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Geometric Calculus of Variations

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Abstract. This is a summary of the geometric calculus of variations as an approach of modern analysis and topology. The paper deals with important issues of geometric variational problems such as: mathematical models, the development of basic conceptions of surface and minimality, a modern analysis of classical geometric optics, the multidimensional Huyghens principle, calibrations and volume-minimizing surfaces. In the paper are proposed also some open problems.

1. Mathematical Models of Minimal Surfaces

1.1. Practical Stimuli

Natural phenomena occur in accordance with the optimality law:

• Every thing has the most economical geometric form, construction and situation in space (for instance, having least length, area or volume).

• Any mechanical, physical, biological state or process takes place according to the principle of the smallest action or spending least energy.

• Examples:

- *Light rays,* trajectory of a free motion are length-minimizing.

- *Soap films* minimize the area of membranes with a given boundary, i.e. are solutions to *the classical Plateau problem.*

- *Soap bubbles* maximize the area among membranes (with boundary) enveloping the same volume, i.e. are solutions to a *classical isoperimetric problem.*

- A bee's nest, and some marine animals are area-minimizing branched membranes.

- Membranes, which are minimizing the flow of a mechanical or physical field (temperature flow, fluid flow of stable current, energy flow of an electric magnetic field...).

1.2. Mathematical Stimuli

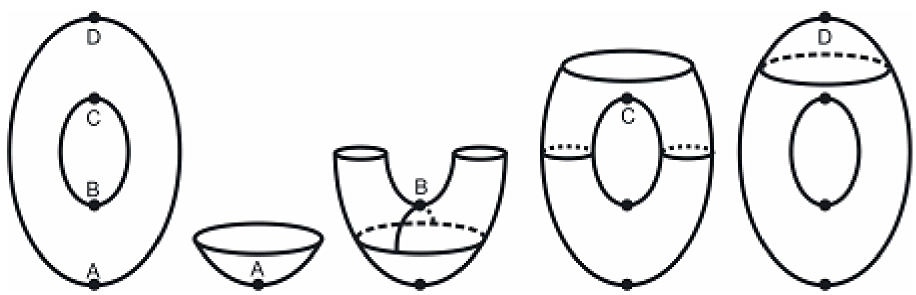
The study of minimal surfaces was stimulated also by deep mathematical ideas on the decisive role of extrema of functions in geometry.

• *Extrema (maxima and minima)* of a function f, defined by the conditions $f' = 0$ and $f'' > 0$ (for a minimum) or $f'' < 0$ (for a maximum) determine the behaviour of the graph of f.

• *Critical points* of a function f on a manifold are defined by the condition: $\mathrm{grad} f = 0$. The *index* of a critical point x is the index of the Jacobian matrix of f at x.

According to Morse theory, the topological type of a manifold M can be reconstructed from critical points (with their indices) of a Morse function on M.

Example. The height function on a two-dimensional torus has 4 critical points: a minimum A with index 0, two saddles B, C with index 1 and a maximum D with index 2. The reconstruction of the torus is illustrated in pic. 1.



Pic 1.

1.3. The Conception of Surface

• A k-surface S is a set of points provided with a corresponding *tangent field* $\vec{S}$: each point $x \in S$ is furnished with a k-dimensional tangent space $\vec{S}_x \subset \vec{S}$. We emphasize the difference between two approaches to the basic conception of k-surface:

- In the geometric approach a k-surface is identified with its set S of points (i.e. the set of points is the original objective) and the corresponding k-dimensional tangent spaces $\vec{S}_x$ are determined by the given set S.

- The variational approach regards both of the set S of points and the tangent field $\vec{S}$ as original objectives and defines a k-surface S as the integration along the given tangent field $\vec{S}$.

• *The boundary* of a k-surface may be realized as the usual *geometric boundary* or generalized as an *algebraic boundary* (by Adams, Reifenberg, Fomenko) as follows: a compact subset $A \subset S$ is called a "boundary" for a k-surface S if A realizes a given homology (homotopy, cohomology...) class of S and S nullifies a given homology (homotopy, cohomology...) class of A, i.e. let $i : A \hookrightarrow S$ be the embedding, $i_* : H_*(A) \to H_*(S)$ - the induced homomorphism, $\alpha \subset H_*(S)$ and $\beta \subset H_*(A)$-given subgroups, then $\alpha \subset \mathrm{Im} i_*$ and $\beta \subset \mathrm{Ker} i_*$.

