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Introduction to Quantum Cohomology

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Abstract. We begin with some general remarks on ordinary cohomology theory, which is a basic concept in modern geometry with many applications. We review briefly its origin and purpose, and the way in which it is used by researchers. Quantum cohomology is much more recent and less developed, but it already has deep relations with physics, differential geometry, and algebraic geometry, as well as topology. We give an informal definition of quantum cohomology, with some concrete examples, and conclude with a discussion of the quantum differential equations which are responsible for this deeper structure.

We begin with some brief remarks on homology and cohomology, as motivation for readers who do not use homology and cohomology as everday tools. We shall focus on those aspects which will reappear when we study quantum cohomology.

The basic process of algebraic topology, and one of the great mathematical ideas of the 20th century, can be described as

$$\text{geometric object} \longrightarrow \text{algebraic object}$$

where, for example, the "geometric object" could be a surface in three-dimensional Euclidean space and the "algebraic object" could be a number or a collection of numbers. In more abstract terms, the geometric object could be a manifold and the algebraic object could be a group. Many such constructions have been discovered, but in each case the construction is "topologically invariant" in the sense that topologically equivalent geometric objects produce algebraically equivalent algebraic objects (where the concept of equivalence is defined appropriately in each case).

One of the most famous examples is the Euler characteristic. If the geometric object M is a polyhedral surface constructed from v vertices, e edges and f faces, then the Euler characteristic $\chi(M)$ is the (positive or negative) integer

$$\chi(M) = v - e + f.$$

It can be proved that this number is the same even if the surface is subdivided in a different way (unlike, for example, $v - 2e + 5f$, which does not have this property). Furthermore, it can be proved that this number is the same even if the surface is "deformed" continuously "without tearing" (for example, if a two-dimensional polyhedral cylinder is squashed into a one-dimensional polyhedral ring).

Another famous example is the fundamental group; to any topological space M one can associate a group called $\pi_1(M)$. It is a well known theorem that if two such topological spaces are homeomorphic (or homotopy equivalent), then their fundamental groups are isomorphic.

The rationale for studying such topological invariants is this: it is important in many parts of mathematics to distinguish between geometric objects (ideally, one attempts to classify all geometric objects of a certain type); on the other hand, algebraic objects are much easier to distinguish (and classify) than geometric objects; therefore, a topological invariant gives a way of converting a difficult geometric problem to an easier algebraic problem.

For example, in the case of (closed orientable) surfaces, the sphere and the torus are distinguished by the Euler characteristic in the sense that $\chi = 2$ for the sphere but $\chi = 0$ for the torus. This example illustrates the main idea of "geometric" algebraic topology very simply but it also illustrates its unavoidable awkwardness. One should be careful from the start about the precise definition of geometrical object, such as polyhedral surface or topological space or manifold. But the primary source of awkwardness is the experimental fact that any construction which is reasonably understandable seems to need a lot of additional (but ultimately irrelevant) data, such as the choice of subdivision into vertices, edges and faces. In subsequent developments of the theory, one is continually burdened with the task of proving that the choice of additional data is really irrelevant.

1. Simplicial Homology

Let us illustrate the above remarks in the case of simplicial homology theory. This is a family of topological invariants which is directly related to the Euler characteristic described earlier; the point is that, underlying the number $\chi(M)$, there is a more sophisticated object, the homology group H_*M. The definition of the group H_*M involves three steps.

Step 1. We assume that the geometric object M is a particular kind of topological space called a simplicial complex. The definition of simplicial complex can be found in books on algebraic topology; it formalizes the idea of dividing into vertices, edges and faces. The basic unit is an i-dimensional simplex, or i-simplex (a vertex would be a 0-simplex, an edge would be a 1-simplex, and so on). Roughly speaking, a simplicial complex is an object formed when simplices of various dimensions are attached together in a certain specified way.

Step 2. For each i, the i-th chain groups C_iM are constructed. An element of

C_iM is simply a linear combination $\sum_j n_j A_j$ where the n_j are integers and the A_j are i-simplices of M. This is just an abstract concept but one should have in mind the set-theoretic union, where A_j is counted n_j times (although this is not quite as simple as it seems, since n_j could be negative).

Step 3. The i-th homology groups H_iM are constructed. By definition, H_iM is the quotient group of

(a) the subgroup of C_iM consisting of linear combinations with zero boundary (these are called "cycles")

by

(b) the subgroup of C_iM consisting of elements which are boundaries of elements of $C_{i+1}M$ (these are called "boundaries").

Roughly speaking, the rank of the group H_iM (called the i-th Betti number of M, and denoted by b_i) is the number of $i+1$-dimensional "holes" in M. A more accurate description is the number of (essentially different) i-dimensional cycles which are not boundaries of $i+1$-dimensional chains.

The total homology group H_*M is defined to be the group-theoretic direct sum

$$H_*M = H_0M \oplus H_1M \oplus \dots$$

of the individual homology groups.

