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A Stochastic EOQ Policy of Cold-Drink-For a Retailer

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Abstract. This paper extends a stochastic **EOQ** (*economic order quantity*) model both for discrete and continuous distribution of demands of cold-drink. A general characterization of the optimal inventory policy is developed analytically. An optimal solution is obtained with proper numerical illustration.

1. Introduction

A well-known stochastic extension of the classical **EOQ** (*economic order quantity*) model bases the re-order decision or the stock level (see Hadley and Whitin [4], Wagner [13]). Models of storage systems with stochastic supply and demand have been widely analysed in the models of Faddy [3], Harrison and Resnick [5], Miller [8], Moran [9], Pliska [10], Puterman [11], Meyer, Rothkopf and Smith [7], Teisberg [12], Chao and Manne [1], Hogan [6] and Devarangan and Weiner [2].

In this paper, an optimal inventory policy is characterised by conditions: (a) demand rate is stochastic that depends upon temperature as random variable; (b) supply rate is instanteneously infinite and order is placed in the begining of the cycle; (c) inventory cost is a linear function of temperature.

2. Fundamental Assumptions and Notations

1. Model is developed on single-item products.

2. Lead time is negligible.
3. Demand is uniform over the period and a function of temperature that follows a probability distributions.
4. Production rate is instanteneously infinite.
5. Reorder-time is fixed and known. Thus the set-up cost is not included in the total cost.

Let the holding cost per item per unit time be C_h, the shortage cost per item per unit time be C_s, the inventory level be Q of item, r is the demand over the period, T is the cycle length.

3. The Model

In this model, we consider demand rate of the product (r) and inventory holding cost per item per unit time (C_h) are:

$$r = a\tau$$

and

$$C_h = C_1 + C_2(\tau - \mu).$$

where,
$a = \frac{dr}{d\tau}(\geq 0)$ = marginal response of cold-drink consumption to a change in τ(temperature)
C_1 = opportunity cost of money tied up in inventory.
C_2 = rate of change of inventory cost with respect to temperature.
μ = optimum temperature for a buyer, according to their demand. Generally μ is $5°C$.

Now, the governing equations are as follows:

Case 1. When Shortage does not occur

$$\frac{dQ}{dt} = -\frac{r}{T},\ 0 \leq t \leq T \tag{1}$$

with $Q(0) = Q_0$.
From Eq. (1), we have

$$Q(t) = Q_0 - \frac{r}{T}t,\ 0 \leq t \leq T.$$

Here $Q(T) \geq 0 \Rightarrow Q_0 - \frac{r}{T}T \geq 0 \Rightarrow Q_0 \geq r$. Therefore, the inventory is

$$\int_0^T (Q_0 - \frac{r}{T}t)dt = (Q_0 - \frac{r}{2})T,\ \text{for } r \leq Q_0.$$

Case 2. When Shortage occurs:

$$\frac{dQ}{dt} = -\frac{r}{T},\ 0 \leq t \leq t_1 \tag{2}$$

with $Q(0) = Q_0$, and $Q(t_1) = 0$,

and

$$\frac{dQ}{dt} = -\frac{r}{T},\ t_1 \le t \le T \tag{3}$$

with $Q(T) < 0$.
From Eq. (2), we have

$$Q(t) = Q_0 - \frac{r}{T}t,\ 0 \le t \le t_1.$$

Now $Q(t_1) = 0 \Rightarrow t_1 = \dfrac{Q_0T}{r}$. The Eq. (3) gives us

$$Q(t) = -\frac{r}{T}(t - t_1),\ t_1 \le t \le T.$$

So $Q(T) < 0 \Rightarrow -\frac{r}{T}(T - t_1) < 0 \Rightarrow T > t_1 \Rightarrow T > \frac{Q_0T}{r} \Rightarrow Q_0 < r$. Therefore, the inventory during $(0, t_1)$ is

$$\int_0^{t_1} (Q_0 - \frac{r}{T}t)dt = Q_0t_1 - \frac{r}{2T}t_1^2 = \frac{1}{2}\frac{Q_0^2}{r}T\ .$$

The shortage during (t_1, T) is

$$\int_{t_1}^{T} -Q(t)dt = \frac{r}{2T}(T - t_1)^2$$

$$= \frac{1}{2}rT\Big(1 - \frac{Q_0}{r})^2,\ r > Q_0.$$

Since, $Q_0 \ge r \Rightarrow Q_0 \ge a\tau \Rightarrow \tau \le \frac{1}{a}Q_0 = \tau^*$ (say). i.e., $Q_0 = a\tau^*$. Also, $Q_0 < r \Rightarrow \tau > \tau^*$.

Case I. Uniform demand and discrete units.
τ is random variable with probability $p(\tau)$ such that $\sum_{\tau=\tau_0}^{\infty} p(\tau) = 1$ and $p(\tau) \ge 0$.

