Vietnam Journal of Mathematics **33**:4 (2005) 463–468

Vietnam Journal
of
MATHEMATICS

On Efficient Sets in $\mathbb{R}^2$

Hoang Xuan Phu

Institute of Mathematics, 18 Hoang Quoc Viet Road, 10307 Hanoi, Vietnam

Received May 4, 2004

Abstract. Let $A \subset \mathbb{R}^2$ be a nonempty closed convex subset and $C \subset \mathbb{R}^2$ be a nonempty nontrivial convex cone. Due to Luc (1985 and 1989), if A is compact and if the closure $\overline{C}$ is pointed, then the efficient set $E(A|C)$ of A w.r.t. C is homeomorphic to a nonempty closed interval of $\mathbb{R}^1$, whose proof was completed by Huy, Phuong, and Yen (2002). Huy (2003) extended this result by replacing the compactness of A with the compactness of $A \cap (\{a\} - \overline{C})$, for all $a \in A$. In this paper, we show the same conclusion in a much shorter way and under essentially weaker assumption, namely C is pointed and there exists an $a \in A$ such that $A \cap (\{a\} - C)$ is bounded. Moreover, the weakly efficient set $E^w(A|C)$ w.r.t. any convex cone C having nonempty interior is homeomorphic to a closed interval in $\mathbb{R}^1$ even if C is not pointed.

1. Some Theorems on Efficient Sets

Let A be a nonempty subset of a real topological vector space X, which is partially ordered by a convex cone $C \subset X$. $a \in A$ is said to be an *efficient point* of A w.r.t. C provided

$$\text{there exists no } \ b \in A \setminus \{a\} \text{ such that } a - b \in C. \tag{1}$$

This definition was often applied (e.g., in [1] and [2]) and it is used now throughout this paper. Another definition of efficient point a claims that

$$\text{if } a - b \in C \text{ for some } b \in A \text{ then } b - a \in C, \tag{2}$$

as done, e.g., in [4, p. 39]. In fact, (1) and (2) are equivalent if C is a *pointed cone* defined by $C \cap (-C) = \{0\}$ (see [4, p. 40]). But they are quite different if the ordering cone C is not pointed. So one must be careful when applying results in [4] for efficient points in the sense of (1) (as done in [2, p. 291]). For instance, Corollary 3.11 [4, p. 50] says that if X is finite-dimensional then each compact nonempty set A possesses efficient points w.r.t. a convex cone C in the sense of (2), which is no more true in the sense of (1). For example, let

$$A = \{(0, y) \in \mathbb{R}^2 : 0 \leqslant y \leqslant 1\}, \quad C = \{(x, y) \in \mathbb{R}^2 : x \geq 0\},$$

then each point of A is efficient w.r.t. C in the sense of (2), while A has no efficient point w.r.t. C in the sense of (1).

The set of efficient points of A w.r.t. C is called *efficient set* and denoted by $E(A|C)$.

If the interior of C is nonempty and $C' = \operatorname{int} C \cup \{0\}$, then each $a \in E(A|C')$ is named a weakly efficient point of A w.r.t. C, and the set of such points is called *weakly efficient set* and denoted by $E^w(A|C)$, i.e., $E^w(A|C) = E(A|C')$ (see [1, 2, 4]).

The purpose of [2] is to complete the proof of the following result, which was stated formerly in [3] and [4, p. 144].

Theorem 1. [2, p. 291] *Let $A \subset \mathbb{R}^2$ be a nonempty compact convex subset and $C \subset \mathbb{R}^2$ be a nonempty convex cone whose closure $\overline{C}$ is a pointed cone. Then the set $E(A|C)$ is homeomorphic to a 0-dimensional or an 1-dimensional simplex.*

Actually, the assertion of Theorem 1 fails if C is a trivial cone, because $C = \{0\} \subset \mathbb{R}^2$ is a pointed convex cone but $E(A|C) = A$, i.e., $\dim E(A|C) = \dim A = 2$ is possible.

