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The Model of Stochastic Control and Applications

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Abstract. In this paper, we present some results for a class of the jump homogeneous controllable stochastic processes on infinite time interval, in particular

· Conditions for the existence of optimal strategy (Theorem 3.1).

· Construction of optimal strategy and defining the cost optimal (Theorem 4.1 and Theorem 4.2).

Introduction

In recent years, controlled Markov models are paid a great attention. Those models with the different assumptions on state spaces, on control spaces and on cost functions have been considered by many authors such as: Arapostathis, Kumar, and Tangiralla [6, 8]; Bokar [7]; Xi-Ren Cao [9]; Chang, Fard, Marcus, and Shayman [11]; Liu [4]. Some applications of controlled Markov processes to different economic, scientific fields have also investigated by Sennott [5]; Karel Sladký [10]...

In this paper we present some results on optimal solution concerning controlled Semi-Markov process with Poisson jumps depending on controlled process states on infinite time interval. That process describes the oscillation of some object on half-line. The controlling cost at each step is unbounded and is defined by conditional expectation of the cost caused by the number of jumps and of the integral of the square of the difference of the state and control processes.

The goal of controlling is to minimize the cost evarage on infinite time interval.

The main result obtained in this paper is to show the existence of optimal control, the method for establishing optimal strategy and for defining minimum cost.

These results can be applied to queueing system and to renewal theory.

This paper is organized as follows:

Section 1: Defining control model.

Section 2: Formulas for the transition probabilities and for the cost.

Section 3: Existence of optimal strategy.

Section 4: Finding optimal strategy and optimal cost.

1. Defining Control Model

1.1. Constructure of the Model

Suppose there exist two sequences of independent random variables $\{\eta_n | n = 1, 2, \dots\}$ and $\{\xi_n | n = 1, 2, \dots\}$ defined on probability spase $(\Omega, \mathcal{A}, P)$. Those sequences are independent and satisfy the following conditions:

- $\xi_n > 0; n = 1, 2, ...(\bmod P)$
- $\begin{cases} E|\xi_n|^p < +\infty, & n = 1, 2, ..., p \geq 3 \\ E|\eta_n|^q < +\infty, & n = 1, 2, ..., q \geq 2. \end{cases}$

Let us consider a stochastic control system with state process $\{x_n | n = 1, 2, ...\}$ and with control process $\{u_n = u(\mu_n) | n = 1, 2, ...\}$ described as follows:

For an initial state of elementary process $x_1 = x (x \in \mathbb{R})$, at the first step, a sequence of controlling variables

$$u_1 = u(\mu_1) := \left\{\xi'_{1,j} | \quad j = 1, 2, ..., \nu_{\mu_1}(\xi_1) + 1\right\}$$

is defined, where $\xi'_{1,j}$ are the exponentially distributed independent random variables with the parameter $\mu_1 (\mu_1 > 0)$, and $\nu_{\mu_1}(\xi_1)$ is the random variable defined as follows:

$$\sum_{j=1}^{\nu_{\mu_1}(\xi_1)} \xi'_{1,j} \leqslant \xi_1 < \sum_{j=1}^{\nu_{\mu_1}(\xi_1)+1} \xi'_{1,j} \quad \text{a.s.}$$

The values μ_1 are called controlling parameter at the first step.

By induction, suppose that at the n-th step $(n \geq 1)$ if controlled process is at the state x_n and controlling variables $u_n = u(\mu_n)$ selected corresponding to the parameter $\mu_n (\mu_n > 0)$, then the state x_{n+1} will be defined by the following formula

$$x_{n+1} = \eta_n + x_n - \nu_{\mu_n}(\xi_n),$$

whereas the controlling variable is defined by

$$u_{n+1} = u(\mu_{n+1}) := \left\{\xi'_{n+1,j} | j = 1, 2, \dots \nu_{\mu_{n+1}}(\xi_{n+1}) + 1\right\},$$

where $\xi'_{n+1,j}$ is the sequence of the exponentially distributed independent random variables with the parameter $\mu_{n+1} (\mu_{n+1} > 0)$, and $\nu_{\mu_{n+1}}(\xi_{n+1})$ is random variable defined by

$$\sum_{j=1}^{\nu_{\mu_{n+1}}(\xi_{n+1})} \xi'_{n+1,j} \leqslant \xi_{n+1} < \sum_{j=1}^{\nu_{\mu_{n+1}}(\xi_{n+1})+1} \xi'_{n+1,j} \quad \text{a.s.}$$

μ_{n+1} is called controlling parameter at the $(n+1)$-th step.
$U = \{u_n = u(\mu_n) | n = 1, 2, \dots\}$ is called a controlling strategy.

