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Stress Field in an Elastic Strip in Frictionless Contact with a Rigid Stamp*

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Dedicated to Prof. Nguyen Van Dao on the occasion of his seventieth birthday

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Abstract. The author considers an elastic strip of thickness h represented in Cartesian coordinates by

$$-\infty < x < \infty, \quad 0 \leqslant y \leqslant h$$

The strip is clamped at the bottom $y = 0$, the upper side is in contact with a rigid stamp and is assumed to be free from shear and the normal stress $\sigma_y = 0$ on $y = h$ away from the bottom of the stamp. The purpose of this note is to determine the stress field in the elastic strip given the normal displacement $v(x)$ and the lateral displacement $u(x)$ under the stamp.

Consider an elastic strip of thickness h represented in Cartesian coordinates by

$$-\infty < x < \infty, \quad 0 \leqslant y \leqslant h \tag{1}$$

The strip is clamped at the bottom $y = 0$. The upper side of the strip is in contact with a rigid stamp and is assumed to be free from shear, and furthermore, away from the bottom of the stamp, the normal stress σ_y vanishes, i.e.,

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$$\sigma_y = 0 \quad \text{on} \quad y = h, \quad x \in R \backslash D, \tag{2}$$

where D is the breadth of the bottom of the stamp. The normal component of the displacement of the strip under the stamp is given

$$v = g \quad \text{on} \quad y = h, \quad x \in D. \tag{3}$$

The paper consists of two parts. In the first part, Part A, we compute the normal stress σ_y under the stamp. The second part, Part B, is devoted to a determination of the stress field in a rectangle of the elastic strip situated under the bottom of the stamp from the data given in Part A and a specification of the displacement $u = u(x)$ under the stamp.

Part A. The Contact Problem

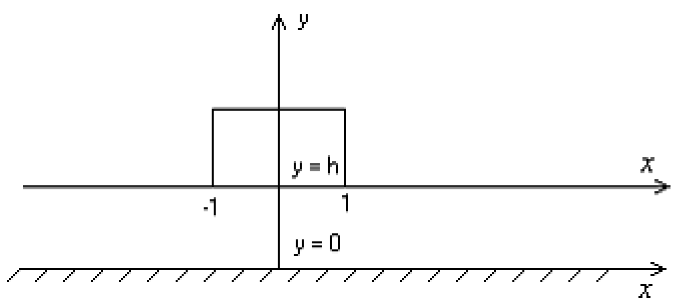

Fig.1

We propose to compute the normal stress $\sigma_y(x) = f(x), x \in D \;\; y = h$. As shown in [1,2] the problem reduces to solving the following integral equation in f

$$\int_D k(x-y) f(y) dy = g(x), \quad x \in D \tag{4}$$

where

$$k(t) = \int_0^\infty 2K(u)\cos(ut)du \tag{5}$$

with

$$K(u) = \frac{2c(2\mu sh(2hu) - 4h(u))}{2u(2\mu ch(2hu) + 1 + \mu^2 + 4h^2u^2)} \tag{6}$$

$\mu = 3 - 4\nu, \nu$ being Poisson's ratio, c: a positive constant.

To be specific, we assume that D is a finite interval, in fact, the interval $[-1, 1]$, although it could be a finite union of intervals.

With $D = [-1, 1]$ as assumed above, Eq. (4) becomes

$$\int_{-1}^{1} k(x-y) f(y) dy = g(x), \quad x \in [-1, 1] \tag{7}$$

where $K(u)$ is given by (6).

