^p(\Omega)$, the last term on the right-hand side is integrable on Ω. On the other hand, $\nabla w_{m_j}(x) \to \nabla w(x)$ for a.e. $x \in \Omega$, so that $a(x, \nabla w_{m_j}(x)) \cdot \nabla v(x) \to a(x, \nabla w(x)) \cdot \nabla v(x)$ for a.e. $x \in \Omega$ by the Carathéodory property of a. According to the Lebesgue Dominated convergence theorem, we have

$$\int_\Omega a(x, \nabla w_{m_j}) \cdot \nabla v \, dx \to \int_\Omega a(x, \nabla w) \cdot \nabla v \, dx, \quad \text{i.e.,} \quad DP(w_{m_j})(v) \to DP(u)(v).$$

This contradiction implies that the map $u \mapsto DP(u)(v)$ is continuous on Y. Thus, P belongs to $C^1_w(Y)$.

(v) By (i) we obtain the convexity of P, which implies

$$\begin{aligned} &\frac{P(u_m + t(u - u_m)) - P(u_m)}{t} = \frac{P((1-t)u_m + tu) - P(u_m)}{t} \\ &\leqslant \frac{(1-t)P(u_m) + tP(u) - P(u_m)}{t} = P(u) - P(u_m) \ \ \forall\, t \ \in \ (0, 1). \end{aligned}$$

Letting $t \to 0$, we have $DP(u_m)(u - u_m) \leqslant P(u) - P(u_m)$. This proves (v) and concludes the proof of our lemma. ∎

The following lemma concerns the coercivity of J.

Lemma 3.5.

(i) *There exist* $k_3 > 0$ *and* $c_3 > 0$ *such that* $J(u) \geq \|u\|^p_Y (k_3 - c_3 \|u\|^{q-p}_Y)$ $\forall u \in Y$.

(ii) *There exist* $c_4 > 0$, $k_4 \in \mathbb{R}$ *such that*

$$J(u) \geq \|u\|_Y \left(c_4 \|u\|^{p-1}_Y - \frac{1}{\theta}\|DJ(u)\|\right) + k_4 \quad \forall u \in Y.$$

Proof.

(i) By $(F3)$ we have for a.e. $x \in \Omega$.

$$f(x, s) \begin{cases} \leqslant \mu s^{p-1} \text{ if } s \in (0, \delta), \\ \geq -\mu |s|^{p-1} \text{ if } s \in (-\delta, 0). \end{cases}$$

It follows that $F(x, s) \leqslant \dfrac{\mu}{p}|s|^p \ \ \forall s \in (-\delta, \delta), a.e. \ x \in \Omega.$

On the other hand, by (ii) of Lemma 3.3 there exists a positive constant c_2' such that

$$|F(x, s)| \leqslant c_2' |s|^q \quad \forall s \in \mathbb{R} \setminus (-\delta, \delta), \quad \text{a.e} \quad x \in \Omega.$$

Hence $F(x,s) \leqslant \dfrac{\mu}{p}|s|^p + c_2'|s|^q \;\; \forall s \in \mathbb{R}$, a.e $x \in \Omega$.

Furthermore, by the Sobolev embedding theorem there is a positive real number c_3 such that

$$T(u) = \int_\Omega F(x,u)dx \leqslant \frac{\mu}{p}\int_\Omega |u|^p dx + c_2'\int_\Omega |u|^q dx \leqslant \frac{\mu}{p\lambda_1}\|u\|_Y^p + c_3\|u\|_Y^q \;\; \forall u \in Y.$$

Put $k_3 = k_0 - \dfrac{\mu}{p\lambda_1} > 0$, for any $u \in Y$ we have

$$\begin{aligned} J(u) = P(u) - T(u) &= \int_\Omega A(x,\nabla u)dx - \int_\Omega F(x,u)dx \\ &\geq k_0 \int_\Omega h(x)|\nabla u|^p dx - \frac{\mu}{p\lambda_1}\|u\|_Y^p - c_3\|u\|_Y^q = k_3\|u\|_Y^p - c_3\|u\|_Y^q. \end{aligned}$$

(ii) Put $c_4 := k_0(1 - \dfrac{p}{\theta}) > 0$. From (iv) of Lemma 3.4 and $(A3)$, we infer

$$\begin{aligned} P(u) - \frac{1}{\theta}DP(u)(u) &= \int_\Omega A(x,\nabla u)dx - \frac{1}{\theta}\int_\Omega a(x,\nabla u)\cdot\nabla u dx \\ \geq \Big(1 - \frac{p}{\theta}\Big)\int_\Omega A(x,\nabla u)dx &\geq \Big(1 - \frac{p}{\theta}\Big)k_0\int_\Omega h|\nabla u|^p dx = c_4\|u\|_Y^p \quad \forall u \in Y. \end{aligned}$$

We put $\Omega_u = \{x \in \Omega : |u(x)| > s_0\}$ for any $u \in Y$. By $(F1)$, $(F2)$ and (ii) of Lemma 3.3, there exists a constant M such that

$$\frac{1}{\theta}f(x,u)u - F(x,u) \geq 0 \quad \text{a.e.} \quad x \in \Omega_u,$$

and $|\dfrac{1}{\theta}f(x,u)u - F(x,u)| \leqslant M$, a.e. $x \in \Omega \setminus \Omega_u$.

