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Central Limit Theorem for Functional of Jump Markov Processes

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Abstract. In this paper some conditions are given to ensure that for a jump homogeneous Markov process $\{X(t), t \geq 0\}$ the law of the integral functional of the process: $T^{-1/2}\int_0^T \varphi(X(t))dt$, converges to the normal law $N(0, \sigma^2)$ as $T \to \infty$, where φ is a mapping from the state space E into $\mathbb{R}$.

1. Introduction

The central limit theorem is a subject investigated intensively by many well-known probabilists such as Linderberg, Chung,.... The results concerning central limit theorems, the iterated logarithm law, the lower and upper bounds of the moderate deviations are well understood for independent random variable sequences and for martingales but less is known for dependent random variables such as Markov chains and Markov processes.

The first result on central limit for functionals of stationary Markov chain with a finite state space can be found in the book of Chung [5]. A technical method for establishing the central limit is the regeneration method. The main idea of this method is to analyse the Markov process with arbitrary state space by dividing it into independent and identically distributed random blocks between visits to fixed state (or atom). This technique has been developed by Athreya - Ney [2], Nummelin [10], Meyn - Tweedie [9] and recently by Chen [4].

The technical method used in this paper is based on central limit for martingales and ergodic theorem. The paper is ogranized as follows:

In Sec. 2, we shall prove that for a positive recurrent Markov sequence

$\{X_n, n \geq 0\}$ with Borel state space $(E, \mathcal{B})$ and for $\varphi : E \to \mathbb{R}$ such that

$$\varphi(x) = f(x) - Pf(x) = f(x) - \int_E f(y)P(x, dy)$$

with $f : E \to \mathbb{R}$ such that $\int_E f^2(x)\Pi(dx) < \infty$, where $P(x, .)$ is the transition probability and $\Pi(.)$ is the stationary distribution of the process, the distribution of $n^{-1/2}\sum_{i=1}^n \varphi(X_i)$ converges to the normal law $N(0, \sigma^2)$ with $\sigma^2 = \int_E(\varphi^2(x)+ 2\varphi(x)Pf(x))\Pi(dx)$.

The central limit theorem for the integral functional $T^{-1/2}\int_0^T \varphi(X(t))dt$ of jump Markov process $\{X(t), t \geq 0\}$ will be established and proved in Sec. 3.

Some examples will be given in Sec. 4.

It is necessary to emphasize that the conditions for normal asymptoticity of $n^{-1/2}\sum_{i=1}^n \varphi(X_i)$ is the same as in [8] but they are not equivalent to the ones established in [10, 11]. The results on the central limit for jump Markov processes obtained in this paper are quite new.

2. Central Limit for the Functional of Markov Sequence

Let us consider a Markov sequence $\{X_n, n \geq 0\}$ defined on a basic probability space $(\Omega, \mathcal{F}, P)$ with the Borel state space $(E, \mathcal{B})$, where $\mathcal{B}$ is the σ-algebra generated by the countable family of subsets of E. Suppose that $\{X_n, n \geq 0\}$ is homogeneous with transition probability

$$P(x, A) = P(X_{n+1} \in A | X_n = x),\ A \in \mathcal{B}.$$

We have the following definitions

Definition 2.1. *Markov process $\{X_n, n \geq 0\}$ is said to be irreducible if there exists a σ- finite measure μ on $(E, \mathcal{B})$ such that for all $A \in \mathcal{B}$*

$$\mu(A) > 0 \text{ implies } \sum_{n=1}^{\infty} P^n(x, A) > 0,\ \forall x \in E$$

where

$$P^n(x, A) = P(X_{m+n} \in A | X_m = x).$$

The measure μ is called irreducible measure.

By Proposition 2.4 of Nummelin [10], there exists a maximum irreducible measure μ^* possessing the property that if μ is any irreducible measure then $\mu \ll \mu^*$.

Definition 2.2. *Markov process $\{X_n, n \geq 0\}$ is said to be recurrent if*

$$\sum_{n=1}^{\infty} P^n(x, A) = \infty,\ \forall x \in E, \forall A \in \mathcal{B} : \mu^*(A) > 0.$$

The process is said to be Harris recurrent if

$$P_x(X_n \in A\ i.o.) = 1.$$

Let us notice that a process which is Harris recurrent is also recurrent.

Theorem 2.1. *If* $\{X_n, n \geq 0\}$ *is recurrent then there exists a uniquely invariant measure* $\Pi(.)$ *on* $(E, \mathcal{B})$ *(up to constant multiples) in the sense*

$$\Pi(A) = \int_E \Pi(dx)P(x, A), \ \forall A \in \mathcal{B}, \tag{1}$$

or equivalently

$$\Pi(.) = \Pi P(.). \tag{2}$$

(*see Theorem* 10.4.4 *of Meyn-Tweedie,* [9]).

Definition 2.3. *A Markov sequence* $\{X_n, n \geq 0\}$ *is said to be positive recurrent* (*null recurrent*) *if the invariant measure* Π *is finite* (*infinite*).

For a positive recurrent Markov sequence $\{X_n, n \geq 0\}$, its unique invariant probability measure is called stationary distribution and is denoted by Π. Hereafter we always denote the stationary distribution of Markov sequence $\{X_n, n \geq 0\}$ by Π and if ν is the initial distribution of Markov sequence then $P_\nu(.), E_\nu(.)$ are denoted for probability and expectation operator responding to ν. In particular, $P_\nu(.), E_\nu(.)$ are replaced by $P_x(.), E_x(.)$ if ν is the Dirac measure at x.

We have the following ergodic theorem:

Theorem 2.2. *If Markov sequence* $\{X_n, n \geq 0\}$ *possesses the unique invariant distribution* Π *such that*

$$P(x, .) \ll \Pi(.), \ \forall x \in E, \tag{3}$$

then $\{X_n, n \geq 0\}$ *is metrically transitive when initial distribution is the stationary distribution. Further, for any measurable mapping* $\varphi : E \times E :\to \mathbb{R}$ *such that* $E_\Pi|\varphi(X_0, X_1)| < \infty$*, with probability one*

$$\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} \varphi(X_k, X_{k+1}) = E_\Pi \varphi(X_0, X_1) \tag{4}$$

and the limit does not depend on the initial distribution. (*See Theorem* 1.1 *from Patrick Billingsley* [3]).

