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On the Generalized Convolution with a Weight - Function for Fourier, Fourier Cosine and Sine Transforms

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Abstract. A generalized convolution for Fourier, Fourier cosine and sine transforms is introduced. Its properties and applications to solving systems of integral equations are presented.

1. Introduction

The convolution for integral transforms were studied in the 19$^{\text{th}}$ century, at first the convolutions for the Fourier transform (see, e.g. [3, 20]), for the Laplace transform (see [18, 20] and the references therein) for the Mellin transform [18] and after that the convolutions for the Hilbert transform [4, 21], the Hankel transform [7, 22], the Kontorovich–Lebedev transform [7, 26], the Stieltjes transform [19, 23], the convolutions with a weigh-function for the Fourier cosine transform [14].

The convolutions for different integral transforms have numerous applications in several contexts of science and mathematics [5, 6, 11, 18, 21, 25].

The convolution of two functions f and g for the Fourier integral transform F is defined by [3, 20]

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x-y)g(y)dy, \quad x \in R, \tag{1}$$

for which the factorization property holds

$$F(f * g)(y) = (Ff)(y)(Fg)(y), \quad \forall y \in R. \tag{2}$$

Here the integral Fourier transform takes the form

$$(Ff)(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x) e^{-iyx} dx.$$

The convolution of two functions f and g for the Fourier cosine transform F_c is also given [3, 20]

$$(f \underset{F_c}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(y)\big[g(|x-y|) + g(x+y)\big] dy, \quad x > 0, \tag{3}$$

with the factorization property

$$F_c(f \underset{F_c}{*} g)(y) = (F_c f)(y).(F_c g)(y), \ \ \forall y > 0, \tag{4}$$

where the integral Fourier cosine transform is [3, 20]

$$(F_c f)(y) = \sqrt{\frac{2}{\pi}} \int_{-\infty}^{+\infty} f(x) \cos(yx) dx.$$

The convolutions of two functions f and g for the Laplace integral transform L has the form [18, 23]

$$(f * g)(x) = \int_0^x f(x-t) g(t) dt, \quad x > 0, \tag{5}$$

which satisfies the factorization equality

$$L(f * g)(y) = (Lf)(y)(Lg)(y), \quad y = c + it, \ \ t \in R, \tag{6}$$

where the Laplace integral transform is defined by [18, 23]

$$(Lf)(y) = \int_0^{+\infty} e^{-yx} f(x) dx.$$

The generalized convolution for the Fourier sine and cosine transforms was first introduced by Churchill in 1941 [3]

$$(f \underset{1}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^{+\infty} f(y)\big[g(|x-y|) - g(x+y)\big] dy, \quad x > 0 \tag{7}$$

for which the factorization property holds

$$F_s(f \underset{1}{*} g)(y) = (F_s f)(y)(F_c g)(y), \quad \forall y > 0. \tag{8}$$

In the 90s of the last century, Yakubovic published some papers on special cases of genneralized convolutions for integral transforms according to index [17, 24, 26]. In 1998, Kakichev and Thao proposed a constructive method of defining the generalized convolution for any integral transforms K_1, K_2, K_3 with the weight-function $\gamma(y)$ [8] of functions f, g for which we have the factorization property

$$K_1(f \overset{\gamma}{*} g)(y) = \gamma(y)(K_2 f)(y)(K_3 g)(y).$$

In recent years, there have been published some works on generalized convolution, for instance: the generalized convolution for integral transforms Stieltjes, Hilbert and the cosine-sine transforms [12], the generalized convolution for H-transform [9], the generalized convolution for I-transform [16]. For example, the generalized convolution for the Fourier cosine and sine has been defined [13] by the identity:

$$(f \underset{2}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int\limits_0^{+\infty} f(y)\big[\mathrm{sign}(y-x)g(|y-x|) + g(y+x)\big]dy, \quad x > 0 \tag{9}$$

for which the factorization property holds

$$F_c(f \underset{2}{*} g)(y) = (F_s f)(y)(F_s g)(y), \quad \forall y > 0. \tag{10}$$

In this article we will give a notion of the generalized convolution with a weight-function of functions f and g for the Fourier, Fourier sine and cosine integral transforms. We will prove some of its properties as well as point out some of its relationships to several well-known convolutions and generalized convolutions. Also we will show that there does not exist the unit element for the calculus of this generalized convolution as well as there is not aliquote of zero. Finally, we will apply this notion to solving systems of integral equations.

2. Generalized Convolution for the Fourier, Fourier Cosine and Sine Transforms

Definition 1. *Genneralized convolution with the weight-function $\gamma(y) = sign\, y$ for the Fourier, Fourier cosine and sine transforms of functions f and g is defined by*

$$(f \overset{\gamma}{*} g)(x) = \frac{i}{\sqrt{2\pi}} \int\limits_0^{+\infty} \big[f(|x-u|) - f(|x+u|)\big]g(u)du, \quad x \in R \tag{11}$$

Denote by $L(R_+)$ the set of all functions f defined on $(0, \infty)$ such that $\int\limits_0^{+\infty} \big|f(x)\big|$ $dx < +\infty$.