Using the preceding one, the next extended result was proven in [1], where $A \subset \mathbb{R}^2$ is not necessarily compact but *has compact sections* w.r.t. $\overline{C}$, i.e.,

$$A \cap (\{a\} - \overline{C}) \text{ is compact for all } a \in A. \tag{3}$$

Theorem 2. [1, p. 47] *Suppose that $C \subset \mathbb{R}^2$ is a convex cone whose interior is nonempty and whose closure is pointed, and $A \subset \mathbb{R}^2$ is nonempty convex set having compact sections w.r.t. $\overline{C}$. Then the set $E(A|C)$ (resp., $E^w(A|C)$) is homeomorphic to a 0-dimensional or an 1-dimensional polyhedral convex set. This amounts to saying that $E(A|C)$ (resp., $E^w(A|C)$) is homeomorphic to one of the following four subsets of the real line: $\{0\}$, $[0, 1]$, $[0, +\infty)$, $(-\infty, +\infty)$.*

By the way, it is unnecessarily complicated to use 0-dimensional and 1-dimensional simplex or polyhedral convex set for describing intervals in $\mathbb{R}$. Therefore, we will not follow to use these notions.

The pointedness of $\overline{C}$ as required in Theorems 1–2 is often assumed in the literature, sometimes by saying that C has a convex bounded base (see [4, p. 4]). This assumption is, from some points of view, a very strict restriction. Let us illustrate by considering the vector optimization problem

$$\text{minimize } F(x) \text{ subject to } x \in A,$$

where $F : A \subset \mathbb{R}^n \to \mathbb{R}^m$. If we use the concept of efficient points to investigate image set $F(A)$, then $\overline{C} = \mathbb{R}^m_+$, which is pointed, but there is no need to study other forms of cone C. If F is a linear mapping and we use this concept to A, i.e., to investigate the optimal solution set, then $\overline{C}$ is the intersection of

m halfspaces, which contains a line and is therefore not pointed, as long as $m < n$. For instance, consider for the special case $n = 2$ and $m = 1$ the linear optimization problem

$$\text{minimize } \xi_1 x_1 + \xi_2 x_2 \text{ subject to } (x_1, x_2) \in A \subset \mathbb{R}^2,$$

where $(\xi_1, \xi_2) \neq (0, 0)$. Then, corresponding to (1) or (2), the ordering cone C must be chosen by

$$C = \{(c_1, c_2) \in \mathbb{R}^2 : \xi_1 c_1 + \xi_2 c_2 > 0\} \cup \{(0, 0)\} \tag{4}$$

or

$$C = \{(c_1, c_2) \in \mathbb{R}^2 : \xi_1 c_1 + \xi_2 c_2 \geq 0\},$$

respectively. In both cases, the closure $\overline{C} = \{(c_1, c_2) \in \mathbb{R}^2 : \xi_1 c_1 + \xi_2 c_2 \geq 0\}$ is not pointed at all.

The main result of this paper is the following.

Theorem 3. *Assume that*
(A_1) $A \subset \mathbb{R}^2$ *is a nonempty closed convex subset,*
(A_2) $C \subset \mathbb{R}^2$ *is a nonempty nontrivial pointed convex cone,*
(A_3) *there exists an* $a \in A$ *such that* $A \cap (\{a\} - C)$ *is bounded.*
Then the efficient set $E(A|C)$ *is homeomorphic to a nonempty closed interval in* $\mathbb{R}^1$*, which is bounded if* A *is bounded.*

The replacement of the pointedness of $\overline{C}$ (as required in Theorems 1–2) by the pointedness of C (as assumed in Theorem 3) is an essential extension. From the application point of view, it enables to cover some standard problems, which is impossible when requiring the pointedness of $\overline{C}$, as explained above. Obviously, the cone C given in (4) is pointed, but its closure not. We will come back to the technical point of view of this extension at the end of this paper.