It turns out that the homology groups H_iM are topological invariants in the same sense as the Euler characteristic (although the chain groups C_iM are not). Moreover, they are related to the Euler characteristic by the formula

$$\chi(M) = b_0 - b_1 + b_2 - \dots$$

In fact, there is another formula

$$\chi(M) = c_0 - c_1 + c_2 - \dots$$

where c_i is the rank of the group C_iM. From the definition of C_iM, c_i is simply the number of i-simplices in M, so this formula directly generalizes the earlier naive definition of the Euler characteristic.

To prove all these statements involves quite a lot of work, and this "simplicial homology theory" was an early triumph of algebraic topology. Nevertheless, the theory is rather intuitive, and (at least for simple examples) it is easy to work with. For example, the two-dimensional torus has Betti numbers given by $b_0 = 1$, $b_1 = 2$, $b_2 = 1$, and $b_i = 0$ for $i \geq 3$. The torus has essentially one 2-dimensional cycle, namely the torus itself, and two essentially different kinds of 1-dimensional cycles, namely the two essentially different circles (e.g. $S^1 \times 1$ and $1 \times S^1$ if the torus is regarded as $S^1 \times S^1$). There is only one kind of 0-dimensional cycle, namely a point, and all such cycles are equivalent since any two points can be regarded as the boundary of a path on the torus between them.

2. Simplicial Cohomology

We have seen how the Euler characteristic can be enhanced to a more sophisticated invariant, the homology groups. There is no reason to stop here, though,

as more refined algebraic objects give stronger invariants. The next step is to introduce a product structure, which corresponds geometrically to intersection of cycles. Roughly speaking, we obtain a map

$$H_{m-i}M \times H_{m-j}M \to H_{m-(i+j)}M, \quad (A, B) \mapsto A \cap B$$

where A, B are, respectively, cycles of dimension $m - i, m - j$. More precisely, A, B are equivalence classes of cycles, since homology groups are quotients of cycle groups, so it has to be verified that $A_1 \cap B_1$ is equivalent to $A_2 \cap B_2$ if A_1 is equivalent to A_2 and B_1 is equivalent to B_2.

However, there is a much more serious problem, since there is absolutely no guarantee that the intersection $A \cap B$ will have dimension $m - (i + j)$, and in fact it is easy to produce counterexamples. Resolving this problem leads to even more technical difficulties, and this is the end of the road for simplicial homology theory as a practical tool – while simplicial homology is still a good way to introduce homology theory, introductory textbooks no longer attempt to develop the product structure this way. Nevertheless, the intuition behind the above definition is absolutely correct, as the codimension of $A \cap B$ will be equal to the sum of the codimensions of A and B *providing* that A and B intersect "transversally". This is the reason for writing the dimensions of A, B as $m - i, m - j$; if m is the dimension of M then i, j are the codimensions of A, B and the codimension of the transversal intersection $A \cap B$ is $i + j$.

It is easier from the technical point of view to use "dual" homology groups, or cohomology groups, denoted H^iM. Therefore, in the usual approach, one introduces a product

$$H^iM \times H^jM \to H^{i+j}M$$

in simplicial cohomology theory, and (after a considerable amount of work) one ends up with an algebraic invariant

$$H^*M = H^0M \oplus H^1M \oplus \dots$$

which is a ring or an algebra (not just a group).

3. Other Versions of Homology and Cohomology

Despite the technical difficulties of setting up the rigorous theory of simplicial cohomology, the ideas involved are sound, and more efficient ways have been found to implement them. One of the standard approaches is singular (homology or) cohomology. Instead of constructing a topological space directly from simplices and then considering the structure of all i-simplices in that space, one takes any topological space M and considers all possible continuous maps of a fixed i-simplex into M. Such a map is regarded as a singular i-simplex (the image may not look like an i-simplex at all; if the map is constant, then its image is just a single point). This situation is somewhat harder to think about, since the number of such singular simplices is uncountably infinite, but the constructions of the simplicial theory can still be carried out, and many of them become easier. For example it is easy to show that a continuous map between topological

spaces induces a homomorphism of chain groups, whereas in the simplicial theory a messy argument (the simplicial approximation theorem) is needed in order to convert the continuous map to an approximating map which takes simplices to simplices.