The following notations will be used in this paper: For a measurable mapping $\varphi : E \to \mathbb{R}$ we denote

$$\Pi\varphi = \int_E \varphi(x)\Pi(dx), \ P\varphi(x) = \int_E \varphi(y)P(x, dy) = E(\varphi(X_{n+1})|X_n = x),$$
$$P^n\varphi(x) = \int_E \varphi(y)P^n(x, dy) = E(\varphi(X_{n+m})|X_m = x).$$

For the countable state space $E = \{1, 2, ...\}$ we denote

$$P_{ij} = P(i, \{j\}) = P(X_{n+1} = j | X_n = i),\ P_{ij}^{(n)} = P^n(i, \{j\}) = P(X_{m+n} = j | X_n = i)$$
$$\pi_j = \Pi(\{j\}),\ P = [P_{ij}, i, j \in E],\ P^{(n)} = [P_{ij}^{(n)}, i, j \in E] = P^n.$$

Then

$$\Pi\varphi = \sum_{j\in E} \varphi(j)\pi_j,\ P\varphi(j) = \sum_{k\in E} \varphi(k)P_{jk},\ P^n\varphi(j) = \sum_{k\in E} \varphi(k)P_{jk}^{(n)}.$$

If the distribution of random variable Y_n converges to the normal distribution $N(\mu, \sigma^2)$ then we denote $\xrightarrow{\mathcal{L}} N(\mu, \sigma^2)$. The indicator function of a set A is denoted by $\mathbf{1}_A$, where

$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{else.} \end{cases}$$

Finally, the mapping $\varphi : E = \{1, 2, ...\} \longrightarrow \mathbb{R}$ is denoted by column vector $\varphi = (\varphi(1), \varphi(2), ...)^T$.

The main result of this section is to establish the conditions for

$$n^{-1/2} \sum_{k=1}^{n} \varphi(X_k) \xrightarrow{\mathcal{L}} N(\mu, \sigma^2).$$

We need a central limit theorem for martingale differences as follows

Theorem 2.3. (Central limit theorem for martingale differences) *Suppose that* $\{u_k, k \geq 0\}$ *is a sequence of martingale differences defined on a probability space* $(\Omega, \mathcal{F}, P)$ *corresponding to a filter* $\{\mathcal{F}_k, k \geq 0\}$, *i.e.,* $E(u_{k+1}|\mathcal{F}_k) = 0, k = 0, 1, 2, \cdots$ *Further, assume that the following conditions are satisfied*

(A_1) $n^{-1} \sum_{k=1}^{n} E(u_k^2 | \mathcal{F}_{k-1}) \xrightarrow{P} \sigma^2$,

(A_2) $n^{-1} \sum_{k=1}^{n} E(u_k^2 \mathbf{1}_{[|u_k| \geq \varepsilon\sqrt{n}]} | \mathcal{F}_{k-1}) \xrightarrow{P} 0$, *for each* $\varepsilon > 0$ (*the conditional Linderberg's condition*).

Then

$$n^{-1/2} \sum_{k=1}^{n} u_k \xrightarrow{\mathcal{L}} N(0, \sigma^2). \tag{5}$$

(*see Corollary of Theorem* 3.2, [7]).

Remark 1. Theorem 2.3 remains valid for $\{u_k, k \geq 0\}$ being a m-dimensional martingale differences where the condition (A_1) is replaced by

$$n^{-1} \sum_{k=1}^{n} \text{Var}\ (u_k | \mathcal{F}_{k-1}) \xrightarrow{P} \sigma^2 = [\sigma_{ij}, i, j = 1, 2, \cdots, m]$$

with

$$\text{Var }(u_k|\mathcal{F}_{k-1}) = [E(u_{ik}u_{jk}|\mathcal{F}_{k-1}), i, j = 1, 2, \cdots, m].$$

We shall prove the following theorem.

Theorem 2.4. (Central limit theorem for functional of Markov sequence) *Suppose that the following conditions hold:*

(H_1) *The Markov sequence* $\{X_n, n \geq 0\}$ *is positive recurrent with the transition probability* $P(x, .)$ *and the unique stationary distribution* $\Pi(.)$ *satisfying the condition* (3).

(H_2) *The mapping* $\varphi : E \to \mathbb{R}$ *can be represented in the form*

$$\varphi(x) = f(x) - Pf(x), \ x \in E, \tag{6}$$

where $f : E \to \mathbb{R}$ *is measurable and* $\Pi f^2 < \infty$.

Then

$$n^{-1/2} \sum_{k=1}^{n} \varphi(X_k) \xrightarrow{\mathcal{L}} N(0, \sigma^2) \tag{7}$$

for any initial distribution, where

$$\sigma^2 = \Pi(f^2 - (Pf)^2) = \Pi(\varphi^2 + 2\varphi Pf). \tag{8}$$

Proof. We have

$$\begin{aligned} & n^{-1/2} \sum_{k=1}^{n} \varphi(X_k) = n^{-1/2} \sum_{k=1}^{n} [f(X_k) - Pf(X_k)] \\ & = n^{-1/2} \sum_{k=1}^{n} [f(X_k) - Pf(X_{k-1})] + n^{-1/2} \sum_{k=1}^{n} Pf(X_{k-1}) - n^{-1/2} \sum_{k=1}^{n} Pf(X_k) \\ & = n^{-1/2} \sum_{k=1}^{n} u_k + n^{-1/2} [Pf(X_0) - Pf(X_n)], \end{aligned}$$

where

$$u_k = f(X_k) - Pf(X_{k-1}) = f(X_k) - E(f(X_k)|X_{k-1})$$

are martingale differences with respect to $\mathcal{F}_k = \sigma(X_0, X_1, \cdots, X_k)$, whereas

$$n^{-1/2}[Pf(X_0) - Pf(X_n)] \xrightarrow{P} 0$$

by Chebyshev's inequality. Thus, it is sufficient to prove that

$$Y_n := n^{-1/2} \sum_{k=1}^{n} u_k \xrightarrow{\mathcal{L}} N(0, \sigma^2)$$

and the convergence does not depend on the initial distribution. For this purpose, we shall show that the martingale differences $\{u_k, k \geq 1\}$ satisfy the conditions $(A_1), (A_2)$.