If C is closed, then assumption (A_3) is equivalent to condition (3) required in Theorem 2, because

$$\begin{array}{l}\text{if a closed convex set } S \subset \mathbb{R}^n \text{ contains}\\ \text{some halfline with direction } d\text{, then it contains}\\ \text{every halfline with direction } d \text{ whose initial point is in } S\end{array} \tag{5}$$

(see Proposition 2.5.1. [5, p. 75]). If C is not closed, then (A_3) is weaker than (3). To see it, just choose

$$A = \{(x, y) \in \mathbb{R}^2 : x \geq 0,\ y \leqslant 0\},$$
$$C = \{(x, y) \in \mathbb{R}^2 : x > 0,\ y \geq 0\} \cup \{(0, 0)\}.$$

Then $A \cap (\{a\} - C) = \{a\}$ is bounded for all $a \in \{(x, y) \in \mathbb{R}^2 : x = 0,\ y \leqslant 0\} \subset A$, while $A \cap (\{a\} - \overline{C})$ is unbounded for all $a \in A$. An important case for nonclosed C is given by (4).

Note that a nonempty closed interval in $\mathbb{R}^1$ mentioned in Theorem 3 is homeomorphic to one of the four subsets mentioned at the end of Theorem 2.

If the nontrivial convex cone $C \subset \mathbb{R}^2$ is not pointed, i.e., it is a closed halfspace, then the efficient set $E(A|C)$ (in the sense of (1)) is either empty or a singleton.

Since $E^w(A|C) = E(A|C')$ and $C' = \operatorname{int} C \cup \{0\}$ is pointed for every nontrivial convex C with $\operatorname{int} C \neq \emptyset$, applying Theorem 3 to C' yields immediately the following.

Corollary 4. *Suppose* (A_1), (A_3), *and*
(A_2') $C \subset \mathbb{R}^2$ *is a nontrivial convex cone whose interior is nonempty.*
Then the weakly efficient set $E^w(A|C)$ *is homeomorphic to a nonempty closed interval in* $\mathbb{R}^1$, *which is bounded if* A *is bounded.*

This result on the weakly efficient set $E^w(A|C)$ is stronger than the one in Theorem 2, because (3) is now replaced by (A_3), and C is no more claimed to be pointed.

2. Proof of Theorem 3

Step 1: Existence of efficient points. By (A_3), there exists an $a \in A$ such that $A \cap (\{a\} - C)$ is bounded, and following, $B = \overline{A \cap (\{a\} - C)} \subset \mathbb{R}^2$ is nonempty and compact. Therefore, Corollary 3.11 [4, p. 50] implies that there exists at least an efficient point of B w.r.t. C in the sense of (2), which is equivalent to (1) for pointed C. It remains to show that $E(B|C) \subset E(A|C)$. Assume $b \in E(B|C)$. Then $B \subset A \cap \overline{\{a\} - C}$ implies $b \in A$ and $b - a \in \overline{\{a\} - C} - \{a\} = -\overline{C}$. For C is a pointed convex cone in $\mathbb{R}^2$, it is easy to deduce from the latter inclusion that $b - a - c \in -C$, or equivalently,

$$b - c \in \{a\} - C, \text{ for all } c \in C \setminus \{0\}. \tag{6}$$

Hence, if $b \notin E(A|C)$ then there exists $c \in C \setminus \{0\}$ such that $b - c \in A$, which yields along with (6) that $b - c \in A \cap (\{a\} - C) \subset B$, a contradiction to $b \in E(B|C)$. Therefore, $b \in E(A|C)$ must follow from $b \in E(B|C)$, i.e., $\emptyset \neq E(B|C) \subset E(A|C)$.

Step 2: Homeomorphism F. By (A_3) and by rotation if necessary, we can assume that $(0,1) \in C$, $a \in A$, and $A \cap \{a - (0,\mu) : \mu \geq 0\}$ is bounded. Let

$$f(x) := \inf\{y \in \mathbb{R} : (x,y) \in A\},$$

where $\inf \emptyset = +\infty$. Then (5) yields that $f(x) > -\infty$ for all $x \in \operatorname{dom} f := \{x \in \mathbb{R} : f(x) < +\infty\}$. Since A is closed, $(x, f(x)) \in A$ for all $x \in \operatorname{dom} f$. Moreover, both f and $\operatorname{dom} f$ are convex because A is convex.