• *A variational class* is a set of k-surfaces, satisfying:

- Certain requirements on the topological structure a k-surface such as: differentiability, parametrizability, topological type (fixed or varying)...

- Boundary conditions: fixed or varying boundary, certain constraints on the situation of the boundary A: A is situated in a compact set $K; A$ realizes a given homology (homotopy, cohomology...) class of set $K; K$ nullifies a given homology (homotopy, cohomology...) class of A.

- Certain constraints on the situation of k-surface S in the space: S is situated in a subspace of $M; S$ realizes a given homology (homotopy, cohomology) class of $M; M$ nullifies a given homology (homotopy, cohomology...) class of S.

- Constraints of isoperimetric type and other types.

1.4. The Conception of Minimality

• *The minimality* of a surface is considered with respect to a functional $J = J(S)$, which depends on a variable point $x \in S$ and the corresponding variable tangent space $\vec{S_x}$ (i.e. on tangent field $\vec{S}$): $J(S) = J(\vec{S}) = J(x, \vec{S_x})$.

• A k-surface is called an *extremal* or locally minimal with respect to a functional J if the *first variation* of J taken at this k-surface, is trivial: $\delta J = 0$.

If in addition the *second variation* of J is negative: $\delta^2 J < 0$, then the k-surface is called *stably minimal*.

• A k-surface is called *globally minimal* with respect to J in a variational class if it minimizes J in the given variational class. The global minimality may be *absolute, relative, homological or homotopical...* depending on the choice of a respective variational class.

2. The Existence of Minimal Surfaces and the Development of Basic Conceptions of Surfaces and Minimality

2.1. Hilbert's Remark

Every variational problem has *solutions*, but it is necessary to give to the term *"solution"* a suitable meaning, i.e. to extend in a reasonable way the basic conceptions of k-surface, minimality and concerning notions of tangent space, boundary, volume... The development of the conceptions of k-surface and minimality is closely connected with the greatest philosophical and mathematical ideas, leading to the most important achievements not only in calculus of variations but in the progress of mathematics in general.

Usually, generalized surfaces are obtained as limits of traditional surfaces with respect to a suitable topology.

2.2. Regular Surfaces

A *regular k-surface* S in a manifold M is a submanifold with the following usual notions:

• The k-dimensional tangent subspace $\vec{S}_x \subset T_xM$ to S (as a submanifold), determined at every point $x \in S$, where T_xM is the tangent space to M at x.

• The boundary δS of S (as a submanifold).

• The k-volume $\mathrm{vol}_k(S)$ of S (as a submanifold)

2.3. Functionals of Volume Type

Let $\mathcal{G}_kM$ be the *Grassmannian bundle* of tangent k-spaces of M, $l : \mathcal{G}_kM \to R$ a continuous function. The functional J, defined by

$$J(S) = \int_S l(\vec{S}_x)dx$$

is called a *functional of volume type* or *an integrand,* given by a *Lagrangian* l.

Examples.

• The case $l \equiv 1$ gives the k-dimensional vol_k (arc length for $k = 1$, area for $k = 2$, volume for $k = 3$...)

• The integration of a differential k-form ω:

$$J(S) = \int_S \omega = \int_S \omega(\vec{S}_x)dx \quad (l = \omega).$$

• The flow of a vector field ξ is an integrand, defined by

$$J(S) = \int_S (\xi, n^S)dx$$

where n^S denotes the normal vector to a hypersurface S.

2.4. Singular Surfaces

A *singular k-surface* S has a k-dimensional tangent subspace almost everywhere, except at several *singular points* (singular set S^0). The regular part $S \backslash S^0$ is a submanifold. The boundary δS of S can be defined in the same way as for the case of a regular k-surface. Since $\mathrm{vol}_k(S^0) = 0$ we can identify $\mathrm{vol}_k(S)$ with $\mathrm{vol}_k(S \backslash S^0)$ and $J(S)$ with $J(S \backslash S^0)$ for any integrand J.

Examples.

• *Minimal cones* (the singular set is a point).

• *Branched* minimal k-surfaces, minimal k-surfaces with *self-intersections* (the singular set has dimension $< k$.)