There are other, quite different approaches, for example Cech cohomology, and de Rham cohomology. These are all very convenient in their own way, but less intuitive than the simplicial theory. In the case of de Rham cohomology, one starts with a manifold M, rather than a general topological space, and then defines an i-cochain to be a (smooth) differential form of degree i, i.e. something which can be written using a local coordinate system $x_1, \ldots, x_m$ in the form

$$\sum_{i_1,\ldots,i_k} f_{i_1,\ldots,i_k}(x_1,\ldots,x_m) dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

Instead of the boundary operator (which assigns to a simplicial chain its boundary chain), one has the exterior derivative operator. A differential form with exterior derivative zero is called a cocycle, and a differential form which is equal to the exterior derivative of another one is called a coboundary. The i-th de Rham cohomology group is defined to be the quotient of the group of i-cocycles by the group of i-coboundaries. It is denoted by exactly the same notation as before, $H^i M$. To be more precise, we should write $H^i(M;\mathbf{R})$ to indicate that the "scalars" are the real numbers; in the case of simplicial cohomology the scalars are initially regarded as the integers, so we should write $H^i(M;\mathbf{Z})$ in that case.

The reason for using the same notation is that the simplicial cohomology groups turn out to be isomorphic to the de Rham cohomology groups, at least if the manifold M is compact (any compact manifold turns out to be homeomorphic to a simplicial complex, so the simplicial cohomology groups of M make sense). This is clearly a very remarkable fact, and its proof is nontrivial. However, there is a simple and fundamental connection between the two theories, because a de Rham i-cocycle can be *integrated* over a simplicial i-chain, so a differential form can be regarded as a function on the space of cycles, and this is exactly what a cocycle should be. More precisely, there is a natural "pairing"

$$_{\text{de Rham}}H^i(M;\mathbf{R}) \times {}_{\text{simplicial}}H_i(M;\mathbf{R}) \to \mathbf{R}$$

given by integration. For a cycle which is the boundary of another chain, the integral of any differential form is zero, by Stokes' theorem. Conversely, it is possible to show that any cycle over which the integrals of all differential forms are zero must be a boundary. Hence the pairing is nondegenerate, and it gives rise to an isomorphism between the vector space ${}_{\text{de Rham}}H^i(M;\mathbf{R})$ and the dual of the vector space ${}_{\text{simplicial}}H_i(M;\mathbf{R})$. The latter is simply ${}_{\text{simplicial}}H^i(M;\mathbf{R})$, so we obtain the desired isomorphism.

Even more remarkably, all other sensible definitions of homology and cohomology theory give the same answers, at least when M has a certain specified type. This indicates that the concept is more general than the original geometrical definition suggests, and it suggests an axiomatic approach to the subject. It can be shown that any (co)homology theory satisfying the famous Eilenberg-Steenrod axioms leads to isomorphic groups $H_i M$, $H^i M$. Thus, in order to show that simplicial and de Rham cohomology (or any other of the standard

theories) give the same answers it suffices to show that both theories satisfy the Eilenberg-Steenrod axioms.

4. How to Think About Homology and Cohomology

Relying on the above foundations of algebraic topology, most mathematicians nowadays think of homology and cohomology in a rather general way, without specifying the precise theory that they have in mind. This causes no difficulties so long as the spaces involved belong to well defined types (such as compact manifolds, or simplicial complexes, or CW-complexes) to which the Eilenberg-Steenrod axioms apply. For example, one says that the first cohomology group of the torus is $\mathbf{Z}^2$ (or, with real coefficients, $\mathbf{R}^2$), without really thinking about simplices or differential forms. This flexibility is very helpful, because one has the freedom to revert to a specific theory when working on a specific problem.

For example, the product structure of de Rham cohomology theory is very easy to describe: it is given by the exterior product of differential forms. It is easier to work with the exterior product $\alpha \wedge \beta$ of differential forms on M than to work with the intersection $A \cap B$ of cycles in M, because one does not have to worry about specifying that the cycles are transversal. If M is a compact orientable manifold, the Poincaré duality theorem says that

$$H^i M \cong H_{m-i} M,$$

a fact which is not at all clear from the point of view of simplicial theory. From the point of view of de Rham theory, the theorem says that the natural pairing

$$H^i M \times H^{m-i} M \to \mathbf{R}, \quad ([\alpha], [\beta]) \mapsto \int_M \alpha \wedge \beta$$

is nondegenerate, which is (also not obvious but) more plausible than the ingenius explanations of Poincaré duality which can be found in textbooks on simplicial theory.

Another example is the way that one thinks about cycles in a manifold M as submanifolds (or, at least, submanifolds with singular points). To be completely rigorous one should specify a theory such as simplicial theory, then regard the submanifold as approximated by small simplices, and one should also keep in mind that a homology class is an equivalence class of such objects. But it is convenient to think of the 1-dimensional homology group $\mathbf{Z}$ of the cylinder $S^1 \times [0,1]$ as generated by the 1-cycle represented by the submanifold $S^1 \times \frac{1}{2}$, for example. Similarly, the circles $S^1 \times 1$ and $1 \times S^1$ represent generators of the 1-dimensional homology group $\mathbf{Z}^2$ of the torus, and the product of these homology classes is represented by the intersection of these two circles, namely a single point, which in turn represents a generator of the 2-dimensional homology group $\mathbf{Z}$.

