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Characteristic Classes and Singular Varieties

Jean-Paul Brasselet

Directeur de recherche CNRS
Institut de Mathématiques de Luminy - Marseille - France

Abstract. In this paper, we give a survey on the definitions of characteristic classes for singular varieties.

In the smooth case, characteristic classes measure the obstruction to the construction of linearly independent sections of the tangent bundle to the manifold. In the singular case, there is no more tangent bundle. Different authors proposed different definitions of characteristic classes, in the complex analytic (or algebraic) case.

In fact, each of them corresponds to a particular substitute for the tangent bundle.

That is the case for the Schwartz classes (identical to the MacPherson's ones), for the Mather classes (one of the ingredients for the MacPherson's definition) and the Fulton's ones (at least in the case of hypersurfaces or ICIS). One shown that under suitable hypothesis, all known characteristic classes contain a cycle defined by the same formula, issued from the Schwartz's definition by obstruction theory.

1. Introduction

The Euler-Poincaré characteristic has been the first characteristic class to be introduced. For a triangulated (possibly singular) compact variety X without boundary, it has been defined, as

$$\chi(X) = \sum (-1)^i n_i \,,$$

where n_i is the number of i-dimensional simplices of the triangulation of X. It is also equal to $\sum(-1)^i b_i$ where b_i is the i-th Betti number, rank of $H_i(X)$. The Poincaré-Hopf theorem says that, if M is a (compact) manifold and v a continuous vector field with a finite number of isolated singularities a_k with indices $I(v, a_k)$, then

$$\chi(M) = \sum I(v, a_k).$$

This means that the Euler-Poincaré characteristic is a measure of the obstruction to the construction of a non-zero vector field tangent to M.

During several years, the attractiveness of the axiomatic properties of Chern classes caused the viewpoint of obstruction theory to be somewhat forgotten. It is interesting to see that this viewpoint came back on the scene with the question of defining characteristic classes for singular varieties.

There are various definitions of characteristic classes for singular varieties. In the real case, there is a combinatorial definition, which simplifies the problem. In the complex case, the situation is more complicated (and certainly more interesting!), due to the fact that there is no combinatorial definition of Chern classes. Thinking of the obstruction theory point of view, one has to find a substitute to the tangent bundle; in fact there are various candidates to substitute the tangent bundle and to each of them corresponds a different definition of Chern class for singular varieties.

If X is a singular complex analytic variety, equipped with a Whitney stratification and embedded in a smooth complex analytic manifold M one can consider the union of tangent bundles to the strata, that is a subspace $\mathcal{T}$ of the tangent bundle to M. The space $\mathcal{T}$ is not a bundle but it generalizes the notion of tangent bundle in the following sense: A section of $\mathcal{T}$ over X is a section v of $TM|_X$ such that in each point $x \in X$, then $v(x)$ belongs to the tangent space of the stratum containing x. Such a section is called a stratified vector field over X. To consider $\mathcal{T}$ as the substitute to the tangent bundle of X and to use obstruction theory is the Schwartz point of view (1965, [9]), in the case of analytic complex varieties. Another possibility is to consider the space of all possible limits of tangent vector spaces $T_{x_i}(X_{\mathrm{reg}})$ where x_i is a sequence of points in the regular part X_{reg} of X converging in $x \in X$. That point of view leads to the notion of Mather classes, which are an ingredient in the MacPherson definition in the case of algebraic complex varieties (1974, [8]). The other main ingredient for these classes is the notion of Euler local obstruction. Finally, when there exists a normal bundle N to X in M, for example in the case of local complete intersections, one can consider the virtual bundle $TM|_X \setminus N$ as a substitute to the tangent bundle of X. That point of view is the one of Fulton (1980, [7]).

There are many relations between the classes obtained by the previous constructions. First of all, the Schwartz and MacPherson classes coincide, via Alexander duality (1979, [3]).

The relation between Mather classes on the one side and Schwartz-MacPherson classes on the other side follows from the MacPherson's definition itself: His construction uses Mather classes, taking into account the local complexity of the singular locus along Whitney strata. This is the role of the local Euler obstruction.

A natural question arised to compare the Schwartz-MacPherson and the Fulton-Johnson classes. A result of Suwa [13] shows that in the case of isolated singularities, the difference of these classes is given by the sum of the Milnor numbers in the singular points. It was natural to call Milnor classes the difference arising in the general case. This difference has been described by several authors by different methods (P. Aluffi, J.P. Brasselet-D. Lehmann-J. Seade-T. Suwa, A.

Parusiński-P. Pragacz and S. Yokura).

In this paper, one explicits the fact that all known characteristic classes for singular varieties contain (under suitable hypothesis) a cycle defined by the same formula issued from obstruction theory:

$$\boxed{\sum_{\substack{\sigma\subset X\\ \dim\sigma=2(r-1)}} \alpha(\sigma) I(v^{(r)},\hat{\sigma})\ \sigma}$$

for a suitable constructible function α (depending on the characteristic class) and with notations defined in the following.

2. Review of Obstruction Theory

Let us recall the idea of constructing characteristic classes by obstruction theory, following Steenrod [12, Part III].

The Poincaré-Hopf Theorem says that $\chi(M)$ is a measure of the obstruction for the construction of a vector field tangent to the manifold M. In the same way, the objective of the obstruction theory is to define invariant objects which evaluate the measure of the obstruction to the construction of linearly independent sections of vector bundles. In a more precise way, the objective is to answer to questions like:

Let E be a vector bundle of rank n on a variety X and fix r such that $1 \leqslant r \leqslant n$, is it possible to construct r sections of E, linearly independent everywhere?