1.2. Definition of the Cost

If at the n-th step, the state of elementary process is x and we selected a control with the parameter $\mu(\mu > 0)$ then we define the cost at this step by formula

$$r_n(x, \mu) = E\Big\{a\big[\nu_{\mu_n}(\xi_n) + 1\big] + \int_0^{\xi_n} \big[\eta_n + x_n - \nu_{\mu_n}(t)\big]^2 dt\big|_{x_n = x, \mu_n = \mu}\Big\},$$

where a is a positive constant, $\nu_\mu(t)$ is the number of independent random variables, possessing the exponential distribution with parameter $\mu(\mu > 0)$ and such that their sum is less than or equal to $t(t > 0)(\nu_\mu(t)$ have Poisson's distribution with parameter $\mu t)$.

1.3. Definition of the Cost Function

If $U = \{u_n = u(\mu_n) | n = 1, 2, \dots\}$ is a controlling strategy of the stochastic process $X = \{x_n,\ n = 1, 2, \dots\}$, with initial state $x_1 = x$ then the cost function defined by

$$\Psi_x(U) = \lim_{n\to\infty} E_x^U \Big\{\frac{1}{n}\sum_{k=1}^{n} r_k(x_k, \mu_k)\Big\},$$

where $E_x^U(\cdot)$ denotes the mathematical expectation operator with respect to the initial state $x_1 = x$, and to controlling strategy $U = \{u_n = u(\mu_n) | n = 1, 2, \dots\}$.

Let us denote by $\mathcal{M}$ the set of all strategies U such that the following limit exists:

$$\lim_{n\to\infty} E_x^U \Big\{\frac{1}{n}\sum_{k=1}^{n} r_k(x_k, \mu_k)\Big\}, \quad \forall x \in \mathbb{R}.$$

1.4. Definition of Optimal Controlling Strategy

The funtion $\rho(x) = \inf\limits_{U\in\mathcal{M}} \psi_x(U),\ \ \forall x \in \mathbb{R}$ is called the optimal cost.

The strategy U^* satisfying

$$\psi_x(U^*) = \min_{U\in\mathcal{M}} \psi_x(U), \quad \forall x \in \mathbb{R}$$

is called the optimal strategy, if it exists.

2. Formulas for the Transition Probabilities and for the Cost

2.1. Defining Transition Probability $P_{n+1}(x, dy, \mu)$

It is easy to see that $\{x_n, n = 1, 2, \dots\}$ is a Markov chain. Let us consider

$$\begin{aligned}
P_{n+1}(x, y, \mu) &= P\big[x_{n+1} < y|_{x_n = x; \mu_n = \mu}\big] \\
&= P[\eta_n + x - \nu_\mu(\xi_n) < y] \\
&= P\Big\{\bigcup_{k=0}^{\infty} \big[\eta_n + x - \nu_\mu(\xi_n) < y\big] \cap \big[\nu_\mu(\xi_n) = k\big]\Big\} \\
&= \sum_{k=0}^{\infty} P\big\{\big[\eta_n + x - \nu_\mu(\xi_n) < y\big] \cap \big[\nu_\mu(\xi_n) = k\big]\big\} \\
&= \sum_{k=0}^{\infty} P\big\{\nu_\mu(\xi_n) = k\big\} P\big\{\eta_n + x - \nu_\mu(\xi_n) < y\big|_{\nu_\mu(\xi_n) = k}\big\} \\
&= \sum_{k=0}^{\infty} \Big(\int e^{-\mu t} \frac{(\mu t)^k}{k!} F_{\xi_n}(dt)\Big) P\big\{\eta_n + x - k < y\big\} \\
&= \sum_{k=0}^{\infty} \Big(\int e^{-\mu t} \frac{(\mu t)^k}{k!} F_{\xi_n}(dt)\Big) F_{\eta_n}(y - x + k) \\
\Rightarrow P_{n+1}(x, dy, \mu) &= \sum_{k=0}^{\infty} \Big(\int e^{-\mu t} \frac{(\mu t)^k}{k!} F_{\xi_n}(dt)\Big) F_{\eta_n}(dy - x + k)
\end{aligned}$$