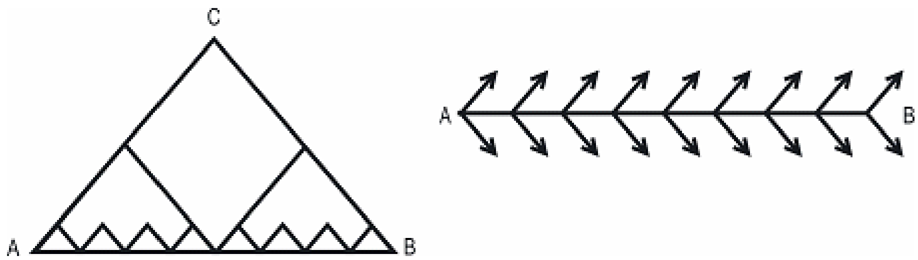
2.5. Generalized Surfaces (Varifolds) (L. Young, F.J. Almgren)

• Examples of generalized curves.

- A motion at one place can be regarded as a motion with compensated velocities.

- Limiting *zigzag line*. We consider a sequence of broken lines γ_n, joining two points A and B (see pic. 2). Under a limit γ_n tends to the segment AB, but the corresponding tangent vectors to γ_n tend to two limits, coinciding with the two common directions $\vec{AC}$ and $\vec{CB}$ of the broken lines γ_n. Thus, a limiting zigzag line is a straight line with *"double tangent vectors"*. Since the length of γ_n is constant and equal to $AC + CB$ the length of $\lim \gamma_n$ is equal to $AC + CB > AB$.

We note that optimal motions of a boat against the current and of climbing up to a mountain are zigzag lines



Pic 2

- Varifolds

A k-varifold (or a *generalized k-surface)* on a manifold M is a measure with compact support on the Grassmannian bundle $\mathcal{G}_k M$. The support spt S of a varifold S is the projection of support *supp* S of the measure S to M.

The random tangent space $\vec{S}_x$ to S at a point $x \in M$ is the restriction of the measure S to $\mathcal{G}_k M_x$.

The boundary of a $k-$varifold S can be understood as the boundary of spt S.

Given a k-varifold S we can define a *mass (volume) measure* $\|S\|$ on M by the following formula

$$\|S\|(\phi) = \sup\{S(f) : f(\xi) \leqslant \phi(x), \quad \forall \xi \in \mathcal{G}_k M_x\}.$$

We have

$$S(f) = \int \vec{S}_x(f) d\|S\|(x)$$

for any function f on $\mathcal{G}_k M$.

Mass (volume) $M(S)$ of a k-varifold S is the norm of the measure S, i.e. $M(S) = \|S\|(1)$

Any compact k-surface S can be considered as a k-varifold, determined by the formula

$$S(f) = \int_S f(\vec{S}_x) dx$$

for any function f on $\mathcal{G}_k(M)$.

In this case notions of tangent space, boundary and mass (volume) coincide with the traditional ones.

An integrand J, given by a lagrangian l, is defined as

$$J(S) = \int \vec{S}_x(l) d\|S\|(x)$$

for any k-varifold S on M.

2.6. De Rham Currents

A *k-current* on a manifold M is a continuous functional on the space $\Lambda^k M$ of differential k-forms.

For any k-current S we can define a *mass (volume) measure* $\|S\|$ on M by the following formula:

$$\|S\|(f) = \sup\{S(\phi) : \|\phi_x\|^* \leqslant f(x), \quad \forall \phi \in \Lambda^k M$$

where $\|\phi_x\|^*$ denotes the *comass* of the k-covector ϕ_x.

There exists almost everywhere in the sense of $\|S\|$ a k-vector $\vec{S}_x$ such as:

$$S(\phi) = \int \phi(\vec{S}_x) d\|S\|(x).$$

$\vec{S}_x$ is called the *tangent k-vector* to S at x.

The boundary of a k-current S is the $(k-1)$-current ∂S, determined by the following formula

$$\partial S(\phi) = S(d\phi), \quad \forall \phi \in \Lambda^{k-1} M.$$

The mass (volume) $M(S)$ of S is $\|S\|(1)$.

An integrand J, given by a lagrangian l, is defined as:

$$J(S) = \int l(\vec{S}_x) d\|S\|(x)$$

for any k-current S on M.

Any compact k-surface S can be considered as a k-current, identified with integration along S. In this case notions of tangent space, boundary and mass (volume) coincide with the traditional ones.