It is obviously possible to define such sections on the 0-skeleton of a triangulation of X. So, the question becomes the following:

Performing the construction of r independent sections by increasing dimension of simplices of a triangulation of X, up to what dimension can we proceed? When arriving to the obstruction dimension, is it possible to evaluate this obstruction?

In a first step, we study the case of the (real) tangent bundle to a differentiable smooth manifold or the (complex) tangent bundle to an analytic complex manifold. We will denote by $\mathbb{K}$ the field $\mathbb{R}$ or $\mathbb{C}$, according to the situation.

Let M be a manifold of dimension m, over $\mathbb{K}$, endowed with an euclidean (or hermitian) metric. The tangent bundle to M, denoted by TM, is a vector bundle over $\mathbb{K}$ of rank n, whose fiber in a point x of M is the tangent vector space to M in x, denoted by $T_x(M)$ and is isomorphic to $\mathbb{K}^m$. The vector bundle TM is locally trivial, i.e. there is a covering of M by open subsets such that the restriction of TM to U is isomorphic to $U \times \mathbb{K}^m$.

The objective is to evaluate the obstruction to the construction of r sections of TM linearly independent (over $\mathbb{K}$) in each point, i.e. an r-frame.

Let us consider the fiber bundle $T^r(M)$, with basis M, associated to TM and whose fiber in the point x of M is the set of r-frames of $T_x(M)$. This bundle is no more a vector bundle. The "typical" fiber of $T^r(M)$ is the set of all r-frames of $\mathbb{K}^m$, i.e. the Stiefel manifold $V_{m,r}(\mathbb{K})$. To construct r linearly independent sections of TM over a subset A of M is equivalent to construct a section of $T^r(M)$ over A.

Let us consider a triangulation (K) of M sufficiently small so that every simplex σ is contained in an open subset U over which $T^r(M)$ is trivial.

We are interested by the following question:

Let us suppose that there is a section v^r of $T^r(M)$ on the boundary $\partial\sigma$ of the k-dimensional simplex σ. Is it possible to extend this section in the interior of σ ? Is the answer is no, what is the obstruction for such an extension?

The section v^r on the boundary of σ defines a map

$$\partial\sigma \xrightarrow{v^r} T^r(M)|_U \cong U \times V_{m,r}(\mathbb{K}) \xrightarrow{pr_2} V_{m,r}(\mathbb{K})$$

where pr_2 is the projection on the second factor. We obtain a map

$$S^{k-1} \cong \partial\sigma \stackrel{pr_2\circ v^r}{\longrightarrow} V_{m,r}(\mathbb{K})$$

which induces an element of $\pi_{k-1}(V_{m,r}(\mathbb{K}))$ denoted by $[\gamma(v^r,\sigma)]$. In order to answer to the previous question, we need to know the homotopy groups of $V_{m,r}(\mathbb{K})$.

If one has $[\gamma(v^r,\sigma)] = 0$, then, by classical homotopy theory, the map $S^{k-1} \to V_{m,r}(\mathbb{K})$ can be extended inside the ball B^k. In another words, the map $\partial\sigma \to V_{m,r}(\mathbb{K})$ can be extended inside σ.

$$\begin{array}{ccc} \partial\sigma \cong S^{k-1} & \longrightarrow & V_{m,r}(\mathbb{K}) \\ \downarrow & ? \nearrow & \\ \sigma \cong B^k & & \end{array}$$

In that case, there is no obstruction to the extension of the section v^r inside σ. This happens for exemple in the case $\pi_{k-1}(V_{m,r}(\mathbb{K})) = 0$.

The homotopy groups $\pi_{k-1}(V_{m,r}(\mathbb{K}))$ have been computed by Stiefel and by Whitney (see [11]) in the cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{C}$. One has the following result:

Let $V_{m,r}(\mathbb{R})$ be the Stiefel manifold of r-frames in $\mathbb{R}^m$, one has:

$$\pi_i(V_{m,r}(\mathbb{R})) = \begin{cases} 0 & \text{for } i < m-r \\ \mathbb{Z} & \text{for } i = m-r \text{ even or } i = m-1 \text{ if } r = 1 \\ \mathbb{Z}_2 & \text{for } i = m-r \text{ odd and } r > 1 \end{cases} \tag{2.1}$$

For the Stiefel manifold of r-frames in $\mathbb{C}^m$, one has:

$$\pi_i(V_{m,r}(\mathbb{C})) = \begin{cases} 0 & \text{for } i < 2m-2r+1 \\ \mathbb{Z} & \text{for } i = 2m-2r+1 \end{cases} \tag{2.2}$$

Let us denote $2p = 2(m-r+1)$. A generator of $\pi_{2p-1}(V_{m,r}(\mathbb{C}))$ can be described in the following way: let us choose a $(r-1)$-frame in $\mathbb{C}^m$. It defines a $(r-1)$-subspace of $\mathbb{C}^m$ whose complementary is a complex space $\mathbb{C}^p$. The unit sphere in $\mathbb{C}^p$ denoted by S^{2p-1} is oriented, with orientation induced by the natural one of $\mathbb{C}^p$. Let us consider, for every point of the sphere, a r-frame consisting of the vector w and the fixed $(r-1)$-frame, one obtains an element of

$V_{m,r}(\mathbb{C})$. The induced map from the oriented sphere S^{2p-1} to $V_{m,r}(\mathbb{C})$ defines a generator of $\pi_{2p-1}(V_{m,r}(\mathbb{C}))$.

3. Applications: Chern Classes

We recall briefly the principle of the construction of Chern classes by obstruction theory, according to [12, Part III]. In this section, M is a complex analytic manifold of (complex) dimension m.