Theorem 1. *Let f and g be functions in $L(R_+)$. Then the genneralized convolution with the weight-function $\gamma(y) = sign\, y$ for the Fourier, Fourier cosine and sine transforms of functions f and g has a meaning and belongs to $L(R)$ and the factorization property holds*

$$F(f \overset{\gamma}{*} g)(y) = \operatorname{sign} y (F_c f)(|y|)(F_s g)(|y|), \quad \forall y \in R. \tag{12}$$

Proof. Based on (11) and the hypothesis that f and $g \in L(R_+)$ we have

$$\int_{-\infty}^{+\infty} \big|(f \overset{\gamma}{*} g)(x)\big| dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \int_{0}^{+\infty} |g(u)| \, \big|f(|x-u|) - f(|x+u|)\big| du dx$$

$$\leqslant \frac{1}{\sqrt{2\pi}} \int_{0}^{+\infty} |g(u)| \Big[\int_{-\infty}^{+\infty} \big|f(|x-u|)\big| dx + \int_{-\infty}^{+\infty} \big|f(|x+u|)\big| dx \Big] du$$

$$= 2\sqrt{\frac{2}{\pi}} \int_{0}^{+\infty} |g(u)| du \int_{0}^{+\infty} |f(v)| dv < +\infty.$$

Therefore, $(f \overset{\gamma}{*} g)(x) \in L(R)$.

Further,

$$\operatorname{sign} y (F_c f)(|y|)(F_s g)(|y|) = (F_c f)(y)(F_s g)(y)$$

$$= \frac{2}{\pi} \int_{0}^{+\infty} \int_{0}^{+\infty} \cos(yu) \sin(yv) f(u) g(v) du dv$$

$$= \frac{1}{\pi} \int_{0}^{+\infty} \int_{0}^{+\infty} \big[\sin y(u+v) - \sin y(u-v) \big] f(u) g(v) du dv$$

$$= \frac{1}{\pi} \Big[\int_{0}^{+\infty} \int_{v}^{+\infty} \sin(yt) f(t-v) g(v) dt dv - \int_{0}^{+\infty} \int_{-v}^{+\infty} \sin(yt) f(|t+v|) g(v) dt dv \Big]$$

$$= \frac{1}{\pi} \int_{0}^{+\infty} \int_{0}^{+\infty} \sin(yt) \big[f(|t-v|) - f(|t+v|) \big] g(v) dt dv$$

$$- \frac{1}{\pi} \int_{0}^{+\infty} \int_{0}^{v} \sin(yt) \big[f(|t-v|) - f(|t+v|) \big] g(v) dt dv$$

$$- \frac{1}{\pi} \int_{0}^{+\infty} \int_{-v}^{0} \sin(yt) \big[f(|t-v|) - f(|t+v|) \big] g(v) dt dv.$$

On the other hand,

$$\int_{-v}^{0} \sin(yt) \big[f(|t-v|) - f(|t+v|) \big] dt = - \int_{0}^{v} \sin(yt) \big[f(|t-v|) - f(|t+v|) \big] dt.$$

Therefore,

$$\begin{aligned} &\operatorname{sign} y(F_c f)(|y|)(F_s g)(|y|) \\ &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{2}{\pi}} \int\limits_0^{+\infty} \sin yt \Big\{ \int\limits_0^{+\infty} \big[f(|t-v|) - f(|t+v|)\big] g(v)dv \Big\} dt. \end{aligned} \tag{13}$$

Since, if $h(x)$ is odd,

$$(Fh)(x) = -i(F_s h)(x), \quad x \in R \tag{14}$$

from (13) and (14) we obtain

$$\begin{aligned} &\operatorname{sign} y(F_c f)(|y|)(F_s g)(|y|) \\ &= \frac{1}{\sqrt{2\pi}} \frac{i}{\sqrt{2\pi}} \int\limits_{-\infty}^{+\infty} e^{iyt} \Big\{ \int\limits_0^{+\infty} \big[f(|t-v|) - f(|t+v|)\big] g(v)dv \Big\} dt \\ &= F(f \overset{\gamma}{*} g)(y). \end{aligned}$$

The proof is complete. ∎

Corollary 1. *The generalized convolution (11) can be represented by*

$$(f \overset{\gamma}{*} g)(x) = \frac{i}{\sqrt{2\pi}} \int\limits_0^{+\infty} f(u)\big[\operatorname{sign}(x+u) g(|x+u|) + \operatorname{sign}(x-u) g(|x-u|)\big] du. \tag{15}$$