2.7. Stratified Surfaces and Multivarifolds (A.T. Fomenko, Dao Trong Thi)

• **Stratified Surfaces.**

A *stratified k-surface* is a compact space $S = \cup S_i$, where S^i is an i-surface, $0 \leqslant i \leqslant k$; $\quad \mathrm{Int} S^i \cap \mathrm{Int} S^j = \phi \quad (i \neq j)$. S^i is called the *i-stratum* of S.

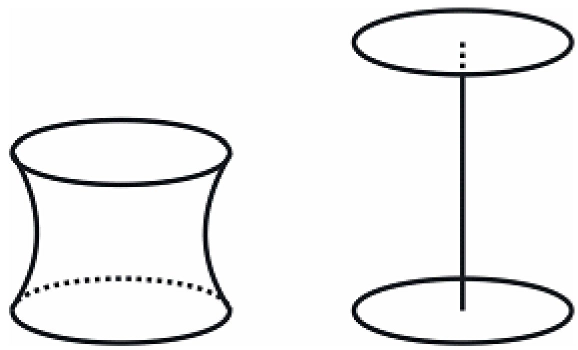
A *parametrized surface,* determined as the image $S = f(W)$ of a manifold W under a locally degenerated mapping f, is a stratified surface. We can not release strata of small dimentions without destroying topological type or even paramatrizability of the surface S.

Example. The degenerated catenoid is two disks with a segment, connecting these disks (pic. 3).

The stratified volume of a stratified k-surface S is $\mathrm{vol}(S) = (\mathrm{vol}_0 S^0, ..., \mathrm{vol}_k S^k)$. The minimization can be taken with respect to the volume of maximal dimension vol_k or to $\mathrm{vol}(S)$ in *the lexicographical order.*

• **Multivarifolds.**

A k-multivarifold (or a *stratified k-varifold*) on a manifold M is a measure with compact support on the *Grassmannian stratified bundle* $G_k M = \cup \mathcal{G}_i M$.



Pic 3.

The restriction S^i of a multivarifold S to the stratum $\mathcal{G}_i M$ is an i-varifold, called the i-stratum of S:

$$S = S^0 + ... + S^k.$$

The *stratified support* $SptS = (SptS^0, ..., SptS^k)$ and the *stratified tangent random space* $\vec{S}_x$ of a multivarifold S are defined by the same way as for a varifold; the *mass (volume) measure* $\|S\|$, the *strafified mass* $M(S) = (M_0(S^0), ..., M_k(S^k))$ and *a stratified integrand* $J(S) = (J_0(S^0), ..., J_k(S^k))$ can be defined similarly for the case of stratified surfaces.

Any stratified k-surface S can be considered as a k-multivarifold, determined by the formula

$$S(f) = \sum_{0 \leqslant i \leqslant k} \int_{S^i} f_i(\vec{S}_x^i) dx$$

for any function f on $G_k M (f_i = f|_{\mathcal{G}_i M})$. In this case the relevant notions such as stratified tangent space, stratified mass (volume) coincide with those of stratified k-surfaces.

3. A Modern Analysis of One-Dimensional Variational Problems

3.1. Classical Geometric Optics

A one-dimensional variational problem can be regarded as a mathematical model of the classical geometric optics.

Light rays $x = x(t)$ in a *nonhomogeneous medium* in accordance with *Fermat′s principle* minimizing the integral

$$J(x) = \int_{t_0}^{t_1} l(x(t), \dot{x}(t)) dt,$$

where l is a positively homogeneuos lagrangian, defined by properties of the medium. By the positive homogenity of l we have

$$\dot{x}l_{\dot{x}} = l.$$

3.2. Euler's Equation and the Weierstrass Condition

• A necessary condition for a curve $x = x(t)$ to be a light path is that its first variation $\delta J = 0$, which is equivalent to *Euler′s equation:*

$$l_x - \frac{d}{dt}l_{\dot{x}} = 0.$$

• The classical Weierstrass condition

$$l(x, \dot{x}) \geq l(x, p) + (\dot{x} - p)l_{\dot{x}}(x, p)$$

means that l is convex with respect to $\dot{x}$ at $\dot{x} = p$. Consider the differential form $\Phi(x, \dot{x}) = \dot{x}l_{\dot{x}}(x, p)$. By the positive homogenity of l we have

$$\begin{aligned}\Phi(x, \dot{x}) &= pl_{\dot{x}}(x, p) + (\dot{x} - p)l_{\dot{x}}(x, p)\\ &= l(x, p) + (\dot{x} - p)l_{\dot{x}}(x, p).\end{aligned}$$

In particular, $\Phi(x, p) = l(x, p)$. Therefore, the classical Weierstrass condition is equivalent to

$$\begin{aligned}\Phi(x, \dot{x}) &\leqslant l(x, \dot{x})\\ \Phi(x, p) &= l(x, p),\end{aligned}$$

i.e. Φ is a supporting differential form to the lagrangian l at $\dot{x} = p$. We call p a *maximal slope* for Φ.