The result of the previous section implies that one can construct an r-frame $v^{(r)}$, i.e. a section of $T^r M$, by induction on the dimension of cells of the given cell decomposition of M without singularity up to the $(2m-2r+1)$-skeleton and with isolated singularities on the $2p = 2(m-r+1)$-skeleton. For each $2p$-cell $d(\sigma)$, the index of the complex r-frame $v^{(r)}$ at its only singular point $\hat{\sigma} = d(\sigma) \cap \sigma$ in d is $I(v^{(r)}, \hat{\sigma}) = [v^{(r)}; \partial d] \in \mathbb{Z}$.

On can define a $2p$-cochain in $C^{2p}(K, \mathbb{Z})$, whose value on each $2p$-cell $d(\sigma)$ is $[v^{(r)}; \partial d]$. The cochain is in fact a cocycle and defines an element in $Z^{2p}(M; \mathbb{Z})$. One has:

Lemma 3.1. *Let us consider two r-fields defined by the previous construction, then the difference of the two corresponding cocycles is a coboundary.*

That justifies the following definition:

Definition 3.2. [5] *The p-th (cohomology) Chern class of M, $c^p(M) \in H^{2p}(M; \mathbb{Z})$ is the class of the obstruction cocycle.*

The Chern classes do not depend of the choices we made.

By Poincaré duality isomorphism, the image of $c^p(M)$ in $H_{2(r-1)}(M)$ is the $(r-1)$-st homology Chern class of M represented by the cycle

$$\boxed{\sum_{\dim \sigma = 2(r-1)} I(v^{(r)}, \hat{\sigma})\ \sigma}$$

In particular, the evaluation of $c^m(M)$ on the fundamental class $[M]$ of M yields the Euler-Poincaré characteristic $\chi(M)$.

4. Singular Case: Substitutes of the Tangent Bundle

In the case of a complex analytic singular variety X, there is no more tangent bundle TX. The different notions of Chern classes, in the singular setting, correspond to different notions of substitute to the tangent bundle. There are (at least) three ways to define such a substitute in the case of a singular variety X embedded in a manifold M:

1. consider the union $\mathcal{T}$ of tangent spaces to the strata of a stratification of X and consider the sections of TM whose images are in $\mathcal{T}$. This is the method used by Schwartz, showing that it is not possible to process obstruction theory, using any section, but that one has to use radial vector fields.

2. consider the set of all possible limits of tangent spaces to sequences of points in the regular part of X, that is the Nash transformation and the Nash bundle on it.
3. consider the virtual bundle. That is the method used by Fulton. If X is smooth, one has an exact sequence

$$0 \to TX \to TM|_X \to N_X M \to 0$$

where $N_X M$ is the normal bundle of X in M. In the case of a singular variety such that the normal bundle $N_X M$ exists (for instance hypersurfaces or local complete intersections), one can define the virtual bundle (in the Grothendieck group $KU(X)$) as

$$\tau_X = TM|_X - N_X M.$$

In order to define characteristic classes, it is necessary to know the local structure of the singular variety. That is given by the structure of stratified space and by suitable definition of triangulation on the variety.

In the case of singular spaces, there are no characteristic classes in cohomology. The different notions define characteristic classes in homology.

5. The Schwartz Classes

In the following, M will be a complex m-dimensional manifold equipped with an analytic stratification $\{V_i\}$: for every stratum V_i, the closure $\bar{V}_i$ and the boundary $\dot{V}_i = \bar{V}_i \setminus V_i$ are analytic sets, union of strata. We denote by $X \subset M$ a compact complex analytic variety stratified by $\{V_i\}$.

The first definition of Chern class for singular varieties was given in 1965 by Schwartz in two "Notes aux CRAS" [9].

Definition 5.1. *A stratified vector field v on X is a (continuous) section of the tangent bundle of M, $T(M)$, such that, for every $x \in X$, one has $v(x) \in T(V_{i(x)})$ where $V_{i(x)}$ is the stratum containing x.*

5.1. Triangulations and Cellular Decompositions

Let $X \subset M$ be a singular n-dimensional complex variety embedded in a complex m-dimensional manifold. Let us consider a Whitney stratification $\{V_i\}$ of M [13] such that X is a union of strata and denote by (K) a triangulation of M compatible with the stratification, i.e. each open simplex is contained in a stratum.

The first nice observation of Schwartz concerns the triangulations:

We denote by (K') a barycentric subdivision of (K) and (D) the associated dual cell decomposition. Each cell of (D) is transverse to the strata. This implies that if d is a cell of dimension $2p = 2(m-r+1)$ and V_i is a stratum of dimension $2s$, then $d \cap V_i$ is a cell such that

$$\dim(d \cap V_i) = 2(s - r + 1).$$

This means that if d is a cell whose dimension is the dimension of obstruction to the construction of an r-frame tangent to M, then $d \cap V_i$ is a cell whose dimension is exactly the dimension of obstruction to the construction of an r-frame tangent to the stratum V_i.

The second nice construction of Schwartz is the construction of radial vector fields that we explicit below.

Let us give an idea of the construction of the radial extension, defined by Schwartz. Given the vector field w on M_i, we can firstly define, using Whitney property (a), a "parallel" extension which is a stratified vector field admitting "disks" of singularities, transverse to V_i in each singular point of w. Then, we can consider the gradient field of the distance to the stratum V_i, using the Whitney property (b), we deduce a "transverse" stratified vector field, which is zero along V_i and growing with the distance to the stratum. The sum of the parallel and of the transverse vector fields is the radial extension v. The radial extension of w is defined in a suitable geodesic tube (see [10] c) of the proof of Lemma 3.1.2). The radial extension v admits same singularities than w in the geodesic tube.