Hence, we have:

$$\int V(y) P_{n+1}(x, dy, \mu) = EV(\eta_n + x - \nu_\mu(\xi_n)), \quad n = 1, 2, \dots \tag{2.1}$$

2.2. Defining $r_n(x, \mu)$

We have

$$r_n(x, \mu) = E\Big\{a\big[\nu_\mu(\xi_n) + 1\big] + \int_0^{\xi_n} \big[\eta_n + x - \nu_\mu(t)\big]^2 dt\Big\}.$$

Since

$$E\nu_\mu(\xi_n) = \mu E\xi_n,$$

$$E\int_0^{\xi_n} \nu_\mu(t)dt = \mu \frac{E\xi_n^2}{2},$$

$$E\int_0^{\xi_n} \nu_\mu^2(t)dt = \frac{E\xi_n^3}{3}.\mu^2 + \frac{E\xi_n^2}{2}.\mu,$$

we have

$$r_n(x,\mu) = \frac{E\xi_n^3}{3}\mu^2 + \Big[aE\xi_n + \frac{E\xi_n^2}{2} - (E\eta_n + x)E\xi_n^2\Big]\mu + \big[a + E\xi_n E(\eta_n + x)^2\big] \quad \forall n \in \mathbb{N}^+ \tag{2.2}$$

In this paper, we present some results for the case in which, $\{\xi_n | n = 1, 2, \dots\}$, $\{\eta_n | n = 1, 2, \dots\}$ are independent identically distributed (i.i.d.) variables as random variables ξ, η, respectively:

$$F_{\xi_n}(t) \equiv F_\xi(t), \quad n = 1, 2, \dots,$$
$$F_{\eta_n}(t) \equiv F_\eta(t), \quad n = 1, 2, \dots,$$

In this case $r_n(x,\mu) \quad \equiv r(x,\mu), \;\; n = 1, 2, \dots$

3. Existence of Optimal Strategy

We obtain the following theorem:

Theorem 3.1. *If there exist a constant S and a function $V(x), x \in \mathbb{R}$ such that*

$$V(x) \leqslant Ax^2 + Bx + C, \quad \forall x \in \mathbb{R} \tag{3.1}$$

and

$$S + V(x) = \inf_{\mu > 0} \Big\{ r(x,\mu) + \int V(y) P(x, dy, \mu) \Big\}, \quad \forall x \in \mathbb{R} \tag{3.2}$$

where A, B, C are constants, then

$$S \leqslant \inf_{U \in \mathcal{M}} \psi_x(U), \quad \forall x \in \mathbb{R} \tag{3.3}$$

Proof. Suppose $U \in \mathcal{M}$ is any strategy, $X = \{x_k | k = 1, 2, \dots, x_1 = x\}$ is the controlled process corresponding to the strategy U, then

$$\frac{1}{n}\sum_{k=1}^{n} r(x_k,\mu_k) = \frac{n-1}{n} \cdot \frac{1}{n-1}\sum_{k=1}^{n-1} r(x_k,\mu_k) + \frac{1}{n} r(x_n,\mu_n),$$

hence

$$E_x^U \Big\{ \frac{1}{n}\sum_{k=1}^{n} r(x_k,\mu_k) \Big\} = \frac{n-1}{n} E_x^U \Big\{ \frac{1}{n-1}\sum_{k=1}^{n-1} r(x_k,\mu_k) \Big\} + \frac{1}{n} E_x^U \{ r(x_n,\mu_n) \}.$$