As a final example, the $2i$-th homology group $\mathbf{Z}$ of the $2n$-dimensional complex projective space $\mathbf{C}P^n$ is generated by the $2i$-cycle represented by the submanifold $\mathbf{C}P^i$ of $\mathbf{C}P^n$. In the case of complex algebraic varieties this way of

thinking can be made into a rigorous theory (see the next section): every cohomology class can be represented by a cycle which is an algebraic subvariety, and all basic operations can be carried out directly with such subvarieties.

5. Notation

In this section we summarize the notation for cohomology theory which we shall use. In view of the previous remarks we shall talk about homology classes as though they are geometrical objects like submanifolds, and we shall use the language of differential forms whenever it is helpful in describing cohomology classes and their properties.

We shall always be concerned with complex manifolds (because quantum cohomology involves holomorphic maps), and always with even-dimensional cohomology groups (because the product of such classes is commutative):

Assumption. M is a connected simply connected compact Kähler manifold, of complex dimension n. The integral cohomology groups of M are even-dimensional and torsion-free, i.e.

$$H^*(M;\mathbf{Z}) = \bigoplus_{i=0}^{n} H^{2i}(M;\mathbf{Z}) \cong \bigoplus_{i=0}^{n} \mathbf{Z}^{m_{2i}}$$

where $m_i = \text{rank } H^i(M;\mathbf{Z})$.

Let

$$\text{PD} : H^i(M;\mathbf{Z}) \to H_{2n-i}(M;\mathbf{Z})$$

be the Poincaré duality isomorphism. Generally we shall use lower-case letters

$$a, b, c, \cdots \in H^*(M;\mathbf{Z})$$

for cohomology classes, and $|a|, |b|, |c|, \ldots$ for their degrees. We shall use upper-case letters

$$A = \text{PD}\,(a), B = \text{PD}\,(b), C = \text{PD}\,(c), \ldots \ \in \ H_*(M;\mathbf{Z})$$

for the Poincaré dual homology classes, and $|A|, |B|, |C|, \ldots$ for their degrees. As explained earlier, we shall often speak of $A, B, C, \ldots$ as though they are cycles (rather than equivalences classes of cycles), and we shall often think of them as submanifolds or subvarieties of M. Similarly, we shall regard $a, b, c, \ldots$ as differential forms on M. The fundamental homology class of the manifold M, i.e. the homology class represented by the cycle M, is an element of $H_{2n}(M;\mathbf{Z})$, but its Poincaré dual cohomology class – the identity element of the cohomology algebra – will be denoted by $1 \in H^0(M;\mathbf{Z})$ (rather than m).

Let

$$\langle\ ,\ \rangle : H^i(M;\mathbf{Z}) \times H_i(M;\mathbf{Z}) \to \mathbf{Z}$$

denote the natural pairing; and also the extended pairing

$$\langle\ ,\ \rangle : H^*(M;\mathbf{Z}) \times H_*(M;\mathbf{Z}) \to \mathbf{Z}$$

given by $\langle a, B\rangle = 0$ when $|a| \neq |B|$. In de Rham notation, $\langle a, B\rangle = \int_B a$. Since there is no torsion, these pairings are nondegenerate.

The *intersection pairing* is defined by

$$(\ ,\ ) : H^*(M;\mathbf{Z}) \times H^*(M;\mathbf{Z}) \to \mathbf{Z}, \quad (a,b) = \langle ab, M\rangle = \int_M a \wedge b.$$

We have $\langle ab, M\rangle = \langle a, B\rangle = \langle b, A\rangle$. It follows that the intersection pairing $(\ ,\ )$ is a nondegenerate symmetric bilinear form.

The product structure of quantum cohomology (quantum product) is a generalization of the product structure of cohomology (cup product), so we shall discuss our notation for this in some detail. Although some mathematicians might find this distasteful, it will be convenient for future purposes to specify the cup product by giving its "structure constants" with respect to a basis. Therefore we choose generators as follows

$$H_*(M;\mathbf{Z}) = \bigoplus_{i=0}^{s} \mathbf{Z}A_i, \quad H^*(M;\mathbf{Z}) = \bigoplus_{i=0}^{s} \mathbf{Z}a_i$$

and we define dual cohomology classes $b_0, \ldots, b_s$ by $(a_i, b_j) = \delta_{ij}$. Then for any i, j we have

$$a_i a_j = \sum_{i,j,k} \mu_{ijk} a_k = \sum \lambda_{ijk} b_k$$

for some $\mu_{ijk}, \lambda_{ijk} \in \mathbf{Z}$. These structure constants are given by

$$\mu_{ijk} = \langle a_i a_j b_k, M\rangle, \quad \lambda_{ijk} = \langle a_i a_j a_k, M\rangle.$$