Proof. Indeed for $x \geq 0$, with the substitution $x + u = v$, we get

$$\begin{aligned} &\int\limits_0^{+\infty} f(u) \operatorname{sign}\,(x+u) g(|x+u|) du = \int\limits_x^{+\infty} f(v-x) \operatorname{sign}\, v\, g(|v|) dv \\ &= \int\limits_0^{+\infty} f(|v-x|) g(v) dv - \int\limits_0^{x} f(|v-x|) \operatorname{sign}\, v\, g(|v|) dv. \end{aligned} \tag{16}$$

Similarly, with the substitution $x - u = -v$, we have

$$\begin{aligned} &\int\limits_0^{+\infty} f(u) \operatorname{sign}\,(x-u) g(|x-u|) du = \int\limits_{-x}^{+\infty} f(x+v) \operatorname{sign}\,(-v) g(|v|) dv \\ &= -\int\limits_0^{+\infty} f(x+v) g(|v|) dv + \int\limits_{-x}^{0} f(x+v) g(|v|) dv. \end{aligned} \tag{17}$$

On the other hand,

$$\int_0^x f(|v-x|)\text{ sign } v\, g(|v|)dv = \int_{-x}^0 f(|v+x|)g(|v|)dv.$$

From this and (16), (17) we have

$$\begin{aligned}&\frac{i}{\sqrt{2\pi}}\int_0^{+\infty} f(u)\big[\text{ sign }(x+u)g(|x+u|)+\text{ sign }(x-u)g(|x-u|)\big]du\\&=\frac{i}{\sqrt{2\pi}}\int_0^{+\infty} g(v)\big[f(|v-x|)-f(|v+x|)\big]dv.\end{aligned}\tag{18}$$

Similarly, for $x<0$, we have

$$\begin{aligned}&\frac{i}{\sqrt{2\pi}}\int_0^{+\infty} f(u)\big[\text{ sign }(x+u)g(|x+u|)+\text{ sign }(x-u)g(|x-u|)\big]du\\&=\frac{i}{\sqrt{2\pi}}\int_0^{+\infty} g(v)\big[f(|v-x|)-f(|v+x|)\big]dv.\end{aligned}\tag{19}$$

The equalities (18) and (19) yield (15). The proof is complete. ∎

Theorem 2. *In the space of functions belonging to* $L(R_+)$ *the generalized convolution* (11) *is not commutative*

$$(f \overset{\gamma}{*} g)(x) = -(g \overset{\gamma}{*} f)(x) + i\sqrt{\frac{2}{\pi}}(f \underset{L}{*} g)(|x|)\text{sign}\, x \tag{20}$$

where $(f \underset{L}{*} g)$ *is defined by* (5).

Proof. Indeed,

(i) for $x \geq 0$, by Definition 1, we have

$$(f \overset{\gamma}{*} g)(x) = \frac{i}{\sqrt{2\pi}}\int_0^{+\infty}\big[f(|u-x|)-f(x+u)\big]g(|u|)du.$$

With the substitutions $u-x=t$, $x+u=t$ we get

$$\begin{aligned}(f \overset{\gamma}{*} g)(x) &= \frac{i}{\sqrt{2\pi}}\Big\{\int_{-x}^{+\infty} f(|t|)g(x+t)dt - \int_x^{+\infty} f(t)g(|t-x|)dt\Big\}\\&=\frac{i}{\sqrt{2\pi}}\Big\{\int_0^{+\infty} f(|t|)g(x+t)dt - \int_0^{+\infty} f(t)g(|t-x|)dt\\&\quad+\int_{-x}^0 f(|t|)g(x+t)dt + \int_0^x f(t)g(|t-x|)dt\Big\}\end{aligned}$$

$$= \frac{i}{\sqrt{2\pi}}\Big\{ -\int\limits_0^{+\infty}\big[g(|x-t|)-g(|x+t|)\big]f(t)dt + \int\limits_0^{x} f(t)g(|t-x|)dt +$$

$$+\int\limits_0^{x} f(u)g(x-u)du\Big\}$$

$$= -(g \overset{\gamma}{*} f)(x) + i\sqrt{\frac{2}{\pi}}(f \underset{L}{*} g)(x). \tag{21}$$

Similarly
ii) for $x<0$ we have

$$(f \overset{\gamma}{*} g)(x) = \frac{i}{\sqrt{2\pi}}\int\limits_0^{+\infty}\big[f(|x-u|)-f(|x+u|)\big]g(u)du,$$

with the substitutions $v=u-x$, $t=x+u$ we get

$$(f \overset{\gamma}{*} g)(x) = \frac{i}{\sqrt{2\pi}}\Big\{\int\limits_{-x}^{+\infty} f(|v|)g(|x+v|)dv - \int\limits_{x}^{+\infty} f(|t|)g(|t-x|)dt\Big\}$$

$$= \frac{i}{\sqrt{2\pi}}\Big\{\int\limits_{0}^{+\infty} f(|v|)g(|x+v|)dv - \int\limits_{0}^{-x} f(v)g(|x+v|)dv$$