According to assumption (H_2) we have

$$E_\Pi[E(u_1^2|\mathcal{F}_0)] = E_\Pi(u_1^2) = E_\Pi[f(X_1) - Pf(X_0)]^2 = E_\Pi f^2(X_1) - E_\Pi[Pf(X_0)]^2,$$

thus

$$E_\Pi(u_1^2) = \Pi f^2 - \Pi(Pf)^2 < \infty. \tag{9}$$

Therefore, by the ergodic Theorem 2.2, for any initial distribution with probability one

$$n^{-1}\sum_{k=1}^{n} E(u_k^2|\mathcal{F}_{k-1}) \longrightarrow E_\Pi u_1^2 = \sigma^2.$$

Thus the condition (A_1) of Theorem 2.3 is satisfied.

On the other hand, by (9) we have

$$E_\Pi(u_1^2\mathbf{1}_{[|u_1|\geq t]}) \longrightarrow 0, \tag{10}$$

as $t \uparrow \infty$. Again by the ergodic Theorem 2.2, for any initial distribution, with probability one

$$n^{-1}\sum_{k=1}^{n} E(u_k^2\mathbf{1}_{[|u_k|\geq t]}|\mathcal{F}_{k-1}) \longrightarrow E_\Pi(u_1^2\mathbf{1}_{[|u_1|\geq t]}) \tag{11}$$

for each $t > 0$. By (11) and then (10) we have with probability one

$$\begin{aligned} 0 &\leq \overline{\lim_{n\to\infty}} n^{-1}\sum_{k=1}^{n} E_\Pi(u_k^2\mathbf{1}_{[|u_k|\geq \varepsilon\sqrt{n}]}) \\ &\leq \lim_{n\to\infty} n^{-1}\sum_{k=1}^{n} E_\Pi(u_k^2\mathbf{1}_{[|u_k|\geq t]}) \\ &= E_\Pi(u_1^2\mathbf{1}_{[|u_1|\geq t]}) \longrightarrow 0 \text{ as } t \uparrow \infty. \end{aligned}$$

Thus condition (A_2) is satisfied, hence by the central limit theorem for martingale differences $\{u_k, k \geq 1\}$ (7) holds. ∎

Remark 2. If the series

$$\sum_{n=0}^{\infty} P^n\varphi(x) = \sum_{n=0}^{\infty}\int_E \varphi(y)P^n(x, dy)$$

converges, then we always have

$$\varphi(x) = f(x) - Pf(x)$$

with

$$f(x) = \sum_{n=0}^{\infty} P^n\varphi(x).$$

In fact, it is obvious that

$$f(x) = \varphi(x) + \sum_{n=1}^{\infty} P^n \varphi(x) = \varphi(x) + P \sum_{n=0}^{\infty} P^n \varphi(x) = \varphi(x) + Pf(x).$$

Furthermore, in this case

$$\sigma^2 = \Pi\Big[\varphi^2 + 2\sum_{n=0}^{\infty} \varphi P^n \varphi\Big].$$

Remark 3. If $\varphi = f - Pf$ holds, then

$$\Pi\varphi = \Pi f - \Pi P f = 0. \tag{12}$$

So the condition (12) is necessary for $\varphi = f - Pf$. Furthermore, in addition if we have

$$\lim_{n\to\infty} P^n f(x) = \Pi f, \ \forall x \in E$$

then $f(x)$ is also given by

$$f(x) = \sum_{n=0}^{\infty} P^n \varphi(x) + \Pi f.$$

In fact, we have

$$\begin{aligned} \varphi(x) &= f(x) - Pf(x) \\ P\varphi(x) &= Pf(x) - P^2 f(x) \\ &\cdots \\ P^n\varphi(x) &= P^n f(x) - P^{n+1} f(x). \end{aligned}$$

Summing the above equalities we obtain

$$\sum_{k=0}^{n} P^k \varphi(x) = f(x) - P^{n+1} f(x) \longrightarrow f(x) - \Pi f.$$

Remark 4. Function f given by (6) is defined uniquely up to an additional constant if $\lim_{n\to\infty} P^n g(x) = \Pi g$ for all g Π- integrable.

In fact, suppose that f_1, f_2 are the functions satisfying (6). Then $g = f_1 - f_2$ is a solution of the equations:

$$g(x) = Pg(x), \ g(x) = P(Pg(x)) = P^2 g(x) = \cdots = P^n g(x), \ \forall x \in E$$

for all $n = 1, 2, \cdots$. Thus there exists the limit

$$g(x) = \lim_{n\to\infty} P^n g(x) = \Pi g \text{ (a constant)}.$$

It also follows from Remark 4 and from (8) that if f satisfies the equation (6) then σ^2 is defined uniquely, i.e., σ^2 does not change if f is replaced by $f + C$ with C being any constant, since

$$\Pi[\varphi^2 + 2\varphi P(f + C)] = \Pi[\varphi^2 + 2\varphi Pf] + 2C\Pi\varphi = \Pi[\varphi^2 + 2\varphi Pf].$$

Remark 5. If $\Pi\varphi \neq 0$ we can replace φ by $\varphi^* = \varphi - \Pi\varphi$.

Corollary 2.1. *Assume that a Markov chain $\{X_n, n \geq 0\}$ is irreducible, ergodic with the countable state space $E = \{1, 2, \cdots\}$ and with the ergodic distribution $\Pi = (\pi_1, \pi_2, \cdots)$ and the following condition is satisfied*
(H_3) *The mapping $\varphi : E \to \mathbb{R}$ takes the form*

$$\varphi(x) = f(x) - Pf(x),\ \forall x \in E$$

with $f : E \to \mathbb{R}$ being measurable such that $\Pi f^2 < \infty$. Put

$$\sigma^2 = \Pi[f^2 - (Pf)^2] = \Pi[\varphi^2 + 2\varphi Pf].$$

Then

$$n^{-1/2}\sum_{k=1}^{n}\varphi(X_k) \xrightarrow{\mathcal{L}} N(0, \sigma^2) \text{ as } n \to \infty.$$

3. Central Limit for Integral Functional of Jump Markov Process

3.1. Jump Markov Process

Let $\{X(t), t \geq 0\}$ be a random process defined on some probability space $(\Omega, \mathcal{F}, P)$ with measurable state space $(E, \mathcal{B})$.