Since $U \in \mathcal{M}$ the limit

$$\lim_{n\to\infty} E_x^U \Big\{ \frac{1}{n}\sum_{k=1}^{n} r(x_k,\mu_k) \Big\}$$

is finite. So we have

$$\lim_{n\to\infty} \frac{1}{n} E_x^U r(x_n,\mu_n) = 0, \tag{3.4}$$

and

$$x_{n+1} = \eta_n + x_n - \nu_{\mu_n}(\xi_n),$$

therefore

$$\eta_n + x_n - x_{n+1} = \nu_{\mu_n}(\xi_n),$$
$$x_n(\eta_n + x_n - x_{n+1}) = x_n.\nu_{\mu_n}(\xi_n),$$
$$(\eta_n + x_n - x_{n+1})^2 = \nu^2_{\mu_n}(\xi_n).$$

Furthermore, according to (2.2) and the following relations

$$E(\eta_n + x_n - x_{n+1}) = E\xi E\mu_n,$$
$$E\big\{x_n(\eta_n + x_n - x_{n+1})\big\} = E\xi E(x_n\mu_n),$$
$$E(\eta_n + x_n - x_{n+1})^2 = E\xi E\mu_n + E\xi^2 E\mu_n^2,$$

we have

$$E_x^U r(x_n, \mu_n) = \alpha_1 Ex^2_{n+1} + \alpha_2 Ex^2_n + \alpha_3 E(x_n x_{n+1}) + \alpha_4 Ex_{n+1} + \alpha_5 Ex_n + \alpha_6 \tag{3.5}$$

where $\alpha_j \neq 0, \ \forall j = 1, \dots, 6; \alpha_1 + \alpha_2 + \alpha_3 = E\xi > 0$.

According to formulas (3.4) and (3.5) we have:

$$\lim_{n\to\infty} \frac{Ex_n^2}{n} = 0,$$
$$\lim_{n\to\infty} \frac{Ex_n}{n} = 0. \tag{3.6}$$

Since $V(x) \leqslant Ax^2 + Bx + C, \ \forall x \in \mathbb{R}$

$$\frac{EV(x_n)}{n} \leqslant \frac{E(Ax_n^2 + Bx_n + C)}{n}. \tag{3.7}$$

Let us denote $\mathcal{F}_n = \sigma(x_1, \mu_1, x_2, \mu_2, \dots, x_n, \mu_n)$, then $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \mathcal{F}_n \subset \mathcal{A}$.

By the Markov property and from Bellman's equation (3.3) we obtain

$$E\big(V(x_k)|\mathcal{F}_{k-1}\big) = \int V(y)P\big(x_{k-1}, dy, \mu_{k-1}\big) \geq S + V(x_{k-1}) - r(x_{k-1}, \mu_{k-1}),$$
$$\Rightarrow S + V(x_{k-1}) \leqslant r(x_{k-1}, \mu_{k-1}) + E(V(x_k)|\mathcal{F}_{k-1}),$$
$$\Rightarrow E_x^U\big[S + V(x_{k-1})\big] \leqslant E_x^U\big[r(x_{k-1}, \mu_{k-1}) + E(V(x_k)|\mathcal{F}_{k-1})\big],$$
$$\Rightarrow S + EV(x_{k-1}) \leqslant E_x^U r(x_{k-1}, \mu_{k-1}) + EV(x_k),$$
$$\Rightarrow \sum_{k=2}^{n}\big[S + EV(x_{k-1})\big] \leqslant \sum_{k=2}^{n}\big[E_x^U r(x_{k-1}, \mu_{k-1}) + EV(x_k)\big],$$
$$\Rightarrow (n-1)S \leqslant \sum_{k=2}^{n} E_x^U r(x_{k-1}, \mu_{k-1}) + EV(x_n) - EV(x_1),$$
$$\Rightarrow S \leqslant E_x^U\Big\{\frac{1}{n-1}\sum_{k=1}^{n-1} r(x_k, \mu_k)\Big\} + \frac{n}{n-1}\frac{EV(x_n)}{n} - \frac{EV(x_1)}{n-1}. \tag{3.8}$$

By the formulas (3.7) and (3.8) we have

$$S \leqslant E_x^U\Big\{\frac{1}{n-1}\sum_{k=1}^{n-1} r(x_k,\mu_k)\Big\} + \frac{n}{n-1}\frac{E(Ax_n^2+Bx_n+C)}{n} - \frac{EV(x_1)}{n-1},$$

$$\Rightarrow S \leqslant \lim_{n\to\infty} E_x^U\Big\{\frac{1}{n-1}\sum_{k=1}^{n-1} r(x_k,\mu_k)\Big\}.$$