Put $k_4 = -M|\Omega|$. From (ii) of Lemma (3.4), it follows that

$$\begin{aligned} \frac{1}{\theta}DT(u)(u) - T(u) = &\int_{\Omega_u}\Big[\frac{1}{\theta}f(x,u)u - F(x,u)\Big]dx + \int_{\Omega\setminus\Omega_u}\Big[\frac{1}{\theta}f(x,u)u - F(x,u)\Big]dx \\ &\geq -M|\Omega\setminus\Omega_u| \geq -M|\Omega| = k_4. \end{aligned}$$

Hence

$$\begin{aligned} J(u) - \frac{1}{\theta}DJ(u)(u) &= [P(u) - \frac{1}{\theta}DP(u)(u)] + [\frac{1}{\theta}DT(u)(u) - T(u)] \geq c_4\|u\|_Y^p + k_4, \\ \text{or} \quad J(u) &\geq c_4\|u\|_Y^p + \frac{1}{\theta}DJ(u)(u) + k_4 \geq c_4\|u\|_Y^p - \frac{1}{\theta}\|DJ(u)\|\|u\|_Y + k_4. \end{aligned}$$

This proof is complete. ■

In order to apply generalized Mountain Pass Theorem (Theorem 1.2), we need to verify the following facts.

Lemma 3.6

(i) $J(0) = 0$.

(ii) *J belongs to $C^1_w(Y)$ and*

$$J(u)(v) = \int_\Omega a(x, \nabla u) \cdot \nabla v \, dx - \int_\Omega f(x, u) v \, dx \; ; \; \forall u, v \in Y.$$

(iii) *J satisfies the Palais-Smale condition on Y.*

(iv) *There exist two positive real number r and α such that*

$$\inf\{ \; J(u) : u \in Y, \;\; \|u\|_Y = r\} \geq \alpha.$$

(v) *There exists $z_0 \in Y$ such that $\|z_0\|_Y > r$ and $J(z_0) \leqslant 0$.*

(vi) *The set $S = \{u \in \hat{Y} | J(u) \geq 0\}$ is bounded whenever $\hat{Y}$ is a finite dimensional subspace of Y.*

Proof.

(i) Since $P(0) = 0$ and $T(0) = 0$, we have $J(0) = 0$.

(ii) This comes from (ii), (iv) of Lemma 3.4.

(iii) Let $\{u_m\}$ be a sequence in Y and c be a real number such that $\lim\limits_{m\to\infty} J(u_m) = c$ and $\lim\limits_{m\to\infty} \|DJ(u_m)\| = 0$.

Suppose by contradiction that $\{\|u_m\|_Y\}$ is not bounded, then there exists a subsequence $\{u_{m_j}\}_j$ of $\{u_m\}$ such that $\|u_{m_j}\|_Y \geq j$ for any j in $\mathbb{N}$. By (ii) of Lemma 3.5 , we have

$$J(u_{m_j}) \geq \|u_{m_j}\|_Y \Big(c_4 \|u_{m_j}\|_Y^{p-1} - \frac{1}{\theta} \|DJ(u_{m_j})\| \Big) + k_4.$$

Letting $j \to \infty$, we deduce $J(u_{m_j}) \to \infty$, which is a contradiction. Hence $\{\|u_m\|_Y\}$ is bounded.

By (i) of Lemma 3.2, the sequence $\{u_m\}$ is also bounded in the usual Sobolev space $W_0^{1,p}(\Omega)$. By the reflexivity of $W_0^{1,p}(\Omega)$, we can (and shall) assume that the sequence $\{u_m\}$ converges weakly to some u in $W_0^{1,p}(\Omega)$. We shall prove that $\{u_m\}$ converges strongly to u in Y.