$$-\int\limits_{0}^{+\infty} f(|t|)g(|t-x|)dt - \int\limits_{x}^{0} f(|t|)g(|t-x|)dt\Big\}$$

$$= \frac{i}{\sqrt{2\pi}}\Big\{ -\int\limits_0^{+\infty}\big[g(|t-x|)-g(|t+x|)\big]f(t)dt - \int\limits_0^{-x} f(v)g(|x+v|)dv$$

$$-\int\limits_{-x}^{0} f(|-u|)g(|-u-x|)(-du)\Big\}$$

$$= -(g \overset{\gamma}{*} f)(x) - i\sqrt{\frac{2}{\pi}}(f \underset{L}{*} g)(-x). \tag{22}$$

The equalities (21) and (22) yield (20).
The proof is complete. ■

Theorem 3. *In the space of functions belonging to $L(R_+)$ the generalized convolution* (11) *is not associative and satisfies the following equalities*

a) $\big[f \overset{\gamma}{*} (g \overset{\gamma}{*} h)\big](x) = \big[g \overset{\gamma}{*} (f \overset{\gamma}{*} h)\big](x)$

b) $\big[f \overset{\gamma}{*} (g \overset{\gamma}{*} h)\big](x) = i\big[(f \underset{F_c}{*} g) \overset{\gamma}{*} h\big](x), \quad \forall x \in R$

where $(f \underset{F_c}{*} g)$ is defined by (2).

Proof. a) From the factorization property

$$F(f \overset{\gamma}{*} g)(y) = \text{sign}y(F_c f)(|y|)(F_s g)(|y|), \quad \forall y \in R.$$

On the other hand, because $(f \overset{\gamma}{*} g)(x)$ is odd,

$$\begin{aligned} F_s(f \overset{\gamma}{*} g)(|y|) &= \operatorname{sign} y F_s(f \overset{\gamma}{*} g)(y) \\ &= \operatorname{sign} y\big[-iF(f \overset{\gamma}{*} g)(y)\big] \\ &= -i(F_c f)(|y|)(F_s g)(|y|). \end{aligned} \tag{23}$$

By (23) we have

$$\begin{aligned} F\big[g \overset{\gamma}{*} (f \overset{\gamma}{*} h)\big](y) &= \operatorname{sign} y(F_c g)(|y|)F_s(f \overset{\gamma}{*} h)(|y|) \\ &= (F_c g)(|y|) \times \big[-i \operatorname{sign} y(F_c f)(|y|)(F_s h)(|y|)\big] \\ &= (F_c f)(|y|) \times \big[-i \operatorname{sign} y(F_c g)(|y|)(F_s h)(|y|)\big] \\ &= (F_c f)(|y|) \times F_s(g \overset{\gamma}{*} h)(|y|)\operatorname{sign} y \\ &= F\big(f \overset{\gamma}{*} (g \overset{\gamma}{*} h)\big)(y), \quad \forall y \in R. \end{aligned}$$

From this we get: $f \overset{\gamma}{*} (g \overset{\gamma}{*} h) = g \overset{\gamma}{*} (f \overset{\gamma}{*} h)$. So the generalized convolution (11) is not associative and satisfies the equality $f \overset{\gamma}{*} (g \overset{\gamma}{*} h) = g \overset{\gamma}{*} (f \overset{\gamma}{*} h)$. The proof for b) is similar to that of a).
The theorem is proved. ∎

Theorem 4. *In the space of functions belonging to* $L(R_+)$ *the operation of the generalized convolution* (11) *does not have the unit element but the left unit element* $e_1 = \dfrac{-i\sin 2x}{\sqrt{2\pi}x}$.

Proof. Suppose that there exists the right unit element e_2 of the operation of the generalized convolution (11) in the space of functions in $L(R_+)$:

$$f(x) \equiv (f \overset{\gamma}{*} e_2)(x), \quad \forall x > 0.$$

Therefore

$$F(f \overset{\gamma}{*} e_2)(y) = (Ff)(y), \quad \forall y \in R, \quad \forall f \in L(R_+).$$

From the factorization property, we have

$$\operatorname{sign}\, y(F_c f)(|y|)(F_s e_2)(|y|) = (Ff)(y), \quad \forall y \in R, \quad \forall f \in L(R_+).$$

It follows that

$$(F_c f)(y)(F_s e_2)(y) = (Ff)(y), \quad \forall y \in R, \quad \forall f \in L(R_+).$$

With an even function f, we get

$$(F_c f)(y).(F_s e_2)(y) = (F_c f)(y), \quad \forall y \in R.$$

Hence

$$(F_s e_2)(y) = 1, \quad \forall y \in R. \tag{24}$$

When $y \geq 0$, we have

$$(F_s e_2)(y) = -(F_s e_2)(-y) = -1$$

This is a contradition with (24).