$\Big($Since $\lim\limits_{n\to\infty}\Big[\dfrac{n}{n-1}\dfrac{E(Ax_n^2+Bx_n+C)}{n} - \dfrac{EV(x_1)}{n-1}\Big] = 0$ by (3.6)$\Big)$.

$$\Rightarrow S \leqslant \psi_x(U), \quad \forall x \in \mathbb{R}.$$

Since U is arbitrary, $S \leqslant \inf\limits_{U\in\mathcal{M}} \psi_x(U) \quad \forall x \in \mathbb{R}$. ■

Corollary 3.2. *If there exist a constant* S *and a function* $V(x), x \in \mathbb{R}$ *such that*

$$|V(x)| \leqslant Ax^2 + Bx + C, \quad \forall x \in \mathbb{R}$$

and

$$\begin{aligned} S + V(x) &= \min_{\mu>0}\Big\{r(x,\mu) + \int V(y)P(x,dy,\mu)\Big\} \\ &= r(x,\mu^*(x)) + \int V(y)P(x,dy,\mu^*(x)), \quad \forall x \in \mathbb{R} \end{aligned}$$

where $A, B, C (A > 0)$ *are the constants, then* $U^* = \big\{u_n^* = u(\mu_n^*)\big|n = 1, 2, ...\big\}$ *is an optimal strategy and* $\psi_x(U^*) = S$.

4. Finding Optimal Strategy and Optimal Cost

Let

$$R_n(x) = \inf_{U\in\mathcal{M}} E_x^U\Big\{\frac{1}{n}\sum_{k=1}^{n} r(x_k,\mu_k)\Big\}, \quad \forall x \in \mathbb{R}, \ \ n = 1, 2, ... \tag{4.1}$$

Lemma 4.1. *The function* $R_n(x)$ *satisfies the following Bellman's equation:*

$$R_{n+1}(x) = \inf_{\mu>0}\Big\{\frac{1}{n+1}r(x,\mu) + \frac{n}{n+1}\int R_n(y)P(x,dy,\mu)\Big\}. \tag{4.2}$$

Proof. We have

$$\begin{aligned}
R_{n+1}(x) &= \inf_{U\in\mathcal{M}} E_x^U\Big\{\frac{1}{n+1}\sum_{k=1}^{n+1} r(x_k,\mu_k)\Big\}\\
&= \inf_{U\in\mathcal{M}} E_x^U\Big\{\frac{1}{n+1} r(x_1,\mu_1) + \frac{n}{n+1}\frac{1}{n}\sum_{k=2}^{n+1} r(x_k,\mu_k)\Big\}\\
&= \inf_{U\in\mathcal{M}} E_x^U\Big\{\frac{1}{n+1} r(x_1,\mu_1) + \frac{n}{n+1}E_{x_2}^U\Big[\frac{1}{n}\sum_{k=2}^{n+1} r(x_k,\mu_k)\Big]\Big\}\\
&= \inf_{\mu>0}\Big\{\frac{1}{n+1} r(x,\mu) + \frac{n}{n+1} R_n(x_2)\Big\}\\
&= \inf_{\mu>0}\Big\{\frac{1}{n+1} r(x,\mu) + \frac{n}{n+1}\int R_n(y)P(x,dy,\mu)\Big\}. \qquad \blacksquare
\end{aligned}$$

Suppose x is an arbitrary random variable, we say that x satisfies condition (I) if:

$$x > \frac{aE\xi}{E\xi^2} + \frac{1}{2} - E\eta \pmod P. \tag{4.3}$$

Lemma 4.2. *If at the n-th step $(n = 1, 2, ...)$, the state x of system satisfies Condition (I) then $\mu^*(x) > 0$, otherwise $\mu^*(x) = 0$, where $\mu^*(x)$ is defined by the equation:*

$$r(x,\mu^*(x)) = \inf_{\mu>0} r(x,\mu).$$

Proof. It follows from

$$r(x,\mu) = \frac{E\xi^3}{3}\mu^2 + \Big[aE\xi + \frac{E\xi^2}{2} - (E\eta + x)E\xi^2\Big]\mu + \big[a + E\xi E(\eta+x)^2\big],$$

that

$$\frac{\partial r(x,\mu)}{\partial\mu} = \frac{2E\xi^3}{3}\mu + aE\xi + \frac{E\xi^2}{2} - (E\eta + x)E\xi^2,$$

and hence

$$\frac{\partial r(x,\mu)}{\partial\mu} = 0 \Leftrightarrow \mu = \frac{(E\eta+x)E\xi^2 - aE\xi - \frac{E\xi^2}{2}}{\frac{2}{3}E\xi^3}.$$