Let $x^- := \inf(\operatorname{dom} f)$ and $x^+ := \sup(\operatorname{dom} f)$. If $x^- < x^+$, then f is continuous in (x^-, x^+) (Theorem 5.5.1, [5, p. 224]). If, in addition, $x^- > -\infty$, then $f(x^-) = \lim_{x \downarrow x^-} f(x)$ follows from the closedness and the convexity of A, i.e., f is continuous on the right at x^-. Similarly, if $x^- < x^+ < +\infty$, then f is continuous on the left at x^+. Hence, f is continuous on the whole effective domain $\operatorname{dom} f$.

Let $Q := \{(x, f(x)) : x \in \operatorname{dom} f\}$. Then the mapping $F : x \mapsto (x, f(x))$ from $\operatorname{dom} f$ onto Q is one-to-one, continuous, and its reverse mapping F^{-1} :

$(x, f(x)) \mapsto x$ is obviously continuous, too. That means F is a homeomorphism.

Step 3: Convexity of $F^{-1}(E(A|C))$. Note that, by definition, $E(A|C) \subset Q$. Assume $x_0,\ x_1 \in F^{-1}(E(A|C))$, $x_0 < x_1$, and $\lambda \in (0,1)$. Let $x_\lambda := (1-\lambda)x_0 + \lambda x_1$. Consider an arbitrary $(c_x, c_y) \in C \setminus \{(0,\mu) : \mu \geq 0\}$ satisfying $|c_x| \leqslant \min\{x_\lambda - x_0,\, x_1 - x_\lambda\}$. Since C is pointed and $(0,1) \in C$, c_x must be nonzero. If $c_x < 0$, then the convexity of f implies

$$\frac{f(x_0 - c_x) - f(x_0)}{-c_x} \leqslant \frac{f(x_\lambda - c_x) - f(x_\lambda)}{-c_x}.$$

On the other hand, $(x_0, f(x_0)) \in E(A|C)$, $(0,-1) \in -C$, and the convexity of $-C$ yield $f(x_0) - c_y < f(x_0 - c_x)$, and therefore,

$$\frac{-c_y}{-c_x} = \frac{(f(x_0) - c_y) - f(x_0)}{-c_x} < \frac{f(x_0 - c_x) - f(x_0)}{-c_x}.$$

It follows from the preceding inequalities that

$$\frac{(f(x_\lambda) - c_y) - f(x_\lambda)}{-c_x} = \frac{-c_y}{-c_x} < \frac{f(x_\lambda - c_x) - f(x_\lambda)}{-c_x}.$$

Hence, $f(x_\lambda) - c_y < f(x_\lambda - c_x)$, and consequently,

$$(x_\lambda, f(x_\lambda)) - (c_x, c_y) = (x_\lambda - c_x, f(x_\lambda) - c_y) \notin A.$$

By using $(x_1, f(x_1)) \in E(A|C)$, we can prove similarly that the latter property is also true for $c_x > 0$. Since $(x_\lambda, f(x_\lambda)) - (0,\mu) = (x_\lambda, f(x_\lambda) - \mu) \notin A$ for $\mu > 0$, A is convex and C is a cone, we obtain

$$(\{(x_\lambda, f(x_\lambda))\} + C) \cap A = \{(x_\lambda, f(x_\lambda))\},$$

which implies

$$(x_\lambda, f(x_\lambda)) \in E(A|C), \quad \text{i.e., } x_\lambda \in F^{-1}(E(A|C)).$$

Hence, $F^{-1}(E(A|C))$ is convex, i.e., it is an interval in $\mathbb{R}$.

Step 4: Closedness of $F^{-1}(E(A|C))$. Denote

$$z^- := \inf F^{-1}(E(A|C)) \quad \text{and} \quad z^+ := \sup F^{-1}(E(A|C)).$$

Since $E(A|C)$ is nonempty, $E(A|C) = \{z^-\}$ holds if $z^- = z^+$. Therefore, only case $z^- < z^+$ must be checked.