The μ's and the λ's are of course equivalent, but it seems more elegant to focus on

$$\lambda_{ijk} = \langle a_i a_j a_k, M\rangle = \int_M a_i \wedge a_j \wedge a_k = \sharp\ A_i \cap A_j \cap A_k.$$

Note that the intersection form itself can be specified in a similar way; it is equivalent to the integers

$$(a_i, a_j) = \int_M a_i \wedge a_j = \sharp\ A_i \cap A_j.$$

To emphasize the geometrical point of view and the obvious symmetry it is natural to modify the notation as follows:

Definition 5.1. *For cohomology classes a, b, c we define $\langle A|B|C\rangle_0 = \langle abc, M\rangle = \int_M a \wedge b \wedge c = \sharp\ A \cap B \cap C$.*

The reason for the suffix zero is that the quantum product will be defined in terms of certain *Gromov-Witten invariants* $\langle A|B|C\rangle_D$, where D is not necessarily zero.

Let us reiterate that $\sharp\, A \cap B \cap C$, the number of points (counted with multiplicity) in the intersection $A \cap B \cap C$, has to be interpreted rather carefully: it is valid providing we use representative cycles A, B, C which intersect transversely. In the complex algebraic category, there is a simple criterion for this transversality condition: it holds automatically whenever there exist representative algebraic subvarieties A, B, C whose intersection is finite (or empty) — see the appendix of [5]. The most famous example where this method works is the case where M is a Grassmannian manifold or flag manifold. Here all the generators of the homology groups are representable by algebraic cycles (Schubert varieties), and for any three such generators a, b, c satisfying the condition $|a| + |b| + |c| = \dim M$ there exist representatives whose intersection is finite (or empty).

Example 5.1. We shall calculate the cohomology algebra $H^*\mathbf{C}P^n$ of complex projective space. We have $H^{2i}\mathbf{C}P^n \cong \mathbf{Z}x_i$ $(0 \leqslant i \leqslant n)$, where the Poincaré dual homology generator X_i (of degree $2n - 2i$) can be represented by $X_i = \boldsymbol{P}(V)$, for any complex linear subspace $V \subseteq \mathbf{C}^{n+1}$ of codimension i. As usual we write $x_0 = 1$.

First, we consider the intersection form. We have $(x_i, x_{n-j}) = \delta_{ij}$, as there exist linear subspaces V, W of $\mathbf{C}^{n+1}$ of codimensions $i, n - j$ such that $\boldsymbol{P}(V) \cap \boldsymbol{P}(W)$ is finite and nonempty if and only if $i = j$, and in this case the intersection consists of a single point (of multiplicity one).

Next, the product $x_i x_j$ must be of the form $\lambda_{ij} x_{i+j}$ when $i + j \leqslant n$ (and zero when $i + j > n$). Using the intersection form, we have $\lambda_{ij} = (x_i x_j, x_{n-(i+j)}) = \langle X_i | X_j | X_{n-(i+j)} \rangle_0$. To calculate this, we represent the three classes respectively by linear subspaces of $\mathbf{C}^{n+1}$ of codimensions $i, j, n - (i + j)$. If the subspaces are in general position, the codimension of the intersection is $i + j + n - (i + j) = n$, so this triple intersection is a line, and $X_i \cap X_j \cap X_{n-(i+j)}$ is a single point of $\mathbf{C}P^n$. Since we are taking intersections of linear subspaces, the multiplicity of this point is one. We conclude that $\lambda = 1$, and so $x_i x_j = x_{i+j}$.

The cohomology algebra of $\mathbf{C}P^n$ is therefore isomorphic to $\mathbf{C}[x_1]/\langle x_1^{n+1} \rangle$.

6. 3-point Gromov-Witten Invariants

We turn now to quantum cohomology. The motivation for quantum cohomology comes from symplectic geometry and from physics (see [4]). We shall just proceed directly to a definition, in which quantum cohomology is simply ordinary cohomology but with a new "quantum product" which extends the usual cup product. The structure constants of the quantum product are numbers denoted by $\langle A | B | C \rangle_D$, where D varies in $H_2(M; \mathbf{Z})$; when $D = 0$ we obtain the structure constants of ordinary cohomology. These numbers are called 3-point Gromov-Witten invariants. They are invariants of the symplectic structure of M, although this aspect will not play a major role here.