Proposition 5.2. [9, 11] *One can construct, on the $2p$-skeleton $(D)^{2p}$, a stratified r-frame $v^{(r)}$, called radial frame, whose singularities satisfy the following properties:*

(i) *$v^{(r)}$ has only isolated singular points, which are zeroes of the last vector v_r. On $(D)^{2p-1}$, the r-frame $v^{(r)}$ has no singular point and on $(D)^{2p}$ the $(r-1)$-frame $v^{(r-1)}$ has no singular point.*

(ii) *Let $a \in V_i \cap (D)^{2p}$ be a singular point of $v^{(r)}$ in the $2s$-dimensional stratum V_i. If $s > r-1$, the index of $v^{(r)}$ at a, denoted by $I(v^{(r)}, a)$, is the same as the index of the restriction of $v^{(r)}$ to $V_i \cap (D)^{2p}$ considered as an r-frame tangent to V_i. If $s = r-1$, then $I(v^{(r)}, a) = +1$.*

(iii) *Inside a $2p$-cell d which meets several strata, the only singularities of $v^{(r)}$ are inside the lowest dimensional one (in fact located in the barycenter of d).*

(iv) *The r-frame $v^{(r)}$ is pointing outwards a (particular) regular neighborhood U of X in M. It has no singularity on ∂U.*

The procedure of the construction of radial frames is made by induction on the dimension of the strata, using the properties of Whitney stratifications for proving the existence of frames pointing outward regular neighborhoods and satisfying property (ii). For strata of dimension $s = r-1$, the obstruction dimension to the construction of a r-frame tangent to the strata is 0. For each vertex of a triangulation of M as constructed earlier, one can construct an r-frame tangent to the $2p$-cell transverse dual of the vertex and pointing outwards the cell along its boundary. This provides an r-frame pointing outwards the $2p$-squeleton of a tubular neighborhood of V_i^{2s}. Let V_j^t be a stratum containing V_i^s in its closure, the r-frame is constructed in the $2p$-squeleton of a tubular neighborhood of the boundary of V_j^t, within V_j^t. One can extend the r-frame already constructed inside V_j^t, more precisely in the $2p$-squeleton of the interior of V_j^t, with isolated singularities on the interior of the $2(t-(r-1))$-simplices of the triangulation of V_j^t.

5.2. Obstruction Cocycles and Classes

Let us denote by $\mathcal{T}$ the tubular neighborhood of X in M consisting of the (D)-cells which meet X. Let us recall that d^* is the elementary (D)-cochain whose value is 1 at d and 0 at all other cells. We can define a $2p$-dimensional (D)-cochain in $C^{2p}(\mathcal{T}, \partial\mathcal{T})$ by:

$$\sum_{\substack{d(\sigma)\in\mathcal{T} \\ \dim d(\sigma)=2p}} I(v^{(r)}, \hat{\sigma})\; d^*(\sigma).$$

This cochain actually is a cocycle whose class $c^p(X)$ lies in

$$H^{2p}(\mathcal{T}, \partial\mathcal{T}) \cong H^{2p}(\mathcal{T}, \mathcal{T}\setminus X) \cong H^{2p}(M, M\setminus X),$$

where the first isomorphism is given by retraction along the rays of $\mathcal{T}$ and the second by excision (by $M \setminus \mathcal{T}$).

Definition 5.3. [9, 11] *The p-th Schwartz class is the class*

$$c^p(X) \in H^{2p}(M, M\setminus X).$$

6. Euler Local Obstruction

6.1. Nash Transformation

Let M be an analytic manifold, of complex dimension m. Let X be an sub-analytic complex variety, $X \subset M$, of complex dimension n. Let us denote by $\Sigma = X_{\text{sing}}$ the singular part of X and by $X_{\text{reg}} = X \setminus \Sigma$ the regular part.

The Grassmannian of complex n-planes in $\mathbb{C}^m$ is denoted by $G(n,m)$. Let us consider the Grassmann bundle of n (complex) planes in $T(M)$, denoted by G. The fibre G_x over $x \in M$ is the set of n-planes in $T_x(M)$, it is isomorphic to $G(n,m)$. An element of G is denoted by (x,P) where $x \in M$ and $P \in G_x$.

On the regular part of X, one can define the Gauss map $\sigma : X_{\text{reg}} \longrightarrow G$ by

$$\sigma(x) = (x, T_x(X_{\text{reg}})).$$

Definition 6.1. *The Nash transformation $\widetilde{X}$ is defined as the closure of the image of σ in G. It is equipped with a natural analytic projection $\nu : \widetilde{X} \longrightarrow X$.*

$$\begin{array}{ccccccc} & & G & \qquad & \widetilde{X} = \overline{\operatorname{Im}\sigma} & \hookrightarrow & G \\ & \nearrow\sigma & \downarrow & & \nu\downarrow & & \downarrow \\ X_{\text{reg}} & \hookrightarrow & M & & X & \hookrightarrow & M \end{array}$$

In general, $\widetilde{X}$ is not smooth, nevertheless, it is an analytic variety and the restriction $\nu : \widetilde{X} \to X$ of the bundle projection $G \to M$ is analytic.