3.3. The Invariant Hilbert Integral

The invariance of *the Hilbert integral*

$$\int_{t_0}^{t_1} [l(x, p) + (\dot{x} - p)l_{\dot{x}}(x, p)]dt$$

means that

$$\Phi(x, \dot{x}) = l(x, p) + (\dot{x} - p)l_{\dot{x}}(x, p)$$

is an exact derivative, i.e. $\Phi = dS$ for a function S.

3.4. The Huyghens Algorithm

Suppose that the Hilbert integral is invariant, i.e. $\Phi = dS$. We have

$$\begin{aligned}J(x) = \int_{t_0}^{t_1} l(x(t), \dot{x}(t))dt &\geq \int_{t_0}^{t_1} \Phi(x(t), \dot{x}(t))dt =\\ = S(x(t_1)) - S(x(t_0)),&\end{aligned}$$

with equality when $\dot{x} = p$. We called p a *geodesic slope.*

The Huyghens algorithm can be formulated as follows: any integral curve of geodesic slope is a light ray.

A *theorem of Malyusz*, which is not easy to prove, asserts that the last condition is equivalent to the requirement that a family of integral curves of $\dot{x} = p$ satisfies Euler's equation.

Thus, for a Lagrangian l, convex with respect to $\dot{x}$ at $\dot{x} = p$, the following statements are equivalent:

- A family of integral curves of $\dot{x} = p$ are light paths.
- A family of integral curves of $\dot{x} = p$ are curves of geodesic slope.
- A family of integral curves of $\dot{x} = p$ satisfies Euler's equation.

4. Globally Minimal Currents

4.1. The Multidimensional Huyghens Principle

The above result on light rays can be extended to the multidimensional case of globally currents:

Let J be a convex integrand, defined by a Lagrangian l and bounded above in a neighborhood of some point. A necessary and sufficient condition for a k-current S to be absolutely (resp. homologically) minimal with respect to J is the existence of an exact (resp. closed) differential k-form ω such that

$$\begin{aligned} l(\xi) &\geq \omega(\xi), \quad \forall \xi \in \mathcal{G}_k M \\ l(\vec{S}_x) &= \omega(\vec{S}_x) \quad \text{for} \quad \|S\|\text{-almost all} \quad x \in M \end{aligned}$$

Proof. Sufficiency: for any k-current $T : \partial T = \partial S$ $(T - S = \partial X)$, we have

$$\begin{aligned} J(T) &= \int l(\vec{T}_x) d\|T\|(x) \geq \int \omega(\vec{T}_x) d\|T\|(x) \\ &= T(\omega) = S(\omega) = \int \omega(\vec{S}_x) d\|S\|(x) \\ &= \int l(\vec{S}_x) d\|S\|(x) = J(S). \end{aligned}$$

The necessity can be established by means of deep results of the modern analysis such as the Hahn-Banach theorem and the well-known property that the space of differential k-forms is reflexive.

4.2. Globally Minimal Currents in Symmetric Problems

- *Problems with invariant integrands*

The action of a Lie group G on a manifold M induces a corresponding action of G over differential k-forms ω, k-currents S, Lagrangians l and integrands J as well as notions of differential k-form, k-current, Lagrangian and integrand invariant under the action of G.

The averages with respect to G of a differential k-form ω and a k-current S are denoted by $\Pi_g\omega$ and $\Pi_G S$ respectively:

$$\Pi_G\omega = \int_G g^*\omega dg; \quad \Pi_G S = \int_G g_* S dg$$

We observe that:

- The operator Π_G commutes with the operators ∂ and d; therefore, Π_G preserves boundaries and homology (cohomology) classes.
- If J is lower semicontinuous, then Π_G decreases the mass and the comass.

Therefore, for a problem with an invariant integrand J we can find globally minimal currents S among invariant currents and supporting forms ω among invariant forms.

• *Problems with covariantly constant Lagrangians.*

A Lagrangian l (resp. a differential k-form ω, a k-current, a field of k-vectors) is said to be *covariantly constant (parallel)* if l is invariant with respect to any parallel translation (along any path).