Thus the generalized convolution (11) does not have the right unit element, and so does not have the unit element. We prove that the generalized convolution (11) have the left unit element.

Indeed, we have $e_1 = \dfrac{-i \sin 2x}{\sqrt{2\pi x}} \in L(R_+)$. We prove

$$(e_1 \overset{\gamma}{*} g)(x) = g(x), \quad \forall x > 0.$$

Putting $l_0 = \dfrac{\sin 2x}{\sqrt{2\pi x}}$, we get

$$\begin{aligned} F(e_1 \overset{\gamma}{*} g)(y) &= \operatorname{sign} y (F_c e_1)(|y|)(F_s g)(|y|) \\ &= \operatorname{sign} y F_c(-i l_0)(|y|)(F_s g)(|y|) \\ &= -i \operatorname{sign} y (F_c l_0)(|y|)(F_s g)(|y|) \\ &= -i \operatorname{sign} y (F_c l_0)(y)(F_s g)(|y|). \end{aligned}$$

On the other hand, since

$$\int_0^{+\infty} \cos(-yx) \sin x \cos x \frac{dx}{x} = \frac{\pi}{2}$$

(the formula 3.382.35 [1, p. 470]), we have $(F_c l_0)(y) = 1, \quad \forall y > 0$.

We obtain

$$F(e_1 \overset{\gamma}{*} g)(y) = -i \text{ sign } y (F_s g)(|y|) = -i (F_s g)(y) = (Fg)(y), \quad \forall y \in R.$$

Therefore $(e_1 \overset{\gamma}{*} g)(x) = g(x), \quad \forall x > 0$. Thus, e_1 is the left unit element belonging to $L(R_+)$.

The theorem is proved. ∎

Set $L(e^x, R_+) = \left\{ f : \int\limits_0^{+\infty} e^x |f(x)| dx < +\infty \right\}$.

Theorem 5. (Titchmarch type - Theorem) *Let* f *and* $g \in L(e^x, R_+)$, *if* $(f \overset{\gamma}{*} g)(x) \equiv 0 \quad \forall x \in R$, *then either* $f(t) = 0$ *or* $g(t) = 0, \quad \forall t > 0$.

Proof. Under the hypothesis $(f \overset{\gamma}{*} g)(x) \equiv 0 \quad \forall x \in R$ it follows that $F(f \overset{\gamma}{*} g)(y) = 0, \quad \forall y \in R$.

By virture of Theorem 1,

$$\operatorname{sign} y(F_c f)(|y|)(F_s g)(|y|) = 0, \quad \forall y \in R. \tag{25}$$

As $(F_c f)(|y|)$ and $(F_s g)(|y|)$ are analytic $\forall y \in R$ from (25) we have $(F_c f)(|y|) = 0,\ \forall y \in R$ or $(F_s g)(|y|) = 0,\ \forall y \in R$. It follows that $f(x) = 0,\ \forall x \in R_+$ or $g(x) = 0,\ \ \forall x \in R_+$.

The theorem is proved. ■

Theorem 6. *The generalized convolution* (11) *relates to the known convolutions as follows:*

a) $$(f \overset{\gamma}{*} g)(x) = i(g \underset{1}{*} f)(|x|)\operatorname{sign} x$$

b) $$(f \overset{\gamma}{*} g)(x) = i\big((f \dot{\circ} h)(y) \underset{F}{*} (g \dot{\circ} h)(y) \operatorname{sign} y\big)(x)$$

where $h(x) = |x|$ *and* $(f \underset{F}{*} g)$ *is defined by* (1).

Proof. From (11), when $x \geq 0$ we have: $(f \overset{\gamma}{*} g)(x) = i(g \underset{1}{*} f)(x)$

For $x \leq 0$

$$(f \overset{\gamma}{*} g)(x) = \frac{i}{\sqrt{2\pi}} \int_0^{+\infty} g(u)\big[f\big(\big|\ |x| + u\big|\big) - f\big(\big|u - |x|\ \big|\big)\big]dx$$

$$= \frac{-i}{\sqrt{2\pi}} \int_0^{+\infty} g(u)\big[f\big(\big|\ |x| - u\big|\big) - f\big(\big|\ |x| + u\big|\big)\big]du = -i(g \underset{1}{*} f)(|x|).$$

Thus, we have a).