Since $\dfrac{E\xi^3}{3} > 0$, $r(x,\mu)$ attains the minimum at

$$\mu = \mu^* = \frac{(E\eta+x)E\xi^2 - aE\xi - \frac{E\xi^2}{2}}{\frac{2}{3}E\xi^3}.$$

Thus

$$\mu^* > 0 \Leftrightarrow (E\eta + x)E\xi^2 - aE\xi - \frac{E\xi^2}{2} > 0, \quad \Leftrightarrow x > \frac{aE\xi}{E\xi^2} + \frac{1}{2} - E\eta.$$

If condition (I) is not satisfied then $\mu^*(x) = 0$, hence

$$\inf_{\mu>0} r(x,\mu) = r(x,0) \quad \text{and} \quad r(x,0) = a + E\xi E(\eta + x)^2.$$

The lemma is proved. ∎

Lemma 4.3. *Suppose that* $U = \{u(\mu_n)|n = 1, 2, \dots\}$ *(where* $\mu_n = \mu_n^*(x)$*) is a controlling strategy of the process* $\{x_n : n = 1, 2, ..., x_1 = x\}$.

Then

1. $\lim_{n\to\infty} Ex_n = A,$
2. $\lim_{n\to\infty} Ex_n^2 = B,$
3. $\lim_{n\to\infty} n\Big\{\frac{1}{n}\sum_{k=1}^{n} Ex_k - A\Big\} = A_1x + B_1,$
4. $\lim_{n\to\infty} n\Big\{\frac{1}{n}\sum_{k=1}^{n} (Ex_k)^2 - A^2\Big\} = A_2x^2 + B_2x + C_2,$
5. $\lim_{n\to\infty} n\Big\{\frac{1}{n}\sum_{k=1}^{n} Ex_k^2 - B\Big\} = A_3x^2 + B_3x + C_3,$

where: $A, B, A_1, B_1, A_2, B_2, C_2, A_3, B_3, C_3$ *are constants.*

Proof. The above relations follow immediately from the following equation

$$x_n = \eta_{n-1} + x_{n-1} - \nu_{\mu^*_{n-1}}(\xi_{n-1}), \quad n = 2, 3, ...$$

Without loss of gererality, let $E\eta > 0$ (in the case of $E\eta < 0$ we obtain similar result).

Let us denote the strategy with control parameters μ_n^* defined in Lemma 4.2 by $U^* := \{u_n^* = u_n(\mu_n^*)|n = 1, 2, ...\}$, and the process controlled by strategy U^* with the initial condition $x_1^* = x$ by $\{x_n^*|n = 1, 2, ...\}$.

If at k-th step, the condition (I) is not sastisfied then

$$x_k^* = \eta + x_{k-1}^*,$$

or equivalently

$$x_n^* = \begin{cases} \eta + x_{n-1}^* - \nu_{\mu^*_{n-1}}(\xi), & \text{if at } n\text{-th step the condition } (I) \text{ (see (4.3)) holds} \\ \eta + x_{n-1}^*, & \text{otherwise} \end{cases}$$

Let us establish the process $\{\overline{x_n^*} : n = 1, 2, ...\}$ defined as follows

$$\begin{cases} \overline{x_n^*} = x_n^*, & \text{if the condition } (I) \text{ holds} \\ \overline{x_n^*} = \ell E\eta + x_n^*, & \text{otherwise,} \end{cases}$$

where ℓ is the nonnegative integer number such that

$$\ell E\eta + x_n^* \leqslant \frac{aE\xi}{E\xi^2} + \frac{1}{2} < (\ell + 1)E\eta + x_n^* \pmod P.$$

According to Lemma 4.3, it is easy to see that sequence of variances

$$\{Dx_n^* = Ex_n^{*2} - (Ex_n^*)^2\} \text{ is uniformly bounded.}$$

Combining with result 1. of Lemma 4.3, by the law of strongly large numbers, with probability 1, we have

$$\lim_{n\to\infty} x_n^* = A > \frac{aE\xi}{E\xi^2} + \frac{1}{2} - E\eta,$$

hence, there exists a positive interger number N such that $\forall n \geq N$ the condition (I) is sastified a.s.