The fiber E_P of the tautological bundle E over G, in a point $(x,P) \in G$, is the set of the vectors v of the n-plane P.

$$E_P = \{v(x) \in T_x M \, : \, v(x) \in P, \quad x = \nu(P)\}$$

Let us define $\widetilde{E} = E|_{\tilde{X}}$, then $\widetilde{E}|_{\tilde{X}_{\rm reg}} = T(X_{\rm reg})$ where $\widetilde{X}_{\rm reg} = \nu^{-1}(X_{\rm reg}) \cong X_{\rm reg}$ and

$$\widetilde{E} = E \times_G \widetilde{X} = \{(v(x), \tilde{x}) \in E \times \widetilde{X} \, : v(x) \in \tilde{x}\}$$

$\tilde{x} \in \widetilde{X}$ is a n-complex plane in $T_x(M)$ and $x = \nu(\tilde{x})$.

One has a diagram:

$$\begin{array}{ccc} \widetilde{E} & \hookrightarrow & E \\ \downarrow & & \downarrow \\ \widetilde{X} & \hookrightarrow & G \\ \nu\downarrow & & \downarrow \\ X & \hookrightarrow & M \end{array}$$

An element of $\widetilde{E}$ is written (x, P, v) with $x \in X$, P is a n-plane in $T_x(M)$ and v is a vector in P.

Let us denote by $(V_\alpha)_{\alpha \in A}$ a complex analytic stratification of X satisfying the *Whitney conditions*.

The following lemma is fundamental for the understanding of the geometrical definition of the local Euler obstruction.

Lemma 6.2. ([3, Proposition 9.1]) *A stratified vector field v on $A \subset X$ admits a canonical lifting $\tilde{v}$ on $\nu^{-1}(A)$ as a section of $\widetilde{E}$.*

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{\nu_*} & TM|_X \\ \tilde{v}\uparrow\downarrow & & v\uparrow\downarrow \\ \widetilde{X} & \xrightarrow{\nu} & X \end{array} \qquad \nu_*(v(x), \tilde{x}) = v(\nu(\tilde{x})) = v(x).$$

Let us recall that a *radial* vector field v in a neighborhood of the point $\{0\} \in X$ is a stratified vector field so that there exists $\varepsilon_0 > 0$ such that for all ε, $0 < \varepsilon < \varepsilon_0$, the vector $v(x)$ is pointing outwards the ball B_ε over the boundary $S_\varepsilon = \partial B_\varepsilon$. By the Bertini-Sard theorem, S_ε is transverse to the strata V_α if ε is small enough, so the definition takes sense.

Theorem 6.3. [Theorem-Definition 3] *Let v be a radial vector field over $X \cap S_\varepsilon$ and $\tilde{v}$ the lifting of v over $\nu^{-1}(X \cap S_\varepsilon)$. The local Euler obstruction $Eu_0(X)$ is the obstruction to extend $\tilde{v}$ as a nowhere zero section of $\tilde{E}$ over $\nu^{-1}(X \cap B_\varepsilon)$, evaluated on the orientation class $\mathcal{O}_{\nu^{-1}(B_\varepsilon), \nu^{-1}(S_\varepsilon)}$:*

$$Eu_0(X) = Obs(\tilde{v}, \widetilde{E}, \nu^{-1}(X \cap B_\varepsilon)).$$

Theorem 6.4. ([3, Théorème 11.1]) (Proportionality Theorem for frames) *Let v^r be a radial r-frame with isolated singularities on the $2k$-cells d_α^{2k} with index $I(v^r, \hat{\sigma}_\alpha)$ in the barycenter $\{\hat{\sigma}_\alpha\} = d_\alpha^{2k} \cap \sigma_\alpha$. Then the obstruction to the extension of $\tilde{v}^r$ as a section of $\widetilde{E}^r$ on $\nu^{-1}(d_\alpha^{2k} \cap X)$ is*

$$Obs(\tilde{v}^r, \widetilde{E}^r, \nu^{-1}(d_\alpha^{2k} \cap X)) = Eu_{\hat{\sigma}_\alpha}(X) \cdot I(v^r, \hat{\sigma}_\alpha).$$

7. MacPherson and Mather Classes

Let us recall firstly some basic definitions.

A *constructible set* in a variety X is a subset obtained by finitely many unions, intersections and complements of subvarieties. A *constructible function* $\alpha : X \to \mathbb{Z}$ is a function such that $\alpha^{-1}(n)$ is a constructible set for all n. The constructible functions on X form a group denoted by $\mathcal{F}(X)$. If $A \subset X$ is a subvariety, we denote by $\mathbf{1}_A$ the characteristic function whose value is 1 over A and 0 elsewhere.

If X is triangulable, α is a constructible function if and only if there is a triangulation (K) of X such that α is constant on the interior of each simplex of (K). Such a triangulation of X is called α-adapted.

The correspondence $\mathcal{F} : X \to \mathcal{F}(X)$ defines a contravariant functor when considering the usual pull-back $f^* : \mathcal{F}(Y) \to \mathcal{F}(X)$ for a morphism $f : X \to Y$. One interesting fact is that it can be made a covariant functor when considering the pushforward defined on characteristic functions by:

$$f_*(\mathbf{1}_A)(y) = \chi(f^{-1}(y) \cap A), \quad y \in Y$$

for a morphism $f : X \to Y$, and linearly extended to elements of $\mathcal{F}(X)$. The following result was conjectured by Deligne and Grothendieck in 1969.

Theorem 7.1. [8] *Let $\mathcal{F}$ be the covariant functor of constructible functions and let $H_*(; \mathbb{Z})$ be the usual covariant $\mathbb{Z}$-homology functor. Then there exists a unique natural transformation*

$$c_* : \mathcal{F} \to H_*(; \mathbb{Z})$$

satisfying $c_*(\mathbf{1}_X) = c^*(X) \cap [X]$ if X is a manifold.