On the other hand, we have

$$i\big((f \circ h)(y) \underset{F}{*} (g \circ h)(y) \text{ sign } y\big)(x) = \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(|u|).f(|x-u|) \text{ sign } u\, du$$

$$= \frac{i}{\sqrt{2\pi}} \int_0^{+\infty} g(u) f(|x-u|)du - \frac{i}{\sqrt{2\pi}} \int_{-\infty}^{0} g(|u|) f(|x-u|)du$$

$$= \frac{i}{\sqrt{2\pi}} \int_0^{+\infty} g(u) f(|x-u|)du - \frac{i}{\sqrt{2\pi}} \int_0^{+\infty} g(|u|) f(|x+u|)du$$

$$= \frac{i}{\sqrt{2\pi}} \int_0^{+\infty} g(u)\big[f(|x-u|) - f(|x+u|)\big]du$$

$$= (f \overset{\gamma}{*} g)(x).$$

The theorem is proved. ■

3. Application to Solving Systems of Integral Equations

a) Consider the system of integral equations

$$f(y) + \lambda_1 \int_0^{+\infty} g(t)\theta_1(y,t)dt = k(y), \quad y > 0$$

$$\lambda_2 \int_0^{+\infty} \theta_2(t)\big[f(|x-t|) - f(|x+t|)\big]dt + g(|x|)\operatorname{sign} x = h(|x|)\operatorname{sign} x, \quad x \in R. \tag{26}$$

Here, λ_1, λ_2 are complex constants and φ, ψ, k are functions of $L(R_+)$, f and g are the unknown functions, and

$$\theta_1(y,t) = \frac{1}{\sqrt{2\pi}}\big[\operatorname{sign}(t-y)\varphi(|t-y|) + \varphi(t+y)\big]$$

$$\theta_2(t) = \frac{i}{\sqrt{2\pi}}\psi(t).$$

Theorem 7. *With the condition*

$$1 - i\lambda_1\lambda_2(F_s\varphi)(y)(F_s\psi)(y) \neq 0, \quad \forall y > 0,$$

there exists a solution in $L(R_+)$ *of* (26) *which is defined by*

$$f(y) = k(y) - \lambda_1(h \underset{2}{*} \varphi)(y) - (k \underset{F_c}{*} l)(y) + \lambda_1\big[(h \underset{2}{*} \varphi) \underset{F_c}{*} l\big](y)$$

$$g(y) = h(y) - \lambda_2(k \overset{\gamma}{*} \varphi)(y) - (h \underset{1}{*} l)(y) + \lambda_2\big[(k \overset{\gamma}{*} \psi) \underset{1}{*} l\big](y).$$

Here, $l \in L(R_+)$ *and defined by*

$$(F_c l)(y) = \frac{-i\lambda_1\lambda_2 F_c(\varphi \underset{2}{*} \psi)(y)}{1 - i\lambda_1\lambda_2 F_c(\varphi \underset{2}{*} \psi)(y)}.$$

Proof. System (26) can be re-written in the form

$$f(y) + \lambda_1(\varphi \underset{2}{*} g)(y) = k(y), \quad y > 0$$

$$\lambda_2(f \overset{\gamma}{*} \psi)(x) + g(|x|)\operatorname{sign} x = h(|x|)\operatorname{sign} x, \quad x \in R.$$

Using the factorization property of the convolution (11) and $(f \underset{2}{*} g)(x)$ we have

$$(F_c f)(y) + \lambda_1(F_s\varphi)(y)(F_s g)(y) = (F_c k)(y), \quad y > 0$$

$$\lambda_2(F_c f)(y)(F_s\psi)(y) - i(F_s g)(y) = -i(F_s h)(y), \quad y > 0.$$

Accordingly, we have

$$\Delta = \begin{vmatrix} 1 & \lambda_1(F_s\varphi)(y) \\ \lambda_2(F_s\psi)(y) & -i \end{vmatrix} = -i\big[1 - i\lambda_1\lambda_2(F_s\varphi)(y)(F_s\psi)(y)\big] \neq 0$$

$$\Delta_1 = \begin{vmatrix} (F_c k)(y) & \lambda_1(F_s\varphi)(y) \\ -i(F_s h)(y) & -i \end{vmatrix} = -i(F_c k)(y) + i\lambda_1(F_s h)(y)(F_s\varphi)(y)$$

Therefore,

$$(F_c f)(y) = \frac{\Delta_1}{-i} - \frac{\Delta_1}{-i}\frac{-i\lambda_1\lambda_2 F_c(\varphi \underset{2}{*} \psi)(y)}{1 - i\lambda_1\lambda_2 F_c(\varphi \underset{2}{*} \psi)(y)}, \quad y > 0.$$

Due to Wiener-Levy's theorem [2], there exists a continuous function $l \in L(R_+)$ such that

$$(F_c l)(y) = \frac{-i\lambda_1\lambda_2 F_c(\varphi \underset{2}{*} \psi)(y)}{1 - i\lambda_1\lambda_2 F_c(\varphi \underset{2}{*} \psi)(y).}$$