By (iii) of Lemma 3.4 we have

$$P(u) \leqslant \lim_{m\to\infty} P(u_m) = \lim_{m\to\infty} (T(u_m) + J(u_m)) = T(u) + c.$$

According to (iv) of Lemma 3.2, we see that $u \in Y$. Hence $\{\|u_m - u\|_Y\}$ is bounded. Since $\{\|DJ(u_m)\|\}$ converges to 0, $\{DJ(u_m)(u - u_m)\}$ converges to 0. Moreover, the sequence $\{u_m\}$ converges strongly to u in $L^q(\Omega)$ by Rellich-Kondrachov theorem (see [11, p. 144]), and $f(x, u_m)$ is bounded in $L^{q'}(\Omega)$ by the condition $(F1)$ with $q' = q/(q-1)$, so that $\lim\limits_{m\to\infty} DT(u_m)(u - u_m) = \lim_{m\to\infty} \int_\Omega f(x, u_m)(u - u_m) dx = 0$.

Hence, $\lim\limits_{m\to\infty} DP(u_m)(u - u_m) = \lim_{m\to\infty} [DJ(u_m)(u - u_m) + DT(u_m)(u - u_m)] = 0$, which together with (v) of Lemma 3.4 imply that

$$P(u) - \lim_{m\to\infty} P(u_m) = \lim_{m\to\infty} [P(u) - P(u_m)] \geq \lim_{m\to\infty} DP(u_m)(u - u_m) = 0.$$

Combining this fact with (iii) of Lemma 3.4, we get

$$\lim_{m\to\infty} P(u_m) = P(u).$$

Suppose by contradiction that $\{u_m\}$ does not converge strongly to u in Y. Then there exist a positive real number ε and a subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that $\|u_{m_j} - u\|_Y \geq \varepsilon$ for any $j \in \mathbb{N}$. By (i) of Lemma 3.4 we have

$$\frac{1}{2}P(u_{m_j}) + \frac{1}{2}P(u) - P(\frac{u_{m_j}+u}{2}) \geq k_1\|u_{m_j} - u\|_Y^p \geq k_1\varepsilon^p.$$

Since $\lim\limits_{j\to\infty} P(u_{m_j}) = P(u)$, the above inequality implies that

$$P(u) - \liminf_{j\to\infty} P\Big(\frac{u_{m_j}+u}{2}\Big) = \limsup_{j\to\infty}\Big[\frac{1}{2}P(u_{m_j}) + \frac{1}{2}P(u) - P\Big(\frac{u_{m_j}+u}{2}\Big)\Big] \geq k_1\varepsilon^p.$$

On the other hand, $P(u) \leqslant \liminf\limits_{j\to\infty} P(\frac{u_{m_j}+u}{2})$ by (iii) of Lemma 3.4, where $\{\frac{u_{m_j}+u}{2}\}$ converges weakly to u. Hence $0 \geq k_1\varepsilon^p$, which is a contradiction. Therefore, $\{u_m\}$ converges strongly to u in Y. Thus, J verifies the Palais-Smale condition on Y .

(iv) Since $k_3 > 0$ and $q > p$, there exists a positive constant r such that

$$\alpha = r^p(k_3 - c_3 r^{q-p}) > 0.$$

Let u be in Y. By (i) of Lemma 3.5 we have

$$J(u) \geq \|u\|_Y^p\left(k_3 - c_3\|u\|_Y^{q-p}\right) = r^p(k_3 - c_3 r^{q-p}) = \alpha \text{ when } \|u\|_Y = r.$$

(v) Let $t > 1$. Choose v_0 in $C_c^\infty(\Omega)$ such that $v_0(x) \geq 0$ for a.e. $x \in \Omega$ and $\Omega_0 = \{x \in \Omega : v_0(x) \geq s_0\}$ has a positive measure. By the condition $(F2)$, $F(x, v_0(x)) > 0$ if $x \in \Omega_0$. Put $\Omega_t := \{x \in \Omega : tv_0(x) \geq s_0\}$. Then, $\Omega_0 \subset \Omega_t$. By (iii) of Lemma 3.3 we have

$$\int_{\Omega_t} F(x, tv_0)dx \geq \int_{\Omega_t} \gamma(x)(tv_0(x))^\theta dx = t^\theta \int_{\Omega_t} \gamma v_0^\theta dx \geq t^\theta \int_{\Omega_0} \gamma v_0^\theta dx = t^\theta C(v_0),$$

where $C(v_0) = \int\limits_{\Omega_0} \gamma v_0^\theta dx > 0$.

Arguing as in [8, Remark 3.3, p.1212], from the condition $(A3)$ we infer that $A(x, t\xi) \leqslant A(x,\xi)t^p$ for every $\xi \in \mathbb{R}^n$, a.e $x \in \Omega$. Moreover, by (ii) of Lemma 3.3 there exists a positive constant M' such that $|F(x,s)| \leqslant M'$ for any $s \in [0, s_0]$, a.e. $x \in \Omega$. Hence

$$\begin{aligned} J(tv_0) &= \int_\Omega A(x, tv_0)dx - \int_{\Omega_t} F(x, tv_0)dx - \int_{\Omega\setminus\Omega_t} F(x, tv_0)dx \\ &\leqslant t^p \int_\Omega A(x, v_0)dx - t^\theta C(v_0) + \int_{\Omega\setminus\Omega_t} M' dx \\ &\leqslant t^p P(v_0) - t^\theta C(v_0) + M'|\Omega|. \end{aligned}$$

Since $\theta > p$, we deduce $J(tv_0) \to -\infty$ as $t \to +\infty$. Hence, there exists t_1 such that $\|t_1 v_0\|_Y > r$ and $J(t_1 v_0) \leqslant 0$. Choose $z_0 = t_1 v_0$, we have $\|z_0\|_Y > r$ and $J(z_0) \leqslant 0$.