The theorem means that for every algebraic complex variety, one has a functor $c_* : \mathcal{F}(X) \to H_*(X; \mathbb{Z})$ satisfying the following properties:

1. $c_*(\alpha + \beta) = c_*(\alpha) + c_*(\beta)$ for α and β in $\mathcal{F}(X)$,
2. $c_* f_*(\alpha) = f_*(c_*(\alpha))$ for $f : X \to Y$ morphism of algebraic varieties and $\alpha \in \mathcal{F}(Y)$,
3. $c_*(\mathbf{1}_X) = c^*(X) \cap [X]$ if X is a manifold.

7.1. Mather Classes

The first approach to the proof of the Deligne-Grothendieck's conjecture is given by the construction of Mather classes. Let $X \subset M$ a possibly singular algebraic complex variety embedded in a smooth one. Let us define the Nash transformation $\widetilde{X}$ of X, as in Subsec. 6.1 and the Nash bundle $\widetilde{E}$ on $\widetilde{X}$.

Definition 7.2. *The Mather class of X is defined by:*

$$c^M(X) = \nu_*(c^*(\widetilde{E}) \cap [\widetilde{X}])$$

where $c^*(\widetilde{E})$ denotes the usual (total) Chern class of the bundle $\widetilde{E}$ in $H^*(\widetilde{X})$ and the cap-product with $[\widetilde{X}]$ is the Poincaré duality homomorphism (in general not an isomorphism).

The Mather classes do not satisfy the Deligne-Grothendieck's conjecture.

7.2. MacPherson Classes

The MacPherson's construction uses both the constructions of Mather classes and local Euler obstruction.

For a Whitney stratification, we have the following lemma:

Lemma 7.3. [8] *There are integers n_α such that, for every point $x \in X$, we have:*

$$\sum_\alpha n_\alpha Eu_x(\overline{V_\alpha}) = 1.$$

Definition 7.4. [8] *The MacPherson class of X is defined by*

$$c_*(X) = c_*(\mathbf{1}_X) = \sum_\alpha n_\alpha \; i_* c_*^M(\overline{V_\alpha})$$

where i denotes the inclusion $\overline{V_\alpha} \hookrightarrow X$.

Theorem 7.5. [3] *The MacPherson class is the image of the Schwartz class by the Alexander duality isomorphism*

$$H^{2(m-r+1)}(M, M \setminus X) \xrightarrow{\cong} H_{2(r-1)}(X).$$

Corollary 7.6. *The Schwartz-MacPherson class $c_{r-1}(X)$ is represented by the cycle:*

$$\boxed{\sum_{\substack{\sigma \subset X \\ \dim \sigma = 2(r-1)}} I(v^{(r)}, \hat{\sigma}) \; \sigma}$$

Theorem 7.7. [3] *The Chern-Mather class $c_{r-1}^M(X)$ is represented by the cycle:*

$$\boxed{\sum_{\substack{\sigma \subset X \\ \dim \sigma = 2(r-1)}} Eu_{\hat{\sigma}}(X) I(v^{(r)}, \hat{\sigma}) \; \sigma}$$

8. Fulton Classes

Definition 8.1. *If X is a local complete intersection, then the normal bundle of $X_{\rm reg}$ in M extends canonically to X as a vector bundle $N_X M$ and*

$$c^F(X) = c(TM|_X) c(N_X M)^{-1} \cap [X] = c(\tau_X) \cap [X]. \qquad (4)$$

Here $\tau_X = TM|_X - N_X M$ denotes the virtual tangent bundle on X, defined in the Grothendieck group of vector bundles on X.

Theorem 8.2. [4] *Let us assume that $X \subset M$ is a hypersurface, defined by a section s of a holomorphic line bundle L over M. Assume further that L also admits a section s_0 which is everywhere transverse to the zero-section. For each point $a \in X$, let F_a denote a local Milnor fiber, and let $\chi(F_a)$ be its Euler-Poincaré characteristic. Then the Fulton-Johnson class $c^{FJ}_{r-1}(X)$ of X of degree $(r-1)$ is represented in $H_{2(r-1)}(X)$ by the cycle*

$$\boxed{\sum_{\substack{\sigma\subset X\\ \dim\sigma=2(r-1)}} \chi(F_{\hat\sigma})\, I(v^{(r)},\hat\sigma)\ \sigma}$$

On the other hand, the question of understanding the difference between the Schwartz-MacPherson and the Fulton-Johnson classes has been addressed by several authors, and this led to the concept of *Milnor classes*, defined by $\mu_*(X) = (-1)^{n+1}\big(c_*(X) - c^{FJ}_*(X)\big)$, $n = \dim X$, see for instance Aluffi, Brasselet, Lehmann, Seade, Suwa, Parusiński, Pragacz and Yokura. Let us define the local Milnor number of X at the point $a \in X$ by $\mu(X,a) = (-1)^{n+1}(1-\chi(F_a))$; it coincides with the usual Milnor number when a is an isolated singularity of X. It is non zero only on the singular set Σ of X. We have the following immediate consequence of the previous Theorem:

Corollary 8.3. *Under the assumptions of the previous Theorem, the Milnor class $\mu_{r-1}(X)$ in $H_{2(r-1)}(X)$ is represented by the cycle*

$$\boxed{\sum_{\substack{\sigma\subset X\\ \dim\sigma=2(r-1)}} \mu(X,\hat\sigma_\alpha)\, I(v^{(r)},\hat\sigma)\ \sigma}$$

9. Schwartz-MacPherson Classes of Thom Spaces Associated to Embeddings

In this section and as a matter of example, we compute the Schwartz-MacPherson classes of the Thom spaces associated to Segre and Veronese embeddings (see [5]).