It follows that

$$\begin{aligned}(F_c f)(y) &= \frac{\Delta_1}{-i} - \frac{\Delta_1}{-i}(F_c l)(y)\\ &= (F_c k)(y) - \lambda_1 F_c(h \underset{2}{*} \varphi)(y) - [(F_c k)(y) - \lambda_1 F_c(h \underset{2}{*} \varphi)(y)](F_c l)(y)\\ &= (F_c k)(y) - \lambda_1 F_c(h \underset{2}{*} \varphi)(y) - F_c(k \underset{F_c}{*} l)(y) + \lambda_1 F_c\big[(h \underset{2}{*} \varphi) \underset{F_c}{*} l\big](y), \quad y > 0.\end{aligned}$$

Hence

$$f(y) = k(y) - \lambda_1(h \underset{2}{*} \varphi)(y) - (k \underset{F_c}{*} l)(y) + \lambda_1\big[(h \underset{2}{*} \varphi) \underset{F_c}{*} l\big](y) \in L(R_+).$$

Similarly,

$$\begin{aligned}\Delta_2 &= \begin{vmatrix} 1 & (F_c k)(y) \\ \lambda_2(F_s\psi)(y) & -i(F_s h)(y)\end{vmatrix} = -i(F_s h)(y) - \lambda_2(F_c k)(y)(F_s\psi)(y)\\ &= -i(F_s h)(y) + i\lambda_2 F_s(k \overset{\gamma}{*} \psi)(y).\end{aligned}$$

It follows that

$$\begin{aligned}(F_s g)(y) &= \frac{\Delta_2}{-i} - \frac{\Delta_2}{-i}(F_c l)(y)\\ &= (F_s h)(y) - \lambda_2 F_s(k \overset{\gamma}{*} \psi)(y) - \big[(F_s h)(y) - \lambda_2 F_s(k \overset{\gamma}{*} \psi)(y)\big](F_c l)(y)\\ &= (F_s h)(y) - \lambda_2 F_s(k \overset{\gamma}{*} \psi)(y) - F_s(h \underset{1}{*} l)(y) + \lambda_2 F_s\big[k \overset{\gamma}{*} \psi) \underset{1}{*} l\big](y).\end{aligned}$$

Hence

$$g(y) = h(y) - \lambda_2(k \overset{\gamma}{*} \psi)(y) - (h \underset{1}{*} l)(y) + \lambda_2\big[k \overset{\gamma}{*} \psi) \underset{1}{*} l\big](y) \in L(R_+).$$

The theorem is proved. ■

b) Consider the system of integral equations

$$\begin{aligned} f(y) + \lambda_1 \int_0^{+\infty} g(t)\theta_1(y,t)dt &= k(y), \quad y > 0\\ \lambda_2 \int_0^{+\infty} f(t)\theta_2(x,t)dt + g(|x|)\operatorname{sign} x &= h(|x|)\operatorname{sign} x, \quad x \in R.\end{aligned} \tag{27}$$

Here, λ_1, λ_2 are complex constants and φ, ψ, k are functions of $L(R_+)$, f and g are the unknown functions, and

$$\theta_1(y,t) = \frac{1}{\sqrt{2\pi}}\big[\varphi(|t-y|) + \varphi(t+y)\big]$$
$$\theta_2(x,t) = \frac{i}{\sqrt{2\pi}}\big[\psi(|x-t|) - \psi(|x+t|)\big].$$

Theorem 8. *With the condition*

$$1 - i\lambda_1\lambda_2(F_c\varphi)(y)(F_c\psi)(y) \neq 0, \quad \forall y > 0$$

there exists a solution in $L(R_+)$ of (27) which is defined by

$$f(y) = k(y) - \lambda_1(h \underset{1}{*} \varphi)(y) + (k \underset{1}{*} l)(y) - \lambda_1(l \overset{\gamma}{*} (\varphi \overset{\gamma}{*} h))(y)$$
$$g(y) = h(y) - \lambda_2(\psi \overset{\gamma}{*} k)(y) - (h \underset{1}{*} l)(y) + \lambda_2(l \overset{\gamma}{*} (k \underset{1}{*} \psi))(y).$$

Here, $l \in L(R_+)$ and defined by $(F_c l)(y) = \dfrac{-i\lambda_1\lambda_2 F_c(\varphi \underset{F_c}{*} \psi)(y)}{1 - i\lambda_1\lambda_2 F_c(\varphi \underset{F_c}{*} \psi)(y)}$.