(vi) Assume that $\hat{Y}$ is a finite dimensional subspace of Y.

Let $u \in \hat{Y}$ be arbitrary. We put $V_< = \{x \in \Omega : |u(x)| < s_o\}$ and $V_> = \Omega \backslash V_<$.

We have $J(u) = \int_\Omega A(x, \nabla u)dx - \int_{V_<} F(x,u)dx - \int_{V_>} F(x,u)dx$.

We now estimate each term in the right-hand side.

Put $c_5 = c_0 \left(\int_\Omega h dx \right)^{\frac{p-1}{p}}$. By (i) of Lemma 3.3, we conclude that

$$\begin{aligned}\int_\Omega A(x,\nabla u)dx &\leqslant c_0 \int_\Omega h(|\nabla u| + |\nabla u|^p)dx = c_0 \int_\Omega h^{\frac{p-1}{p}} h^{\frac{1}{p}} |\nabla u| dx + \int_\Omega h|\nabla u|^p dx \\ &\leqslant c_0 \Big(\int_\Omega h dx \Big)^{\frac{p-1}{p}} \Big(\int_\Omega h|\nabla u|^p dx \Big)^{\frac{1}{p}} + c_0 \int_\Omega h|\nabla u|^p dx \\ &= c_5 \|u\|_Y + c_0 \|u\|_Y^p.\end{aligned}$$

Moreover, from (ii) and (iii) of Lemma 3.3 we obtain that

$$\int_{V_<} F(x,u)dx \geq -c_2 \int_{V_<} (1+|u|^q)dx \geq -c_2 \int_{V_<} (1+s_0^q)dx \geq -c_2|\Omega|(1+s_0^q) \equiv k_2'$$

and $\int_{V_>} F(x,u)dx \geq \int_{V_>} \gamma(x)|u|^\theta dx$.

Hence, J satisfies a following estimate.

$$J(u) \leqslant c_5 \|u\|_Y + c_0 \|u\|_Y^p + k_2' - \int_\Omega \gamma(x)|u|^\theta dx. \tag{3.1}$$

On the other hand, setting

$$\|v\|_\gamma = \Big(\int_\Omega \gamma(x)|v|^\theta \Big)^{\frac{1}{\theta}} \text{ for all } u \in \hat{Y},$$

we see that $\|.\|_\gamma$ is a norm in $\hat{Y}$. Moreover, since the dimension of the space $\hat{Y}$ is finite, the norms are equivalent. So that there exists a positive constant $K = K(\hat{Y})$ such that $\|v\|_Y \leqslant K\|v\|_\gamma$ for all $v \in \hat{Y}$.

Therefore, (3.1) implies that

$$J(u) \leqslant c_5 \|u\|_Y + c_0 \|u\|_Y^p - k_2' - K^{-\theta} \|u\|_Y^\theta.$$

Furthermore, $\{r \in [0,+\infty) : c_5 r + c_0 r^p - k_2' - K^{-\theta} r^\theta > 0\}$ is bounded in $\mathbb{R}$ since $\theta > p$. Hence, $S \equiv \{u \in \hat{Y} : J(u) \geq 0\}$ is bounded . ∎

We are now ready to prove Theorem 3.1 .

3.7. Proof of Theorem 3.1

(i) According to Lemma 3.2 and Lemma 3.6, Theorem 1.2 can be applied to the function $I \equiv J$ with $E \equiv Y$. Hence, J has at least a critical point $\beta > 0$. There

exists $u_o \in Y$ such that $J(u_0) = \beta > 0$. On the other hand, $J(0) = 0$. Hence, u_0 is a nontrivial critical point of J. Furthermore, since $C_c^\infty(\Omega) \subset Y$, the critical point u_0 of J is a nontrivial generalized solution to the problem (P).
(ii) Let us assume that A and F are even with respect to the second variable. Then J is even. According to Lemmata 3.2 and 3.6, Theorem 1.3 can be applied to the function $I \equiv J$ with $E \equiv Y$, $V \equiv \{0\}$. Hence, J possesses an unbounded sequence of critical values. Therefore, J possesses infinitely many critical points in Y.

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