Therefore the expected average cost is

$$\begin{aligned} Eac(\tau^*) &= \frac{1}{T}\sum_{\tau=\tau_0}^{\tau^*} C_h\Big(Q_0 - \frac{r}{2}\Big)Tp(\tau) + \frac{1}{2}\sum_{\tau=\tau^*+1}^{\infty} C_h\frac{Q_0^2}{r}p(\tau)T \\ &\quad + \frac{1}{2}C_s\sum_{\tau=\tau^*+1}^{\infty} rT\Big(1 - \frac{Q_0}{r}\Big)^2 p(\tau) \\ &= \sum_{\tau=\tau_0}^{\tau^*}(C_1 - C_2\mu + C_2\tau)a\Big(\tau^* - \frac{\tau}{2}\Big)p(\tau) \\ &\quad + \frac{1}{2}\sum_{\tau=\tau^*+1}^{\infty} a(C_1 - C_2\mu + C_2\tau)\tau^{*2}\frac{p(\tau)}{\tau} \\ &\quad + \frac{1}{2}C_s\sum_{\tau=\tau^*+1}^{\infty} a\tau\Big(1 - \frac{\tau^*}{\tau}\Big)^2 p(\tau) \end{aligned}$$

Now,

$$Eac(\tau^*+1) = Eac(\tau^*) + (C_1 - C_2\mu + C_s)a\Big(\sum_{\tau=\tau_0}^{\tau^*} p(\tau) + \Big(\tau^* + \frac{1}{2}\Big)\sum_{\tau^*+1}^{\infty}\frac{p(\tau)}{\tau}\Big)$$
$$- C_s + aC_2\Big(\sum_{\tau=\tau_0}^{\tau^*}\tau p(\tau) + \Big(\tau^* + \frac{1}{2}\Big)\sum_{\tau=\tau^*+1}^{\infty} p(\tau)\Big).$$

In order to find the optimum value of Q_0^* i.e., τ^* so as to minimize $Eac(\tau^*)$, the following conditions must hold: $Eac(\tau^*+1) > Eac(\tau^*)$ and $Eac(\tau^*-1) > Eac(\tau^*)$ i.e., $Eac(\tau^*+1) - Eac(\tau^*) > 0$ and $Eac(\tau^*-1) - Eac(\tau^*) > 0$. Now, $Eac(\tau^*+1) - Eac(\tau^*) > 0$ implies

$$\Im(\tau^*) + \frac{C_2}{C_1 - C_2\mu + C_s}\mho(\tau^*) > \frac{C_s}{C_1 - C_2\mu + C_s},$$

where

$$\Im(\tau^*) = \sum_{\tau=\tau_0}^{\tau^*} p(\tau) + (\tau^* + \frac{1}{2})\sum_{\tau^*+1}^{\infty}\frac{p(\tau)}{\tau}$$
$$\mho(\tau^*) = \sum_{\tau=\tau_0}^{\tau^*}\tau p(\tau) + (\tau^* + \frac{1}{2})\sum_{\tau=\tau^*+1}^{\infty} p(\tau).$$

Similarly $Eac(\tau^*-1) - Eac(\tau^*) > 0$ implies

$$\Im(\tau^*-1) + \frac{C_2}{C_1 - C_2\mu + C_s}\mho(\tau^*-1) < \frac{C_s}{C_1 - C_2\mu + C_s}.$$

Therefore for minimum value of $Eac(\tau^*)$, the following condition must be satisfied

$$\begin{aligned}\Im(\tau^*) &+ \frac{C_2}{C_1 - C_2\mu + C_s}\mho(\tau^*)\\ &> \frac{C_s}{C_1 - C_2\mu + C_s}\\ > \Im(\tau^*-1) &+ \frac{C_2}{C_1 - C_2\mu + C_s}\mho(\tau^*-1)\end{aligned} \tag{4}$$

Case II. Uniform demand and continuous units.
When uncertain demand is estimated as a continuous random variable, the cost equation of the inventory involves integrals instead of summation signs. The discrete point probabilities $p(\tau)$ are replaced by the probability differential $f(\tau)$ for small interval. In this case $\int_0^\infty f(\tau)\, d\tau = 1$ and $f(\tau) \geq 0$. Proceeding exactly in the same manner as in *Case I*, The total expected average cost during period $(0,\ T)$ is

$$
\begin{aligned}
Eac(\tau^*) = a\tau^* \int_{\tau=\tau_0}^{\tau^*} (C_1 - C_2\mu + C_2\tau) f(\tau) d\tau - \frac{a}{2} \int_{\tau_0}^{\tau^*} (C_1 - C_2\mu + C_2\tau)\tau f(\tau) d\tau \\
+ \frac{a}{2}\tau^{*2} \int_{\tau^*}^{\infty} (C_1 - C_2\mu + C_2\tau) \frac{f(\tau)}{\tau} d\tau + \frac{a}{2} C_s \int_{\tau^*}^{\infty} (\tau - \tau^*)^2 \frac{f(\tau)}{\tau} d\tau
\end{aligned} \tag{5}
$$