Recall that the definition of the triple product $\langle A | B | C \rangle_0$ is

$$\langle A|B|C\rangle_0 = \langle abc, M\rangle = \int_M a \wedge b \wedge c = \sharp\ A \cap B \cap C.$$

The usual definition of $\langle A|B|C\rangle_D$ extends this, although it is more difficult because the space M has to be replaced by a certain moduli space (of J-holomorphic curves in the symplectic category, or stable curves in the algebraic category). The construction of this moduli space is complicated, and integration is problematical because the moduli space is not in general compact. Instead of this, we shall just use a naive definition of $\langle A|B|C\rangle_D$, which extends the naive geometrical formula

$$\langle A|B|C\rangle_0 = \sharp\ A \cap B \cap C.$$

Let M be a complex manifold. We define

$$\langle A|B|C\rangle_D = \sharp\ \mathrm{Hol}_D^{A,p} \cap\ \mathrm{Hol}_D^{B,q} \cap\ \mathrm{Hol}_D^{C,r}$$

where

$$\mathrm{Hol}_D^{A,p} = \{\text{holomorphic maps } f : \mathbf{C}P^1 \to M | f(p) \in A \text{ and } [f] = D\}.$$

The points p, q, r are three distinct basepoints in $\mathbf{C}P^1$. The notation $[f]$ denotes the homotopy class of f, which is an element of $\pi_2(M) \cong H_2(M; \mathbf{Z})$. As in the case $D = 0$, the definition employs specific cycles A, B, C, although it turns out that the value of $\langle A|B|C\rangle_D$ depends only on the homology class.

Evidently this definition inherits the defects of the naive definition of $\langle A|B|C\rangle_0$, to which it reduces when $D = 0$. We shall assume that these defects may be remedied in a similar way, at least for certain manifolds M. It is certainly plausible that, under mild restrictions, $\mathrm{Hol}_D^{M,p}$ is a complex manifold. Moreover, it should have complex dimension $n + \langle c_1(TM), D\rangle$, because the hypothetical tangent space at $f \in \mathrm{Hol}_D^{M,p}$ may be identified with the space of holomorphic sections of the bundle f^*TM, and by the Riemann-Roch theorem, the complex dimension of this vector space is $n + \langle c_1(TM), D\rangle$. (This argument is correct if M is *convex* in the sense that $H^1(\mathbf{C}P^1, f^*TM) = 0$ for all holomorphic $f : \mathbf{C}P^1 \to M$.) It is also plausible that $\mathrm{Hol}_D^{A,p}$ is a complex submanifold of $\mathrm{Hol}_D^{M,p}$, whose complex codimension in $\mathrm{Hol}_D^{M,p}$ is equal to the complex codimension of A in M. This suggests that, when A, B, C are tranversal, the intersection

$$\mathrm{Hol}_D^{A,p} \cap\ \mathrm{Hol}_D^{B,q} \cap\ \mathrm{Hol}_D^{C,r}$$

will be zero-dimensional when

$$\mathrm{codim}_{\mathbf{R}}\ \mathrm{Hol}_D^{A,p} +\ \mathrm{codim}_{\mathbf{R}}\ \mathrm{Hol}_D^{B,q} +\ \mathrm{codim}_{\mathbf{R}}\ \mathrm{Hol}_D^{C,r} =\ \dim_{\mathbf{R}}\ \mathrm{Hol}_D^{M,p}$$

i.e. when $|a| + |b| + |c| = 2n + 2\langle c_1(TM), D\rangle$. If M is a *Fano* manifold, in the sense that the cohomology class $c_1(TM)$ can be represented by a Kähler 2-form, then $\langle c_1(TM), D\rangle > 0$ for each homotopy class $D \in \pi_2(M)$ which contains a

holomorphic map $\mathbf{C}P^1 \to M$. Thus, for Fano manifolds, only a finite number of triple products $\langle A|B|C\rangle_D$ can be nonzero (when A, B, C belong to a given basis).

As explained in [4] (for example) it turns out that $\langle A|B|C\rangle_D$ can be defined rigorously under very general conditions. The definition has the form

$$\langle A|B|C\rangle_D \;=\; \int\limits_{[\bar{M}(D)]^{\mathrm{virt}}} ev_1^* a \wedge ev_2^* b \wedge ev_3^* c$$

where $M(D)$ is a certain moduli space of "curves", $\bar{M}(D)$ is a compactification of $M(D)$, obtained by adding suitable "boundary components", and $[\bar{M}(D)]^{\mathrm{virt}}$ denotes the "virtual fundamental class" over which integration is permitted. The evaluation map $ev_i : \bar{M}(D) \to M$ assigns to a curve its value at a given i-th basepoint. It has the following properties:
(1) $\langle A|B|C\rangle_D \in \mathbf{Z}$ is well defined for any $A, B, C \in H_*(M; \mathbf{Z})$, $D \in H_2(M; \mathbf{Z})$.
(2) $\langle A|B|C\rangle_D \in \mathbf{Z}$ is $\mathbf{Z}$-linear and symmetric in A, B, C.
(3) $\langle A|B|C\rangle_D \neq 0 \implies |a| + |b| + |c| = 2n + 2\langle c_1(TM), D\rangle$. (This "numerical condition" will be useful when performing concrete calculations.)
Any compact homogeneous Kähler manifold (i.e. generalized flag manifold) is convex and Fano. It can be shown that, in this case, the naive geometrical definition of $\langle A|B|C\rangle_D$ coincides with the rigorous definition. Moreover, the (generalized) Schubert classes provide a natural homology basis. Thanks to the Kleiman transversality theorem all Gromov-Witten invariants involving this basis may be computed as in the case $D = 0$ by using Schubert cycles.