Definition 3.1. *The process $\{X(t), t \geq 0\}$ is called jump homogeneous Markov process with the state space $(E, \mathcal{B})$ if it is a Markov process with transition probability*

$$P(t, x, A) = P(X(t + s) \in A | X(s) = x),\ s, t \geq 0$$

satisfying the following condition

$$\lim_{t\to 0} P(t, x, \{x\}) = 1,\ \forall x \in E. \tag{13}$$

We suppose also that $\{X(t), t \geq 0\}$ is right continuous and the limit (13) is uniform in $x \in E$.

By Theorem 2.4 in [6] the sample functions of $\{X(t), t \geq 0\}$ are step functions with probability one, and there exist two $q-$ functions $q(.)$ and $q(., .)$ being Baire functions where $q(x, .)$ is finite measure on Borel subsets of $E \setminus \{x\}$, $q(x) = q(x, E \setminus \{x\})$ is bounded. Further

$$\lim_{t\to 0} \frac{(1 - P(t, x, \{x\})}{t} = q(x),$$
$$\lim_{t\to 0} \frac{P(t, x, A)}{t} = q(x, A)$$

uniformly in $A \subset E \setminus \{x\}$.

If $q(x) > 0 \ \forall x \in E$ then the process has no absorbing state. We assume also that $q(x)$ is bounded from 0.

Since $\{X(t), t \geq 0\}$ is right continuous and step process, the system starts out in some state Z_1, stays there a length of time ρ_1, then jumps immediately to a new state Z_2, stays a length of time ρ_2, etc. Therefore there exist random variables $Z_1, Z_2, \cdots$ and $\rho_1, \rho_2, \cdots$ such that

$$X(t) = Z_1, \quad \text{if} \quad 0 \leq t < \rho_1,$$
$$X(t) = Z_n, \quad \text{if} \quad \rho_1 + \cdots + \rho_{n-1} \leq t < \rho_1 + \cdots + \rho_n, \ n \geq 2.$$

ρ_n's are all finite because we have assumed that $q(x) > 0 \ \forall x \in E$.

Let $\nu(t)$ be the random variable defined by

$$\nu(t) = \max\{k : \rho_1 + \cdots + \rho_k < t\}$$

then $\nu(t)$ is the number of jumps which occur up to time t.

It follows from the general theory of discontinuous Markov process (see [6], p.266) that $\{Z_n, n \geq 1\}$ is a Markov chain with transition probability

$$P(x, A) = \frac{q(x, A)}{q(x)}, \tag{14}$$

furthermore

$$P(\rho_{n+1} > s | \rho_1, \cdots, \rho_n, Z_1, \cdots, Z_{n+1}) = e^{-q(Z_{n+1})s}, \ s > 0 \tag{15}$$
$$P(Z_{n+1} \in A | \rho_1, \cdots, \rho_n, Z_1, \cdots, Z_n) = P(Z_n, A). \tag{16}$$

The function $q(.,.)$ is called the transition intensity.

It follows from (15), (16) that $\{(Z_n, \rho_n), n \geq 1\}$ is a Markov chain on the cartesian product $E \times \mathbb{R}^+$, where $\mathbb{R}^+ = (0, \infty)$. This chain is called the imbedded chain with the transition probability

$$\begin{aligned} Q(x, s, A \times B) &= P(Z_{n+1} \in A, \rho_{n+1} \in B | Z_n = x, \rho_n = s) \\ &= \int_A P(x, dy) \int_B q(y) e^{-q(y)u} du, \end{aligned}$$

$A \times B \in \mathcal{B} \times \mathcal{B}(\mathbb{R}^+)$, where $\mathcal{B}(\mathbb{R}^+)$ denotes the Borel σ- algebra on $\mathbb{R}^+$. This transition probability does not depend on s and we rewrite it by $Q(x, A \times B)$ or formally by

$$Q(x, dy \times du) = P(x, dy) q(y) \exp(-q(y)u) du.$$

Definition 3.2. *The probability measure* Π^* *on* $(E \times \mathbb{R}^+, \mathcal{B} \times \mathcal{B}(\mathbb{R}^+))$ *is called the stationary distribution of the imbedded chain* $\{(Z_n, \rho_n), n \geq 1\}$ *if*

$$\Pi^*(A \times B) = \int_{E \times \mathbb{R}^+} \Pi^*(dx \times ds) Q(x, A \times B), \ A \times B \in \mathcal{B} \times \mathcal{B}(\mathbb{R}^+). \tag{17}$$

Letting $B = \mathbb{R}^+$, then Π^* is the stationary distribution of the imbedded chain if and only if

$$\Pi(.) = \Pi^*(. \times \mathbb{R}^+) \tag{18}$$

is the one of $\{Z_n, n \geq 1\}$ with the transition probability $P(x, A) = Q(x, A \times \mathbb{R}^+)$ and

$$\Pi^*(A \times B) = \int_E \Pi(dx) Q(x, A \times B).$$

Since $\Pi P(.) = \Pi(.)$, we have

$$\begin{aligned}\Pi^*(A \times B) &= \int_E \Pi(dx) \int_A P(x, dy) \int_B q(y) \exp(-q(y)u) du \\ &= \int_A (\int_E \Pi(dx) P(x, dy)) \int_B q(y) \exp(-q(y)u) du\end{aligned}$$

or

$$\Pi^*(A \times B) = \int_A \Pi(dy) \int_B q(y) \exp(-q(y)u) du \tag{19}$$

or in differential form

$$\Pi^*(dy \times du) = \Pi(dy) q(y) \exp(-q(y)u) du. \tag{20}$$

Thus we have the following proposition:

Proposition 3.1. *If the Markov chain $\{Z_n, n \geq 1\}$ with the transition probability $P(x, A)$ has the stationary distribution Π then the imbedded chain possesses also the stationary distribution Π^* defined by* (19) *or* (20)*.*

Proposition 3.2. *If $P(x, .) \ll \Pi(.)$ $\forall x \in E$, where Π is the stationary distribution of $\{Z_n, n \geq 1\}$ then the transition probability $Q(x, .)$ of the imbedded chain is also absolutely continuous with respect to the stationary distribution Π^*, i.e.*

$$Q(x, .) \ll \Pi^*(.), \ \forall x \in E.$$

(see [3], p.66).