A k-vector (or a k-covector) ξ at a point $x \in M$ is said to be Ψ_x-invariant if ξ is invariant under the action of the holonomy group Ψ_x of M at the point x, i.e. ξ withstands any parallel translation along any loop passing through x.

By using the notions of covariantly constant (parallel) Lagrangians, differential k-forms, fields of k-vector and currents as well as the notions of Ψ_e-invariant k-vectors and k-covectors, we can verify the following statements:

- For any covariantly constant (parallel) field of k-vectors ξ the k-current S_ξ, defined by

$$S_\xi(\varphi) = \int_M (\xi, \varphi) dx, \quad \text{for any} \quad k\text{-form } \varphi,$$

is covariantly constant (parallel), closed and homologically minimal.

- We can choose supporting differential k-forms ω among covariantly constant (parallel) differential k-forms.

5. Calibrations and Volume-Minimizing Surfaces

Suppose that $J = M(= \mathrm{vol}_k)$ is the mass (volume). In this case, the supporting k-form ω is called a *calibration* and the multidimensional Huyghens principle is also called *the calibration principle*. Globally minimal surfaces, determined by a calibration ω, are called ω-*calibrated* surfaces.

For the application of the calibration principle to find globally minimal surfaces, we need to investigate the set of *maximal directions* of a calibration. Usually, every well investigated calibration is closely connected with a *strong structure* of the manifold M.

5.1. The Complex Structure C^n

• *The Kahler calibrations* are exterior powers of the *fundamental Kahler 2-form:*

$$\Omega_k = \frac{1}{k!}(dz_1 \wedge d\bar{z_1} + ... + dz_n \wedge d\bar{z_n})^k.$$

According to *Wirtinger's inequality* the maximal directions of Ω_k are complex k-planes. In particular, complex submanifolds are globally minimal.

• *The special Lagrangian calibration is*

$$RedZ = Re(dz_1 \wedge ... \wedge dz_n).$$

The maximal directions of $RedZ$ are *Lagrangian $n-$planes* (*purely real n-planes.*) In particular, Lagrangian submanifolds are globally minimal

• *The associative and the coassociative calibrations*

$$(Je_n)^* \wedge \Omega_{n-1} + RedZ$$
$$(Je_n)^* \wedge ImdZ + \Omega_n$$

on $[Je_n] \oplus C^{2n-1}$. The maximal directions of the associative (coassociative) calibration are maximal directions of each component.

5.2. The Quaternion Structure H^n

The *quaternion calibrations* are exterior powers of the *fundamental quaternion 4-form Ω:*

$$\Omega_k = \frac{1}{k!}\Omega^k.$$

The maximal directions of Ω_k are *quaternion $k-$planes*. Inparticular, *quaternion submanifolds* are globally minimal.

5.3. The Octonionic Structure 0

The Cayley calibration

$\phi_1(x, y, z, w) =< x, y \times z \times w >$ on 0, which is equal to

$$1^* \wedge [(Je_4)^* \wedge \Omega + RedZ] + [(Je_4)^* \wedge ImdZ - \Omega_2]$$

on $[1] \oplus [Je_4] \oplus C^3 = 0$. The maximal directions are maximal directions of each component.

When $n = 2$ the associative and the coassociative calibrations on $Im0$ can be expressed in the following ways:

$$\phi_2(x \wedge y \wedge z) = \langle x, yz \rangle$$
$$\phi_3(x \wedge y \wedge z \wedge w) = \langle x, [x, y, z] \rangle.$$

5.4. The Structure of a Symmetric Space

Some *biinvariant calibrations* on Lie groups determine homologically minimal *Lie subgroup, Cartan's models* and *Pontriagin's cycles.*

6. Some Open Problems

6.1 Let W be a k-dimensional closed compact smooth manifold, M a closed compact Riemannian manifold. Does there always exist a smooth mapping f in any homotopy class of mappings from W into M such that the k-dimensional volume (stratified volume) of the image $f(W)$ is the smallest?

6.2. To classify (in a suitable meaning) globally minimal Lie subgroups, Cartan's models and Pontriagin's cycles in a Lie group.

6.3. To classify invariant calibrations in a homogeneous space (the case of a Lie group or a symmetric space is already clear: in each homology class there exists a unique invariant calibration ω, corresponding to the homologically minimal invariant current S_ω.

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