9.1. The Projective Cone

Let us consider an n-dimensional projective variety Y in $\mathbb{P}^m = \mathbb{P}C^m$ and let us denote by L the restriction of the hyperplane bundle of $\mathbb{P}^m$ to Y. We denote by E the completed projective space of the total space of L, i.e. $\mathbb{P}(L \oplus 1_Y)$ where 1_Y is the trivial bundle of complex rank 1 on Y. The canonical projection $p\colon E \to Y$ admits two sections, zero and infinite, with images $Y_{(0)}$ and $Y_{(\infty)}$. The projective cone KY is obtained as a quotient of E by contraction of $Y_{(\infty)}$ in a point $\{s\}$. It is the Thom space associated to the bundle L, with basis Y.

Let us consider $p: E \to Y$ as a sphere bundle with fiber S^2, subbundle of a bundle $\bar{p}: \bar{E} \to Y$ with fiber the ball B^3. We denote by $\theta_{\bar{E}} \in H^3(\bar{E}, E)$ the associated Thom class; One has a Gysin exact sequence

$$\ldots \to H_{j+1}(Y) \to H_{j-2}(Y) \xrightarrow{\gamma} H_j(E) \xrightarrow{p_j} H_j(Y) \to \ldots ;$$

in which the gysin map γ is the composition of

$$H_{j-2}(Y) \overset{(\bar{p}_{j-2})^{-1}}{\longrightarrow} H_{j-2}(\bar{E}) \overset{(\cap\theta_{\bar{E}})^{-1}}{\longrightarrow} H_{j+1}(\bar{E}, E) \xrightarrow{\partial} H_j(E)$$

and can be explicited in the following way: If ζ is a cycle representing the class $[\zeta]$ of $H_{j-2}(Y)$, then $\gamma([\zeta])$ is the class of the cycle $p^{-1}(\zeta)$ in $H_j(E)$.

Let π the canonical projection $\pi: E \to KY$.

Proposition 9.1. *The Chern classes of E and Y are related by the formula*

$$c_*(E) = (1 \ + \ \eta_0 \ + \ \eta_\infty) \cap \gamma(c_*(Y)), \tag{9.4}$$

where $\eta_j := c^1\big(\mathcal{O}(Y_{(j)})\big) \in H^2(E)$ for $j = 0,\ \infty$, and $\cap$ denotes the usual cap-product.

Let us remark that if Y is irreducible in each point and if $*$ denotes the intersection product between the intersection homology groups $IH_*^{\bar{o}}(E)$ and $IH_*^{\bar{t}}(E) \cong H_*(E)$, one has

$$c_*(E) = ([E] \ + \ [Y_{(0)}] \ + \ [Y_{(\infty)}]) \ * \ \gamma(c_*(Y)).$$

Proof. The vertical tangent bundle T_v of $p: E \to Y$ is defined by the exact sequence:

$$0 \to T_v \to TE \to p^*TY \to 0.$$

We have, in $H^*(E)$

$$c^*(E) = c^*(T_v) \cup c^*\big(p^*(TY)\big). \tag{9.5}$$

The sheaf of sections of the bundle T_v is the sheaf canonically associated to the divisor $Y_{(0)} + Y_{(\infty)}$, denoted by $\mathcal{O}_E(Y_{(0)} + Y_{(\infty)})$. By Poincaré isomorphism in Y, the divisor $[Y_{(j)}] \in H_{2n}(E)$ is identified to the class $\eta_j \in H^2(E)$. The Chern class of T_v is

$$c^*(T_v) = 1 \ + \ \eta_0 \ + \ \eta_\infty.$$

By definition of the Gysin map γ, one has a commutative diagram

$$\begin{array}{ccc} H^i(Y) & \xrightarrow{\cap[Y]} & H_{2n-i}(Y) \\ \Big\downarrow p^1 & & \Big\downarrow \gamma \\ H^i(Y) & \xrightarrow{\cap[E]} & H_{2n-i}(E) \end{array}$$

and by Poincaré duality

$$c^*(p^*(TY)) \cap [E] = p^*\big(c^*(TY)\big) \cap [E] = \gamma(c^*(TY) \cap [Y]) = \gamma(c_*(Y)). \tag{9.6}$$

Using formulae (9.5) and (9.6), one obtains

$$c_*(E) = (1 \ + \ \eta_0 \ + \ \eta_\infty) \ \cap \ \gamma(c_*(Y)).$$

■

9.2. Schwartz-MacPherson Classes of the Projective Cone

Definition 9.2. *We call homological projective cone and we denote by K the composition $K = \pi_*\gamma : H_{j-2}(Y) \to H_j(KY)$ for $j \geq 2$. We let $K(0) := [s] \in H_0(KY)$ for $0 = H_{-2}(Y)$.*

Let us remark that K is an homomorphism, out of $j = 0$.

Theorem 9.3. *Let $Y \subset \mathbb{P}_N$, be a projective variety and $\imath\colon Y \hookrightarrow KY$ the canonical inclusion in the projective cone KY on Y with vertex $\{s\}$. Let us denote also by $K : H_*(Y) \to H_{*+2}(KY)$ the homological projective cone, one has*

$$c_j(KY) = \imath_* c_j(Y) + K c_{j-1}(Y), \tag{9.7}$$

where $Kc_{-1}(Y)$ denotes the class $[s] \in H_0(KY)$.