Here and after we shall denote by Π, Π^* the stationary distributions of Markov chain $\{Z_n, n \geq 1\}$ and the imbedded chain $\{(Z_n, \rho_n), n \geq 1\}$, respectively.

3.2. Functional Central Limit Theorem

We have the following ergodic theorem for the imbedded chain

Theorem 3.1. (Ergodic theorem for the imbedded process) *If Markov chain $\{Z_n, n \geq 1\}$ with the transition probability $P(x, .)$ having the stationary distribution Π such that*

$$P(x,.) \ll \Pi(.)\ \forall x \in E,$$

and if $\varphi(Z_1, \rho_1; Z_2, \rho_2)$ is the random variable possessing the finite expectation μ w.r.t. the probability measure P_{Π^*}, then for any initial distribution

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \varphi(Z_k, \rho_k; Z_{k+1}, \rho_{k+1}) = \mu \text{ ; a.s.} \tag{21}$$

In particular, if $\Pi q^{-1} < \infty$ then

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \rho_k = \int_E \Pi(dy)(q(y))^{-1} = \Pi q^{-1} \text{ a.s.} \tag{22}$$

Furthermore

$$\lim_{t\to\infty} \frac{\nu(t)}{t} = (\Pi q^{-1})^{-1} =: \alpha > 0 \text{ a.s.} \tag{23}$$

and (21), (22) *remain valid if in the limits n is replaced by $\nu(t)$, then limits are taken as $t \to \infty$.*

Proof. (21) follows from the ergodic theorem for Markov chain $\{(Z_n, \rho_n), n \geq 1\}$, and (23) follows from (22) by the same argument as in the renewal theory. ■

Applying Theorem 2.4 for the imbedded chain $\{(Z_n, \rho_n), n \geq 1\}$ we obtain the following theorem.

Theorem 3.2. (Central limit theorem for the imbedded chain) *Assume that the following conditions $(C_1), (C_2)$ are satisfied:*

(C_1) *The jump Markov process $\{X(t), t \geq 0\}$ has the imbedded chain $\{(Z_n, \rho_n), n \geq 1\}$ such that the Markov chain $\{Z_n, n \geq 1\}$ has the transition probability $P(x,.)$ with the stationary distribution Π satisfying the following condition*

$$P(x,.) \ll \Pi(.)\ \forall x \in E.$$

(C_2) *The function $\psi : E \times \mathbb{R}^+ \to \mathbb{R}$ takes the form*

$$\psi(x,s) = f(x,s) - Qf(x,s),$$

where $f : E \times \mathbb{R}^+ \to \mathbb{R}$ is $\mathcal{B} \times \mathcal{B}(\mathbb{R}^+)$- measurable and

$$Qf(x) = Qf(x,s) = \int_E P(x,dy) \int_{\mathbb{R}^+} f(y,u) q(y) \exp(-q(y)u) du.$$

Furthermore, the function f has the following property

$$\Pi^* f^2 = \int_E \Pi(dy) \int_{\mathbb{R}^+} |f(y,u)|^2 q(y) \exp(-q(y)u) du < \infty. \tag{24}$$

Then we have

$$n^{-1/2} \sum_{k=1}^{n} \psi(Z_k, \rho_k) \xrightarrow{\mathcal{L}} N(0, \sigma^2) \tag{25}$$

for any initial distribution, where

$$\sigma^2 = \Pi^*(f^2 - (Qf)^2) = \Pi^*(\psi^2 + 2\psi Qf). \tag{26}$$

The goal of this section is to investigate the limit law of the integral functional $T^{-1/2}\int_0^T \varphi(X(t))dt$ as $T \to \infty$.

Let us at first notice that

$$\int_0^T \varphi(X(t))dt = \sum_{k=1}^{\nu(T)} \varphi(Z_k)\rho_k + \varphi(Z_{\nu(T)+1})(T - \tau_{\nu(T)}), \tag{27}$$

where

$$\tau_1 = \rho_1,\ \tau_2 = \rho_1 + \rho_2, \cdots,\ \tau_n = \rho_1 + \rho_2 + ... + \rho_n, \cdots$$

are the jump times of the process $\{X(t), t \geq 0\}$.

In what follows we suppose always that the condition (C_1) is satisfied.

We need the following lemmas.

Lemma 3.1. *If* $\Pi\varphi^2 q^{-2} < \infty$ *then*

$$\frac{1}{\sqrt{T}}\varphi(Z_{\nu(T)+1})(T - \tau_{\nu(T)}) \xrightarrow{P} 0 \tag{28}$$

for any initial distribution.

Proof. Noticing that for $\psi(x,s) = \varphi(x)s$ we have

$$\begin{aligned}\Pi^*\psi^2 &= \int_E \Pi(dy)\varphi^2(y)\int_{\mathbb{R}^+} u^2 q(y)\exp(-q(y)u)du \\ &= 2\int_E \varphi^2(y)q^{-2}(y)\Pi(dy) = 2\Pi\varphi^2 q^{-2} < \infty\end{aligned}$$

and $\nu(T) \to \infty$ a.s. as $T \to \infty$ by (23). By those and by the ergodic Theorem 3.1

$$(\nu(T)+1)^{-1}\sum_{k=1}^{\nu(T)+1} |\varphi(Z_k)\rho_k|^2 \longrightarrow \Pi^*\psi^2 \text{ a.s.}$$

as $T \to \infty$. Hence with probability one

$$(\nu(T)+1)^{-1}|\varphi(Z_{\nu(T)+1})\rho_{\nu(T)+1}|^2 \longrightarrow 0$$

and (28) follows from

$$\frac{1}{\sqrt{T}}|\varphi(Z_{\nu(T)+1})(T - \tau_{\nu(T)})| \leq (\frac{\nu(T)+1}{T})^{1/2}(\nu(T)+1)^{-1/2}|\varphi(Z_{\nu(T)+1})\rho_{\nu(T)+1}| \to 0$$