Assume the contrary that $z^- > -\infty$ but $z^- \notin F^{-1}(E(A|C))$, i.e.,

$$(z^-, f(z^-)) \notin E(A|C).$$

Then there exists $(c_x, c_y) \in C \setminus \{(0,0)\}$ such that $(z^-, f(z^-)) - (c_x, c_y) \in A$. By definition, $c_x = 0$ is impossible. $c_x > 0$ can be excluded by the same way as in Step 3 using the fact that $(z, f(z)) \in E(A|C)$ for $z \in (z^-, z^+)$. If $c_x < 0$, then $f(z^- - c_x) \leqslant f(z^-) - c_y$ follows from $(z^- - c_x, f(z^-) - c_y) \in A$. Since f is convex, $f(z^- - tc_x) \leqslant f(z^-) - tc_y$ for all $t \in [0,1]$. If there exists some $t' \in (0,1)$ with $f(z^- - t'c_x) < f(z^-) - t'c_y$, then, by the continuity of f,

$$f((z^- - t''c_x) - (t' - t'')c_x) = f(z^- - t'c_x) < f(z^- - t''c_x) - (t' - t'')c_y$$

holds for sufficiently small $t'' \in (0, t')$, which yields

$$(z^- - t''c_x, f(z^- - t''c_x)) - (t' - t'')(c_x, c_y) \in A,$$

a contradiction to $(z, f(z)) \in E(A|C)$ for $z \in (z^-, z^+)$. Hence, $f(z^- - tc_x) = f(z^-) - tc_y$ must be true for all $t \in [0, 1]$. But this implies

$$f((z^- - t''c_x) - (1 - t'')c_x) = f(z^- - c_x) = f(z^- - t''c_x) - (1 - t'')c_y,$$

i.e.,

$$(z^- - t''c_x, f(z^- - t''c_x)) - (1 - t'')(c_x, c_y) \in A,$$

for all $t'' \in (0, 1)$, also a contradiction to $(z, f(z)) \in E(A|C)$ for $z \in (z^-, z^+)$. Consequently, if $z^- > -\infty$ then $z^- \in F^{-1}(E(A|C))$.

Similarly, it can be proven by the same way that $z^+ \in F^{-1}(E(A|C))$ if $z^+ < +\infty$.

We have shown that $F^{-1}(E(A|C))$ is a closed convex subset of $\mathbb{R}$, i.e., it is a closed interval. Obviously, if A is compact, so are $E(A|C)$ and $F^{-1}(E(A|C))$, i.e., $F^{-1}(E(A|C))$ is a bounded closed interval. The proof of Theorem 3 is complete. ∎

3. Concluding Remarks

In the proof of [1]–[4], only the special case $\overline{C} = \mathbb{R}^2_+$ was investigated, and other cases should be led to this one by nondegenerate linear transformation. Such a technique works only if $\overline{C}$ is pointed, and it fails for pointed cones whose closure is not pointed (e.g., the cone given in (4)). Hence, from the technical point of view, the replacement of the pointedness of $\overline{C}$ by the pointedness of C in Theorem 3 is a substantial extension, too.

Although the assumption of Theorem 3 is essentially weaker than the one of Theorems 1–2, we receive the same conclusion, and our proof is much shorter than the one of Theorems 1–2 given in [1]–[2].

References

1. N. Q. Huy, Topology of the efficient sets of convex sets in $\mathbb{R}^2$, *Vietnam J. Math.* **31** (2003) 45–55.
2. N. Q. Huy, T. D. Phuong, and N. D. Yen, On the contractibility of the efficient and weakly efficient sets in $\mathbb{R}^2$. In P. Daniele, F. Giannessi, and A. Maugeri (Eds.), *Equilibrium Problems and Variational Models*, Kluwer, Netherlands, 2002, 289–303.
3. D. T. Luc, Structure of the efficient point set, *Proc. Amer. Math. Soc.* **95** (1985) 433–440.
4. D. T. Luc, Theory of Vector Optimization, *Lecture Notes in Economics Mathematical Systems* **319**, Springer-Verlag, Berlin, 1989.
5. R. Webster, *Convexity*, Oxford University Press, Oxford–New York–Tokyo, 1994.