Example 6.1. Let us compute the triple products $\langle A|B|C\rangle_D$ for the manifold $M = \mathbf{C}P^n$. In this case $N = n+1$, so a necessary condition for $\langle A|B|C\rangle_D \neq 0$ is $|a| + |b| + |c| = 2n + 2D(n+1)$. Since $0 \leqslant |a|, |b|, |c| \leqslant 2n$, it follows immediately that $D = 0$ and $D = 1$ are the only relevant values, i.e. that $\langle A|B|C\rangle_D = 0$ for $D \neq 0, 1$.

For $D = 0$ we already know that

$$\langle X_i|X_j|X_k\rangle_0 = \begin{cases} 1 & \text{if} \quad i + j + k = n \\ 0 & \text{otherwise.} \end{cases}$$

As we saw in Example 5.1, the calculation reduces to consideration of the intersection of three linear subspaces in $\mathbf{C}^{n+1}$. For $D = 1$ we claim that

$$\langle X_i|X_j|X_k\rangle_1 = \begin{cases} 1 & \text{if} \quad i + j + k = 2n + 1 \\ 0 & \text{otherwise.} \end{cases}$$

To prove this, one shows that, when $i + j + k = 2n + 1$, there exist complex linear subspaces E^i, E^j, E^k of $\mathbf{C}^{n+1}$, of codimensions i, j, k, with the following property: there exist unique complex lines L', L'', L''' such that $L' \subseteq E^i$, $L'' \subseteq E^j$, $L''' \subseteq E^k$ and such that L', L'', L''' span a subspace E of dimension 2. The holomorphic map of degree 1 defined by the inclusion $\boldsymbol{P}(E) \subseteq \mathbf{C}P^{n+1}$ is then the unique point of the triple intersection $\mathrm{Hol}_1^{\boldsymbol{P}(E^i),p} \cap \mathrm{Hol}_1^{\boldsymbol{P}(E^j),q} \cap \mathrm{Hol}_1^{\boldsymbol{P}(E^k),r}$, and we obtain $\langle X_i|X_j|X_k\rangle_1 = 1$. While elementary, this kind of messy calculation can be avoided by appealing to general properties. The basic point is that one

does not need to compute all the Gromov-Witten invariants one by one, as they are not independent. There are relations between them, and it turns out that a small subset of Gromov-Witten invariants generates the rest. The relations exist precisely because Gromov-Witten invariants are structure constants of the quantum product, as we shall see in the next section.

7. The Quantum Product

To define $a \circ_t b$ for $a, b \in H^*M$ and $t \in H^2M$, it suffices to define $\langle a \circ_t b, C\rangle$ for all $C \in H_*M$. The definition is:

Definition 7.1. *Assume that M is a Fano manifold. Then the quantum product $a \circ_t b$ of two cohomology classes $a, b \in H^*M$ is defined by*

$$\langle a \circ_t b, C\rangle = \sum_{D \in H_2(M;\mathbf{Z})} \langle A|B|C\rangle_D \, e^{\langle t, D\rangle}.$$

The Fano condition ensures that the sum is finite. Observe that as "$t \to -\infty$" the right hand side converges to $\langle A|B|C\rangle_0$; hence $a \circ_t b$ converges to the cup product ab. In this sense, the quantum product is a deformation of the cup product.

The main result concerning the quantum product is:

Theorem 7.1. *For each $t \in H^2M$, $\circ_t$ is a commutative, associative product operation on H^*M.*

The most difficult part of this theorem is the associativity; the other properties are obvious (intuitively, at least). Details can be found in [4].

There are various generalizations of the above definition, e.g. where the Fano assumption is weakened or where the parameter t is allowed to vary in a vector space larger than H^2M. We just mention a simple but convenient variant, which is a product operation $\circ$ on

$$H^*M \otimes \Lambda$$

where Λ is the group algebra $\mathbf{Z}[H_2(M;\mathbf{Z})]$. Formally, an element of Λ is a finite sum $\sum_X \lambda_X q^X$, where $\lambda_X \in \mathbf{Z}$, $X \in H_2(M;\mathbf{Z})$, and where the symbols q^D are multiplied in the obvious way, i.e. $q^D q^E = q^{D+E}$. The definition is:

Definition 7.2. *Let M be a Fano manifold. Then*

$$a \circ b = \sum_{D \in H_2(M;\mathbf{Z})} (a \circ b)_D \, q^D,$$

where $(a \circ b)_D$ is defined by $\langle (a \circ b)_D, C\rangle = \langle A|B|C\rangle_D$ for all $C \in H_*M$. The definition extends in a Λ-linear fashion to $H^*M \otimes \Lambda$. The algebra $(H^*M \otimes \Lambda, \circ)$ is called the quantum cohomology algebra, and we shall denote it by QH^*M.