Further, $\forall n \geq N$

$$\overline{x_n^*} = x_n^*, \text{ a.s.}$$

Thus, the results of Lemma 4.3 holds for the process $\{\overline{x_n^*} | n \in \mathbb{N}^+\}$. It is easy to see that

$$\lim_{n\to\infty} E_x^{U^*}\Big\{\frac{1}{n}\sum_{k=1}^{n} r(x_k^*, \mu_k^*)\Big\} = \lim_{n\to\infty} E\Big\{\frac{1}{n}\sum_{k=1}^{n} r(\overline{x_k^*}, \mu_k^*)\Big\},$$

$$\lim_{n\to\infty} n\Big\{E_x^{U^*}\Big[\frac{1}{n}\sum_{k=1}^{n} r(x_k^*, \mu_k^*)\Big]\Big\} - \lim_{m\to\infty} E_x^{U^*}\Big[\frac{1}{m}\sum_{k=1}^{m} r(x_k^*, \mu_k^*)\Big]$$

$$= \lim_{n\to\infty} n\Big\{E\Big[\frac{1}{n}\sum_{k=1}^{n} r(\overline{x_k^*}, \mu_k^*)\Big]\Big\} - \lim_{m\to\infty} E\Big[\frac{1}{m}\sum_{k=1}^{m} r(\overline{x_k^*}, \mu_k^*)\Big].$$

From the above relations we obtain the following Lemmas

Lemma 4.4. *The results of Lemma* 4.3 *hold for the process* $\{\overline{x_n^*} | n = 1, 2, ...\}$, *furthermore* $\{\overline{x_n^*} | n = 1, 2, ...\}$ *satisfies the condition* (I).

Lemma 4.5. *For all* $x \in R$ *we have:*

1. $\lim_{n\to\infty} R_n(x) = S,$
2. $\lim_{n\to\infty} n\big[R_n(x) - S\big] = V(x) = Ax^2 + Bx + C.$

Proof. The proof is carried out similarly as in Lemma 4.3. ∎

Theorem 4.1. *The constant* S *and the function* $V(x)$ *defined in Lemma* 4.5 *satisfy the following Bellman's equation*

$$S + V(x) = \inf_{\mu>0} \big\{r(x, \mu) + \int V(y) P(x, dy, \mu)\big\}, \quad \forall x \in \mathbb{R}.$$

Proof. We have

$$R_{n+1}(x) = \inf_{\mu>0} \Big\{\frac{1}{n+1} r(x, \mu) + \frac{n}{n+1} \int R_n(y) P(x, dy, \mu)\Big\},$$

$$\Rightarrow S + (n+1)[R_{n+1}(x) - S] = \inf_{\mu>0} \Big\{r(x, \mu) + n \int [R_n(y) - S] P(x, dy, \mu)\Big\}.$$

Therefore

$$S + V(x) = \inf_{\mu>0} \Big\{ r(x,\mu) + \int V(y)P(x,dy,\mu) \Big\}.$$

The proof of the theorem is complete. ∎

Theorem 4.2. *If there exists a strategy U^* such that:*

$$\begin{aligned} S + V(x) &= \inf_{\mu>0} \Big\{ r(x,\mu) + \int V(y)P(x,dy,\mu) \Big\} \\ &= \min_{\mu>0} \Big\{ r(x,\mu) + \int V(y)P(x,dy,\mu) \Big\} \\ &= r(x,\mu^*(x)) + \int V(y)P(x,dy,\mu^*(x)), \end{aligned}$$

then U^* is an optimal strategy, $\{\overline{x_n^*} | n = 1, 2, ...\}$ is the corresponding process and the cost $S = \psi_x(U^*)$ is finite, $\forall x \in \mathbb{R}$.

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