a.s. ∎

Lemma 3.2. *Suppose that* $\{u_k, \mathcal{F}_k, k \geq 1\}$ *defined on* $(\Omega, \mathcal{F}, P)$ *are the square integrable martingale differences such that*

$$\sup_{n,m\geq 1}(n^{-1}\sum_{k=m}^{m+n} Eu_k^2) = C < \infty \tag{29}$$

and that $\{\nu(t), t \geq 0\}$ *is a random process valued in* $\{1, 2, \cdots\}$ *such that* $\{\nu(t) = k\} \in \mathcal{F}_k$ $\forall t \geq 0$ *and*

$$\lim_{t\to\infty} \frac{\nu(t)}{t} = \alpha > 0 \text{ a.s.} \tag{30}$$

Then

$$T^{-1/2}\Big|\sum_{k=1}^{\nu(T)} u_k - \sum_{k=1}^{[\alpha T]} u_k\Big| \xrightarrow{P} 0 \text{ as } T \to \infty. \tag{31}$$

Proof. It follows from condition (30) that: for all $\varepsilon > 0$, and T sufficiently large we have

$$P(|\frac{\nu(T)}{T} - \alpha| > \varepsilon^3) < \varepsilon$$

or

$$P((\alpha - \varepsilon^3)T < \nu(T) < (\alpha + \varepsilon^3)T) \geq 1 - \varepsilon. \tag{32}$$

Putting

$$A_\varepsilon = \{\alpha - \varepsilon^3)T < \nu(T) < (\alpha + \varepsilon^3)T\},$$

we have

$$\begin{aligned} P\Big(T^{-\frac{1}{2}}\Big|\sum_{k=1}^{\nu(T)} u_k - \sum_{k=1}^{[\alpha T]} u_k\Big| > \varepsilon\Big) &\leq P(A_\varepsilon^c) + P\Big(\Big\{T^{-\frac{1}{2}}\Big|\sum_{k=1}^{\nu(T)} u_k - \sum_{k=1}^{[\alpha T]} u_k\Big| > \varepsilon\Big\} \cap A_\varepsilon\Big) \\ &\leq \varepsilon + P\Big(T^{-\frac{1}{2}} \max_{|l-[\alpha T]|<\varepsilon^3 T}\Big|\sum_{k=1}^{l} u_k - \sum_{k=1}^{[\alpha T]} u_k\Big| > \varepsilon\Big) \\ &\leq \varepsilon + P\Big(\max_{a\leq l\leq b}\Big|\sum_{k=a}^{l} u_k\Big| > \frac{\varepsilon T^{\frac{1}{2}}}{2}\Big) \end{aligned} \tag{33}$$

where $a = [\alpha T] - [\varepsilon^3 T],\ b = [\alpha T] + [\varepsilon^3 T]$ with $[r]$ denoting the integer part of the number r.

By Kolmogorov's inequality for a martingale

$$P\Big(\max_{1\leq n\leq N}\Big|\sum_{k=1}^{n} u_k\Big| > \lambda\Big) \leq \frac{1}{\lambda^2} E\Big[\sum_{k=1}^{N} u_k\Big]^2 = \frac{1}{\lambda^2}\sum_{k=1}^{N} Eu_k^2,$$

we have

$$P\Big(\max_{a\leq l\leq b}\Big|\sum_{k=a}^{l} u_k\Big| > \frac{\varepsilon T^{\frac{1}{2}}}{2}\Big) \leq 8\varepsilon\frac{1}{2\varepsilon^3 T} E\Big[\sum_{k=a}^{2[T\varepsilon^3]+a} u_k^2\Big] \leq 8\varepsilon C. \tag{34}$$

It follows from (33), (34) that (31) holds. ∎

Corollary 3.1. *Assume that the martingale differences* $\{u_k, k \geq 1\}$ *take the form*

$$u_k = f(X_k) - E(f(X_k)|X_{k-1}), \quad k = 1, 2, \ldots$$

where $\{X_k, k \geq 0\}$ *is a Markov chain with the stationary distribution* Π *such that* $\Pi f^2 < \infty$*. Then* (31) *holds for any initial distribution.*

Proof. It is obvious that

$$E_\Pi u_k^2 \leq E_\Pi f^2(X_{k-1}) = \Pi f^2 < \infty,$$

therefore

$$E_\Pi(n^{-1} \sum_{k=m}^{m+n} u_k^2) \leq \Pi f^2 = C, \ \forall m, n.$$

Denoting the quantity in the left-hand side of (31) by η_T, by Lemma 3.2 we obtain

$$\lim_{T\to\infty} P_\Pi(|\eta_T| \geq \varepsilon) = 0 \ \forall \epsilon > 0$$

or

$$\lim_{T\to\infty} \int_E P_x(|\eta_T| \geq \varepsilon)\Pi(dx) = 0 \quad \forall \epsilon > 0.$$

It follows that there exists a subset $\Lambda \subset E$ such that $\Pi(\Lambda) = 0$ and

$$\lim_{T\to\infty} P_x(|\eta_T| \geq \varepsilon) = 0 \quad \forall x \in E \setminus \Lambda.$$

Since $P(x, .) \ll \Pi(.) \ \forall x, P(x, E \setminus \Lambda) = 1 \ \forall x \in E$.

On the other hand, letting $A_T = \{|\eta_T| \geq \varepsilon\}$, we observe that $A_T \in \cup_{n\geq n_0} \mathcal{F}_n$ with $n_0 > 1$, where $\mathcal{F}_n = \sigma(X_k, k \geq n)$. Then by Markov property:

$$\begin{aligned} P_x(A_T) &= E(1_{A_T}|X_0 = x) = E[E(1_{A_T}|X_1)|X_0 = x] \\ &= \int_E E(1_{A_T}|X_1 = y)P(x, dy) = \int_{E\setminus\Lambda} E_y(1_{A_T})P(x, dy). \end{aligned}$$

Therefore

$$\begin{aligned} 0 \leq \limsup_{T\to\infty} P_x(A_T) &= \limsup_{T\to\infty} \int_{E\setminus\Lambda} P_y(A_T)P(x, dy) \\ &= \int_{E\setminus\Lambda} \lim_{T\to\infty} P_y(A_T)P(x, dy) = 0. \end{aligned}$$

So

$$\lim_{T\to\infty} P_x(A_T) = 0 \ \forall x$$

and hence

$$\lim_{T\to\infty} P_\nu(|\eta_T| \geq \varepsilon) = \lim_{T\to\infty} \int_E P_x(|\eta_T| \geq \varepsilon)\nu(dx) = 0.$$

This implies (31). ∎

Lemma 3.3. *Assume that the following equation has a solution* $g(x)$

$$(I-P)g(x) = P\varphi q^{-1}(x). \tag{35}$$

Then, putting

$$f(x,s) = \varphi(x)s + g(x), \tag{36}$$

we have the representation

$$\varphi(x)s = f(x,s) - Qf(x), \tag{37}$$

where $Qf(x) = g(x)$.