Proof. Let $\mathbf{1}_E$ the constructible function which is the characteristic function of E, one has

$$\pi_*(\mathbf{1}_E)(x) = \begin{cases} \chi(Y), & if \mathrm{x} = \mathrm{s} \\ 1, & elsewhere, \end{cases}$$

i.e.

$$\pi_*(\mathbf{1}_E) = \mathbf{1}_{KY} + (\chi(Y) - 1)\mathbf{1}_{\{s\}}.$$

As one has

$$\pi_* c_*(\mathbf{1}_E) = c_*(\pi_*(\mathbf{1}_E))$$

one obtains

$$\pi_* c_*(E) = c_*(KY) + (\chi(Y) - 1)[s]. \tag{9.8}$$

On another hand, from the formula (9.4) one obtains:

$$\pi_* c_*(E) = \pi_*\gamma(c_{*-1}(Y)) + \pi_*(\eta_0 \cap \gamma(c_*(Y))) + \pi_*(\eta_\infty \cap \gamma(c_*(Y))). \tag{9.9}$$

Let $\iota_0 : Y \hookrightarrow E$ and $\iota_\infty : Y \hookrightarrow E$ be the inclusions of Y as zero and infinite sections of E respectively. By definition of γ, one has for every cycle ζ in Y and for $j = 0$ or ∞

$$\eta_j \cap \gamma([\zeta]) = (\iota_j)_*([\zeta])$$

then

$$\pi_*(\eta_j \cap \gamma(c_*(Y))) = \pi_* \iota_{j*} c_*(\mathbf{1}_Y) = \pi_* c_*(\mathbf{1}_{Y_{(j)}}) = c_* \pi_*(\mathbf{1}_{Y_{(j)}}).$$

Let us denote by $\iota = \pi \circ \iota_0 : Y \hookrightarrow KY$ the natural inclusion of Y in KY, one has

$$\pi_*(\mathbf{1}_{Y_{(0)}}) = \mathbf{1}_{\iota(Y)} \text{ and } \pi_*(\mathbf{1}_{Y_{(\infty)}}) = \chi(Y)\mathbf{f}1_{\{s\}}.$$

One obtains

$$\pi_*(\eta_0 \cap \gamma(c_*(Y))) = c_*(\mathbf{1}_{\iota(Y)}) = \iota_* c_*(Y),$$

and

$$\pi_*(\eta_\infty \cap \gamma(c_*(Y))) = \chi(Y) c_*(\mathbf{1}_{\{s\}}) = \chi(Y)[s],$$

where $[s]$ is the class of the vertex s in $H_0(KY)$. The comparizon of the formulae 9.8 and 9.9 gives:

$$c_*(KY) = \imath_* c_*(Y) + \pi_* \gamma c_{*-1}(Y) + [s],$$

and the Theorem 9.3. ■

9.3. Case of the Segre and Veronese Embeddings

The previous construction associates canonically a thom space to the embedding of a smooth variety Y in $\mathbb{P}^m$. As examples, let us consider the image of the Segre embedding $\mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3$, defined in homogeneous coordinates by

$$(x_0 : x_1) \times (y_0 : y_1) \ \mapsto \ (x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1),$$

and the image of the Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ defined by

$$(x_0 : x_1 : x_2) \ \mapsto \ (x_0^2 : x_0 x_1 : x_0 x_2 : x_1^2 : x_1 x_2 : x_2^2).$$

With the previous construction, KY is the Thom space associated to the fiber bundle L, of complex rank 1 and restriction to Y of the hyperplane bundle of $\mathbb{P}^m$. Chern classes and intersection homology of these exemples have been computed in [1]. In the case of the Segre embedding, let d_1 and d_2 two fixed lines belonging each to a system of generatrices of the quadric $Y = \mathbb{P}^1 \times \mathbb{P}^1$. Let us denote by ω the canonical generator of $H^2(\mathbb{P}^1)$, one has $c^*(\mathbb{P}^1) = 1 + 2\omega$ and

$$c_*(Y) = c_*(\mathbb{P}^1 \times \mathbb{P}^1) = ([Y] + 2[d_1]) * ([Y] + 2[d_2]) = [Y] + 2([d_1] + [d_2]) + 4[a]$$

where a is a point in Y and where $*$ denotes the intersection of cycles or homology classes. One has

$$K(c_*(Y)) = [KY] + 2([Kd_1] + [Kd_2]) + 4[Ka].$$

let us denote by $\sim$ the homology relation of cycles. In KY, one has [1], Sec. 3:

$$Y \sim Kd_1 + Kd_2, \quad d_1 \sim d_2 \sim Ka, \quad a \sim s,$$

and, with 9.3

$$c_*(KY) = \underbrace{[KY]}_{H_6(KY)} + \underbrace{3([Kd_1] + [Kd_2])}_{H_4(KY)} + \underbrace{8[Ka]}_{H_2(KY)} + \underbrace{5[s]}_{H_0(KY)},$$

which is the result of [1].

In the case of the Veronese embedding, let d be a projective line in $Y := \mathbb{P}^2$, one has: $c^*(\mathbb{P}^2) = 1 + 3\omega + 3\omega^2$ where ω is the canonical generator of $H^2(\mathbb{P}_2)$, and is dual, by Poincaré isomorphism of the class $[d] \in H_2(\mathbb{P}_2)$. One has, by Poincaré duality

$$c_*(Y) = [Y] + 3[d] + 3[a]$$

where a is a point in Y. One has

$$K(c_*(Y)) = [KY] + 3[Kd] + 3[Ka]$$

such that, in KY, [1], 3.b, $Y \sim 2Kd$, $d \sim 2Ka$ and $a \sim s$. One has

$$c_*(KY) = \underbrace{[KY]}_{H_6(KY)} + \underbrace{5[Kd]}_{H_4(KY)} + \underbrace{9[Ka]}_{H_2(KY)} + \underbrace{4[s]}_{H_0(KY)}$$

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