In order to give a meaning to Eq. (4), we must decide on a function space for f and g. Physically f is a surface stress under the stamp, and therefore, we can allow it to have a singularity at the sharp edges of the stamp, i.e., at $x = \pm 1, y = h$. It is usually specified that the stresses under the stamp go to infinity no faster than $(1 - x^2)^{-1/2}$ as x approaches ± 1. It is therefore natural and permissible to consider Eq. (4) in $L_p(D), 1 < p < 2$, as is done, e.g. in [1]. We shall consider, instead, the space H_1 consisting of real-valued functions u on D such that

$$\int_{-1}^{1} u^2(x) d\sigma(x) < \infty \tag{8}$$

where

$$d\sigma(x) = dx/r(x)^{2-p}, \quad r(x) = (1 - x^2)^{1/2}. \tag{9}$$

Then H^1 is a Hilbert space with the usual inner product

$$(u, v) = \int_{-1}^{1} u(x)v(x) d\sigma(x) \tag{10}$$

Let E_1 be the linear space consisting of the functions f formed from u in H_1 by the following rule

$$f(x) = u(x)r(x)^{p-2}. \tag{11}$$

Then E_1 is a subset of L_p and furthermore

$$\|f\|_p \leqslant \|u\| \left(\int_{-1}^{1} r(x)^{-p} dx \right)^{(2-p)/2p} \tag{12}$$

where f is as given by (11), $\|.\|_p$ is the L_p-norm and $\|.\|$ is the norm in H. Inequality (12) implies that if $u_n \to u$ in H, then $f_n \to f$ in L_p. It is noted that E_1 contains functions of the form

$$f(x) = \varphi(x).r(x)^{-1} \tag{13}$$

where $\varphi(x)$ is a bounded function.

Now we define the following operator on H_1

$$Av(x) = \int_{-1}^{1} k(x - y)v(y) d\sigma(y). \tag{14}$$

Then A is a bounded linear operator on H_1, which is symmetric and strictly positive, i.e., $(Au, u) > 0$ for each $u \neq 0$. It can be proved that

$$\|Au\| \leqslant \alpha \|K\|_{6/5} \|u\| \tag{15}$$

where $\|.\|$ is the norm in H_1, $\|.\|_{6/5}$ is the norm in $L_{6/5}(R)$ and

$$\alpha = (2\pi)^{1/6} \left(\int_{-1}^{1} r(y)^{3(p-2)/2} dy \right)^{1/3} . \int_{-1}^{1} d\sigma(x)^{1/2}.$$

Details are given in [1].

Note that for each u in H_1, Au is a continuous function on [-1,1], and therefore the range of A is a proper subspace of H_1. Hence A^{-1} is not continuous on the range of H_1. This means that the solution of

$$Au = g \tag{16}$$

whenever, it exists, does not depend continuously on g, and thus the problem is ill-posed. The following proposition gives a regularized solution.

Proposition A. *Let J be a bounded linear, symmetric and strictly positive operator on a real Hilbert space H. Then for each $\beta > 0, (\beta I + J)^{-1}$, I being the identity operator, exists as a bounded linear operator on H and furthermore*

$$\lim_{\beta \to 0} (\beta I + J)^{-1} Jx = x, \quad \forall x \in H \tag{17}$$

The foregoing (known) result is a variant of Theorem 8.1 of Chap. 4 of [3]. A proof is given in [2].

Let us return to Eq. (16) and consider the equation

$$\beta u_\beta + Au_\beta = g, \quad \beta > 0. \tag{18}$$

For a construction of u_β, we follow a trick due to Zarantonello [4] and rewrite (18) as

$$u_\beta = \frac{\lambda(\delta I - A)u_\beta}{1 + \lambda\delta} + \frac{\lambda g}{1 + \lambda\delta} \tag{19}$$
$$\lambda = \beta^{-1}$$

i.e.,

$$\beta u_\beta + Au_\beta = g. \tag{20}$$

In (19), δ is any number $\geq \|A\|$, in fact, using (15), we can take $\delta \geq \alpha \|K\|_{6/5}$. Since A is strictly positive and symmetric, we can take

$$\lambda \|\delta I - A\| \leqslant \lambda\delta. \tag{21}$$

Hence the right hand side of (19) defines a contraction of coefficient

$$\lambda\delta/(1 + \lambda\delta) < 1. \tag{22}$$

Thus, (19) can be solved by successive approximation.