Proof. At first let us notice that for $\psi : E \times \mathbb{R}^+ \to \mathbb{R}$ given by $\psi(x,s) = \varphi(x)s$ we have

$$\begin{aligned} Qg(x) &= \int_E g(y)P(x,dy)\int_{\mathbb{R}^+} q(y)\exp(-q(y)u)du \\ &= \int_E g(y)P(x,dy) = Pg(x), \\ Q\psi(x) &= \int_E \varphi(y)P(x,dy)\int_{\mathbb{R}^+} uq(y)\exp(-q(y)u)du \\ &= \int_E \varphi(y)q^{-1}(y)P(x,dy) = P\varphi q^{-1}(x). \end{aligned}$$

In order to prove (37) we shall prove that if $g(x)$ is a solution of (35) then $g(x) = Qf(x)$. In fact, by (36)

$$Qf(x) = Q\psi(x) + Qg(x) = P\varphi q^{-1}(x) + Pg(x) = g(x).$$ ∎

Remark 6. A necessary condition for the existence of a solution of (35) is

$$\Pi\varphi q^{-1} = 0. \tag{38}$$

In fact, applying operator Π on both sides of (35) we have $\Pi g - \Pi P g = 0 = \Pi\varphi q^{-1}$.

Let us notice that the condition (38) is satisfied if the function φ is represented in the form

$$\varphi(x) = \varphi^*(x) - \alpha\Pi\varphi^* q^{-1}$$

where $\varphi^* : E \to \mathbb{R},\ \alpha$ is given by (23).

Lemma 3.4. *Assume that the following equation has a solution* g

$$(I-P)g(x) = P\varphi q^{-1}(x)$$

and that $\Pi\varphi^2 q^{-2} < \infty, \Pi g^2 < \infty$. *Furthermore, if the condition* (C_1) *of Theorem 3.2 are satisfied then*

$$T^{-1/2}\sum_{k=1}^{\nu(T)}\varphi(Z_k)\rho_k \xrightarrow{\mathcal{L}} N(0,\alpha\delta^2) \tag{39}$$

for any initial distribution, where α *is given by* (23) *and*

$$\delta^2 = 2\Pi(\varphi^2 q^{-2} + \varphi q^{-1} g).$$

Proof. By Lemma 3.3, we have the representation

$$\begin{aligned}\psi(Z_k,\rho_k) = \varphi(Z_k)\rho_k &= f(Z_k,\rho_k) - Qf(Z_k)\\ &= f(Z_k,\rho_k) - Qf(Z_{k-1}) + Qf(Z_{k-1}) - Qf(Z_k)\\ &= u_k + g(Z_{k-1}) - g(Z_k)\end{aligned}$$

where $\{u_k = f(Z_k,\rho_k)-Qf(Z_{k-1}),\ k\geq 1\}$ are martingale differences. Therefore

$$\begin{aligned}T^{-1/2}\sum_{k=1}^{\nu(T)}\varphi(Z_k)\rho_k &= T^{-1/2}\sum_{k=1}^{\nu(T)}u_k + T^{-1/2}\sum_{k=1}^{\nu(T)}(g(Z_{k-1}) - g(Z_k))\\ &= T^{-1/2}\sum_{k=1}^{\nu(T)}u_k + T^{-1/2}(g(Z_0) - g(Z_{\nu(T)})).\end{aligned} \tag{40}$$

Since $\Pi g^2 < \infty$, by the same argument as in the proof of Lemma 3.1, we can show that

$$T^{-1/2}(g(Z_0) - g(Z_{\nu(T)})) \xrightarrow{P} 0 \tag{41}$$

for any initial distribution.

Furthermore, we have by (36)

$$\Pi^* f^2 \leq 2\Pi^*(\psi^2 + g^2) = 2(\Pi\varphi^2 q^{-2} + \Pi g^2) < \infty,$$

hence by Corollary 3.1, (31) holds for any initial distibution.

Applying Theorem 3.2 for the imbedded chain $\{(Z_k,\rho_k), k\geq 1\}$ we obtain

$$T^{-1/2}\sum_{k=1}^{[\alpha T]}u_k \xrightarrow{\mathcal{L}} N(0,\alpha\delta^2) \tag{42}$$

with

$$\begin{aligned}\delta^2 &= \Pi^*(f^2 - (Qf)^2) = \Pi^*(f^2 - g^2)\\ &= \Pi^*(\psi^2 + 2\psi g) = 2\Pi(\varphi^2 q^{-2} + \varphi g q^{-1}).\end{aligned}$$

Finally, it follows from (40), (31), (41). (34), (42) that (39) holds for any initial distribution. ∎

Now we state and prove the main theorem as follows

Theorem 3.3. *Assume that the condition* (C_1) *of Theorem* 3.2 *and the following condition* (C_3) *are satisfied*

(C_3) (i) $\Pi\varphi^2 q^{-2} < \infty$ *and,* (ii) *The following equation has a solution* g

$$(I - P)g(x) = P\varphi q^{-1}(x)$$

with $\Pi g^2 < \infty$.