Proof. Sytems (27) canbe-written in the form

$$f(y) + \lambda_1(g \underset{1}{*} \varphi)(y) = k(y), \quad y > 0$$
$$\lambda_2(\psi \overset{\gamma}{*} f)(x) + g(|x|)\ \mathrm{sign}\ x = h(|x|)\ \mathrm{sign}\ x, \quad x \in R.$$

Using the factorization property of the convolutions (11) and $(f \underset{1}{*} g)(x)$, we have

$$(F_s f)(y) + \lambda_1(F_s g)(y)(F_c\varphi)(y) = (F_s k)(y), \quad y > 0$$
$$\lambda_2(F_c\psi)(y)(F_s f)(y) - i(F_s g)(y) = -i(F_s h)(y), \quad y > 0,$$

$$\begin{aligned}
\Delta &= \begin{vmatrix} 1 & \lambda_1(F_c\varphi)(y) \\ \lambda_2(F_c\psi)(y) & -i \end{vmatrix} \\
&= -i\big[1 - i\lambda_1\lambda_2 F_c(\varphi \underset{F_c}{*} \psi)(y)\big] \neq 0 \\
\Delta_1 &= \begin{vmatrix} (F_s k)(y) & \lambda_1(F_c\varphi)(y) \\ -i(F_s h)(y) & -i \end{vmatrix} \\
&= -i(F_s k)(y) + i\lambda_1 F_s(h \underset{1}{*} \varphi)(y) \\
&= -i(F_s k)(y) + \lambda_1 F_s(\varphi \overset{\gamma}{*} h)(y)
\end{aligned}$$

Therefore

$$(F_s f)(y) == \frac{\Delta_1}{-i}\Big[1 - \frac{-i\lambda_1\lambda_2 F_c(\varphi \underset{F_c}{*} \psi)(y)}{1 - i\lambda_1\lambda_2 F_c(\varphi \underset{F_c}{*} \psi)(y)}\Big]$$

Due to Wiener-Levy's theorem [2] there exists a function $l \in L(R_+)$ such that

$$(F_c l)(y) = \frac{-i\lambda_1\lambda_2 F_c(\varphi \underset{F_c}{*} \psi)(y)}{1 - i\lambda_1\lambda_2 F_c(\varphi \underset{F_c}{*} \psi)(y)}.$$

It follows that

$$\begin{aligned}
(F_s f)(y) &= \frac{\Delta_1}{-i} - \frac{\Delta_1}{-i}(F_c l)(y), \quad y > 0 \\
&= (F_s k)(y) - \lambda_1 F_s(h \underset{1}{*} \varphi)(y) - \big[(F_s k)(y) + i\lambda_1 F_s(\varphi \overset{\gamma}{*} h)(y)\big] \times (F_c l)(y) \\
&= (F_s k)(y) - \lambda_1 F_s(h \underset{1}{*} \varphi)(y) - F_s(k \underset{1}{*} l)(y) - i\lambda_1 F\big(l \overset{\gamma}{*} (\varphi \overset{\gamma}{*} h)\big)(y) \\
&= (F_s k)(y) - \lambda_1 F_s(h \underset{1}{*} \varphi)(y) - F_s(k \underset{1}{*} l)(y) - \lambda_1 F_s\big(l \overset{\gamma}{*} (\varphi \overset{\gamma}{*} h)\big)(y)
\end{aligned}$$

Hence

$$f(y) = k(y) - \lambda_1(h \underset{1}{*} \varphi)(y) - (k \underset{1}{*} l)(y) - \lambda_1\big(l \overset{\gamma}{*} (\varphi \overset{\gamma}{*} h)\big)(y)$$

Similarly,

$$\begin{aligned}
\Delta_2 &= \begin{vmatrix} 1 & (F_s k)(y) \\ \lambda_2(F_c\psi)(y) & -i(F_s h)(y) \end{vmatrix} \\
&= - i(F_s h)(y) - \lambda_2 F_s(k \underset{1}{*} \psi)(y) \\
&= - i(F_s h)(y) - \lambda_2 F(\psi \overset{\gamma}{*} k)(y) \\
&= - i(F_s h)(y) + i\lambda_2 F_s(\psi \overset{\gamma}{*} k)(y) L(R_+).
\end{aligned}$$

Therefore

$$\begin{aligned}
(F_s g)(y) &== \frac{\Delta_2}{-i} - \frac{\Delta_2}{-i}(F_c l)(y) \\
&= (F_s h)(y) - \lambda_2 F_s(\psi \overset{\gamma}{*} k)(y) - \big[(F_s h)(y) - i\lambda_2 F_s(k \underset{1}{*} \psi)(y)\big] \times (F_c l)(y) \\
&= (F_s h)(y) - \lambda_2 F_s(\psi \overset{\gamma}{*} k)(y) - F_s(h \underset{1}{*} l)(y) + i\lambda_2 F\big[l \overset{\gamma}{*} (k \underset{1}{*} \psi)\big](y) \\
&= (F_s h)(y) - \lambda_2 F_s(\psi \overset{\gamma}{*} k)(y) - F_s(h \underset{1}{*} l)(y) + \lambda_2 F_s\big(l \overset{\gamma}{*} (k \underset{1}{*} \psi)\big)(y)
\end{aligned}$$

Hence

$$g(y) = h(y) - \lambda_2(\psi \overset{\gamma}{*} k)(y) - (h \underset{1}{*} l)(y) + \lambda_2(l \overset{\gamma}{*} (k \underset{1}{*} \psi))(y) L(R_+).$$

The theorem is proved. ∎

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