Part B. The Stress Field Under the Bottom of the Stamp

In this Part B, we propose to determine the stress field in a rectangle of the elastic strip from the results of Part A plus a specification of displacement $u = u(x)$ for $x \in [-a, a] \subset (-1, 1)$ under the stamp. (cf. Fig. 2)

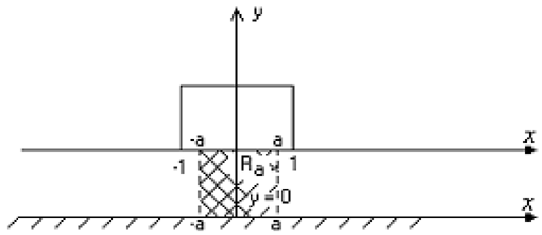

Fig. 2

Considering the plane stress case, we have

$$\frac{\partial u}{\partial x} = \varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \quad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$
$$\gamma_{xy} = \frac{1+\nu}{E}\sigma_{xy} = 0 \text{ on } (-1,1). \tag{23}$$

We restrict ourselves to a subinterval $[-a, a]$ of $(-1, 1)$ away from the ends in order to avoid stress singularities so that now we deal with a σ_y with finite values and furthermore the σ_x computed from the given displacement $u = u(x)$ has finite values (note that a is any number in $(-1, 1)$).

Considering the plane stress case, we have

$$\frac{\partial u}{\partial x} = \varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$
$$\varepsilon_y = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$
$$\gamma_{xy} = \frac{1+\nu}{E}\tau_{xy} = 0 \quad \text{on } (-1,1). \tag{24}$$

Then

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{25}$$
$$\frac{\partial \varepsilon_x}{\partial y} = -\frac{\partial^2 v}{\partial x^2}. \tag{26}$$

From (26)

$$\frac{\partial \sigma_x}{\partial y} - \nu\frac{\partial \sigma_y}{\partial y} = -E\frac{\partial^2 v}{\partial x^2} \tag{27}$$

for $-a < x < a$.

Assuming plane stress, we have from the equations of equilibrium (without body forces)

$$\frac{\partial \sigma_y}{\partial y} = -\frac{\partial \tau_{xy}}{\partial x} = 0. \tag{28}$$

From (27) and (28), we have

$$\frac{\partial \sigma_x}{\partial y} = -E\frac{\partial^2 v}{\partial x^2}. \tag{29}$$

Summarizing, we have from (28) and (29) that

$$\frac{\partial \sigma_x}{\partial y} + \frac{\partial \sigma_y}{\partial y} = -E\frac{\partial^2 v}{\partial x^2} \tag{30}$$

$\forall x \in [-a, a], \quad y = h.$

Now, it follows from the equations of compatibility and the equilibrium equations that

$$\Delta(\sigma_x + \sigma_y) = 0. \tag{31}$$

Thus $\sigma_x + \sigma_y$ is a harmonic function in the rectangle R_a with known Cauchy data, more precisely,

$$\Delta(\sigma_x + \sigma_y) = 0 \text{ in } R_a \tag{32}$$

and

$$\frac{\partial}{\partial y}(\sigma_x, \sigma_y) = \Big(-E\frac{\partial^2 v}{\partial x^2}, 0\Big) \quad \forall x \in [-a, a] \;\; y = h. \tag{33}$$

We introduce the Airy stress function ϕ

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2} \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2} \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}. \tag{34}$$

We have

$$\Delta \phi = \sigma_x + \sigma_y \tag{35}$$

where $\sigma_x + \sigma_y$ was constructed as the solution of a Cauchy problem for the Laplace equation in R_a.

From (34), we have

$$\phi = \frac{1}{4\pi} \iint\limits_{R_a} f(\xi, \eta) \ln[(x-\xi)^2 + (y-\eta)^2] d\xi d\eta$$

where

$$f(\xi, \eta) = (\sigma_x + \sigma_y)(\xi, \eta).$$

Then for any (x, y) in the rectangle R_a, the stress field is given by the formulas

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}.$$

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