Then

$$T^{-1/2}\int_0^T \varphi(X(t))dt \xrightarrow{\mathcal{L}} N(0, \alpha\delta^2)$$

for any initial distribution, where

$$\delta^2 = 2\Pi(\varphi^2 q^{-2} + \varphi g q^{-1}).$$

Proof. The conclusion of Theorem 3.3 follows from Theorem 3.2 and Lemmas 3.1, 3.4. ■

4. Examples

Example 1. Assume that the jump Markov process $\{X(t), t \geq 0\}$ with the state space $E = \{1, 2, 3\}$ has the transition intensity matrix

$$Q = \begin{bmatrix} -1 & 0.5 & 0.5 \\ 0.4 & -1 & 0.6 \\ 0.8 & 0.2 & -1 \end{bmatrix}.$$

Then the Markov chain $\{Z_k, k \geq 1\}$ has the transition probability matrix

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.4 & 0 & 0.6 \\ 0.8 & 0.2 & 0 \end{bmatrix}.$$

It is easy to see that $\{Z_k, k \geq 1\}$ possesses the ergodic distribution as follows

$$\Pi = [\,0.38596 \quad 0.26316 \quad 0.35088\,],$$

whereas the sequence $\{\rho_k, k \geq 1\}$ is the sequence of independent, exponentially indentically distributed random variables with the parameter $q = 1$ (i.e., $q(x) = 1$ for all $x \in E$) and hence $\alpha = 1$.

Let us consider $\varphi^* = [1,\ 2,\ 4]^T$, i.e. $\varphi^*(1) = 1, \varphi^*(2) = 2, \varphi^*(3) = 4$. Then

$$\Pi\varphi^* = 2.3158,\ \varphi = \varphi^* - \Pi\varphi^* = [\,-1.3158 \quad -0.3158 \quad 1.6842\,]^T.$$

We shall prove that as $T \to \infty$

$$\frac{1}{\sqrt{T}}\int_0^T (\varphi^*(X(t)) - 2.3158)dt \xrightarrow{\mathcal{L}} N(0, \sigma^2). \tag{43}$$

For this purpose, we try to find a function $g = [g_1, g_2, g_3]^T$ satisfying the following equation

$$(I-P)g = P\varphi q^{-1} = P\varphi$$

or in detail

$$\begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.4 & 1 & -0.6 \\ -0.8 & -0.2 & 1 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix} = \begin{bmatrix} 0.6842 \\ 0.4842 \\ -1.1158 \end{bmatrix}.$$

The above algebraic equation has the following solution

$$g = [\,1.15788 \quad 0.94735 \quad 0\,].$$

Since E has a finite number of elements, Πg^2 and $\Pi\varphi^2$ are finite. By Theorem 3.3 we have (43) with

$$\sigma^2 = \delta^2 = 2\Pi(\varphi^2 + \varphi g) = 2.046.$$

Example 2. Let us consider the integral functional of the jump Markov process with the state space $E = \{1, 2, \cdots\}$ defined by:

$$T_i = \int_0^T \mathbf{1}_{\{X(t)=i\}}, \quad i \in E.$$

This integral is the total time length during which the process visits the state i. Assume that this process satisfies the condition (C_1).

For each state i, put $\varphi^*(x) = \mathbf{1}_{\{x=i\}}$ then $\alpha\Pi\varphi^* q^{-1} = \alpha\pi_i q_i^{-1}$. Let us consider $\varphi(x) = \varphi^*(x) - \alpha\pi_i q_i^{-1}$.

Suppose that the equation

$$(I-P)g(x) = P\varphi q^{-1}(x) \tag{44}$$

has a solution g such that $\Pi g^2 < \infty$. Then by Theorem 3.3

$$\frac{1}{\sqrt{T}}\int_0^T \varphi(X(t))dt = \frac{1}{\sqrt{T}}\int_0^T (\mathbf{1}_{\{X(t)=i\}} - \alpha\pi_i q_i^{-1})dt \xrightarrow{\mathcal{L}} N(0, \alpha\delta^2)$$

where $\delta^2 = 2\Pi(\varphi^2 q^{-2} + \varphi q^{-1} g)$.

In particular, for the case where

$$E = \{1,2\},\ Q = \begin{bmatrix} -q_1 & q_1 \\ q_2 & -q_2 \end{bmatrix},\ P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\ q_1, q_2 > 0,$$

we have the stationary distribution $\Pi = (1/2, 1/2)$ and

$$\alpha = (\Pi q^{-1})^{-1} = \left(\frac{1}{2}\left(\frac{1}{q_1} + \frac{1}{q_2}\right)\right)^{-1} = \frac{2q_1q_2}{q_1+q_2}.$$

Put $\varphi^*(x) = \mathbf{1}_{\{x=1\}}$, then

$$\alpha\Pi\varphi^* q^{-1} = \alpha\pi_1 q_1^{-1} = \frac{q_2}{q_1+q_2},\ \varphi(x) = \mathbf{1}_{\{x=1\}} - \frac{q_2}{q_1+q_2},$$

and

$$\frac{1}{\sqrt{T}}\int_0^T \left(\mathbf{1}_{\{X(t)=1\}} - \frac{q_2}{q_1+q_2}\right)dt \xrightarrow{\mathcal{L}} N(0, \alpha\delta^2) \tag{45}$$

for any initial distribution. In order to find δ^2 we have to solve the equation (44) for $i = 1$, i.e.

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} P\varphi q^{-1}(1) \\ P\varphi q^{-1}(2) \end{bmatrix} \tag{46}$$

with notice that

$$\begin{bmatrix} P\varphi q^{-1}(1) \\ P\varphi q^{-1}(2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} (1 - q_2/(q_1+q_2))q_1^{-1} \\ -(q_2/(q_1+q_2))q_2^{-1} \end{bmatrix} = \begin{bmatrix} -1/(q_1+q_2) \\ 1/(q_1+q_2) \end{bmatrix}.$$

(46) has a solution $g_1 = -1/(q_1+q_2),\ g_2 = 0$. Hence, by Theorem 3.3, we obtain (45) with

$$\delta^2 = 2\Pi(\varphi^2 q^{-2} + \varphi q^{-1} g) = \frac{1}{(q_1+q_2)^2}.$$

We obtain from (45)

$$\sqrt{T}\Big(\frac{T_1}{T} - \frac{q_2}{(q_1+q_2)}\Big) \xrightarrow{\mathcal{L}} N\Big(0, \frac{2q_1q_2}{(q_1+q_2)^3}\Big).$$

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