Now,

$$
\begin{aligned}
\frac{dEac(\tau^*)}{d\tau^*} = a\Big\{ (C_1 - C_2\mu + C_s)\Big(\int_{\tau_0}^{\tau^*} f(\tau) d\tau + \tau^* \int_{\tau^*}^{\infty} \frac{f(\tau)}{\tau} d\tau \Big) \\
+ C_2 \Big(\int_{\tau_0}^{\tau^*} \tau f(\tau) d\tau + \tau^* \int_{\tau^*}^{\infty} f(\tau) d\tau \Big) - C_s \Big\}
\end{aligned} \tag{6}
$$

and

$$
\frac{d^2 Eac(\tau^*)}{d\tau^{*2}} = a\Big\{ (C_1 - C_2\mu + C_s) \int_{\tau^*}^{\infty} \frac{f(\tau)}{\tau} d\tau + C_2 \int_{\tau^*}^{\infty} f(\tau) d\tau \Big\} > 0. \tag{7}
$$

For minimum value of $Eac(\tau^*)$, $\frac{dEac(\tau^*)}{d\tau^*} = 0$ and $\frac{d^2 Eac(\tau^*)}{d\tau^{*2}} > 0$ must be satisfied. The equation, $\frac{dEac(\tau^*)}{d\tau^*} = 0$ being nonlinear can only be solved by any numerical method (*Bisection Method*) for given parameter values.

4. Numerical Examples

Example 1. For discrete case:

In this case, we consider $C_1 = 0.135,\ C_2 = 0.001,\ C_s = 5.0,\ \mu = 5.0,\ a = 0.8$ in appropriate units and also consider the probability of temperature in a week such that

τ in $^\circ C$:	35	36	37	38	39	40	41
$p(\tau)$:	0.05	0.15	0.14	0.10	0.25	0.10	0.21

Then the optimal solution is $\tau^* = 38^o C$ i.e., $Q^* = a\tau^* = 30.4$ units.

Example 2. For continuous case:

We take the values of the parameters in appropriate units as follows:

$$
\begin{aligned}
f(\tau) &= 0.04 - 0.0008\tau,\ 0 \le \tau \le 50 \\
&= 0,\ \text{elsewhere}
\end{aligned}
$$

$C_1 = 0.135,\ C_2 = 0.001,\ C_s = 5.0\ ,\ \mu = 5.0,\ a = 0.8$. Then the optimal solution is: $\tau^* = 29.30°C$ i.e., $Q^* = a\tau^* = 23.44$ units.

5. Conclusion

From physical phenomenon, it is common belief that the consumption of cold drinks depend upon temperature. Temperature is also random in character. Generally the procurement cost of cold drinks is smaller than their selling price. Consequently, supply of cold drinks to the retialer is sufficiently large. Inventory holding cost is broken down into two components: (i) the first is the opportunity cost of money tied up in inventory that is considered here as C_1 (ii) the 2nd is $C_2(\tau - \mu)$, where μ is optimum temperature for a buyer according to their demand. Generally, μ is $5°C$. So the cost of declining temperature $(\tau - \mu)$ has a remarkable effect on the inventory cost. In reality, the discrete case is more realistic than the continuous one. But we discuss both the cases. As far as the authors are informed, no stochastic **EOQ** model of this type has yet been discussed in the inventory literature.

References

1. H. Chao and A. S. Manne, It oil stock-piles and import reductions: A dynamic programming approach, *Opns. Res.* **31** (1983) 632–651.
2. S. Devarangan and R. Weiner, *Stockpile Behavior as an International Game,* Harvard University, 1983.
3. M. J. Faddy, Optimal control of finite dams, *Adv. Appl. Prob.* **6** (1974) 689–710.
4. G. Hadly, and T. Whitin, Analysis of Inventory System. Prentice-Hall, Englewood Cliffs, NJ, 1963.
5. J. M. Harrison and S. I. Resnick, The stationary distribution and first exit probabilities of a storage process with general release rules, *Math. Opns. Res.* **1** (1976) 347–358.
6. W. W. Hogan, Oil stockpiling: help thy neighbor, *Energy. J.* **4** (1983) 49–71.
7. R. R. Meyer, M. H. Rothkopf, and S. A. Smith, Reliability and inventory in a production-storage system, *Mgmt. Sci.* **25** (1979) 799–807.
8. R. G. Miller, Jr., Continuous time stochastic storage processes with random linear inputs and outputs, *J. Math. Mech.* **12** (1963) 275–291.
9. P. A. Moran, *The Theory of Storage,* Metuen, London, 1959.
10. S. R. Pliska, A diffusion process model for the optimal operations of a reservoir system, *J. Appl. Prob.* **12** (1975) 859–863.
11. M. L. Puterman, A diffusion process model for the optimal operations of a reservoir system, *North-Holland/Times studies in the management service* **1** (1975) 143–159.
12. T. J. Teisberg, A dynamic programming model of the U. S. strategic petroleum reserve, *Bell. J. Econ.* **12** (1981) 526–546.
13. H. M. Wagner, Statistical Management of Inventory Systems, John Wiley & Sons, 1962.