By the Fano assumption, the sum on the right hand side of the definition of $a \circ b$ is finite.

It is convenient to introduce a grading on QH^*M by

$$|aq^D| = |a| + 2\langle c_1(TM), D\rangle.$$

This extends the usual grading on H^*M, and the quantum product preserves the grading in the sense that $|a \circ b| = |a| + |b|$.

For explicit calculations we shall choose bases as in our discussion of cohomology. It will be convenient to use bases for cohomology and homology which are dual in the sense that

$$\langle b_i, A_j\rangle = \int_{A_j} b_i = \delta_{ij}, \quad 0 \leqslant i, j \leqslant s.$$

Since the second (co)homology group plays a prominent role in quantum cohomology, we shall declare that

$$\dim H^2M = r,$$

and we make the following addition to the notation of Sec. 5:

Assumption. For the bases $A_0, \ldots, A_s$ of H_*M, and $b_0, \ldots, b_s$ of H^*M, as in Sec. 5, we have

$$H_0M = \mathbf{Z}A_0, \quad H^0M = \mathbf{Z}b_0$$

and

$$H_2M = \bigoplus_{i=1}^{r} \mathbf{Z}A_i, \quad H^2M = \bigoplus_{i=1}^{r} \mathbf{Z}b_i.$$

A general element of H^2M will be written $t = \sum_1^r t_i b_i \in H^2M$, and a general element of H_2M will be written $D = \sum_{i=1}^r D_i A_i$. If we introduce $q_i = q^{A_i}$, then $q^D = q_1^{D_1} \ldots q_r^{D_r}$.

For Fano manifolds the products $\circ_t$ and $\circ$ are equivalent, and there is no reason to prefer one or the other. For non-Fano manifolds, similar definitions can be made, but it will be necessary to consider whether the infinite series in the definition are convergent, and this is a nontrivial problem. To avoid a direct assault, various strategies can be used. For example, using $\circ$ rather than $\circ_t$, one can introduce the Novikov ring of series in q or one can simply consider all series as formal series. To avoid such complications we shall focus on the Fano case in our exposition, pointing out generalizations only when they are absolutely necessary. In practical terms, the relation between $\circ$ and $\circ_t$ is that the latter is obtained from the former by "replacing q^D by $e^{\langle t, D\rangle}$", or, after choosing a basis as above, by "putting $q_i = e^{t_i}$".

8. Examples of the Quantum Cohomology Algebra

We shall compute some standard examples in this section (more detailed versions of some of these computations can be found in [9]). In each case the procedure is as follows:

(i) Specify the cohomology algebra.
(ii) Choose a basis $A_0, \dots, A_s$ of H_*M. We shall always do this in such a way that a homology classes $D = \sum_1^r D_i A_i \in H_2 M$ is representable by a holomorphic map $f : \mathbf{C}P^1 \to M$ only if $D_1, \dots, D_r \geq 0$. In our examples, this property will be verifiable by direct computation.
(iii) Identify the dual basis $b_0, \dots, b_s$ of H^*M such that $\langle b_i, A_j \rangle = \delta_{ij}$.
(iv) Compute $|q_i| = 2\langle c_1(TM), A_i \rangle$. For this we need to know the first Chern class $c_1(TM)$ in terms of $b_1, \dots, b_r$.
(v) Compute the Gromov-Witten invariants and the quantum product.

Example 8.1. Using the notation from Example 6.1 for $M = \mathbf{C}P^n$, let us choose

$$A_i = X_{n-i}, \quad \text{hence} \quad b_i = x_i.$$

Instead of $q_1 = q^{A_1}$ we shall just write q.

We shall calculate the quantum products $x_i \circ x_j$. We use $\circ$ rather than $\circ_t$ because the grading tells us immediately that the form of the answer is

$$x_i \circ x_j = (x_i \circ x_j)_0 + (x_i \circ x_j)_1 q.$$

(For dimensional reasons, no higher powers of q can occur.) The values of $\langle X_i | X_j | X_k \rangle_D$ from Example 6.1 allow us to calculate $(x_i \circ x_j)_0$ and $(x_i \circ x_j)_1$:

$$\langle (x_i \circ x_j)_0, X_k \rangle = \langle X_i | X_j | X_k \rangle_0 = \begin{cases} 1 & \text{if} \quad i+j+k = n \\ 0 & \text{otherwise.} \end{cases}$$
$$\langle (x_i \circ x_j)_1, X_k \rangle = \langle X_i | X_j | X_k \rangle_1 = \begin{cases} 1 & \text{if} \quad i+j+k = 2n+1 \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$x_i \circ x_j = \begin{cases} x_{i+j} & \text{if} \quad 0 \leqslant i+j \leqslant n \\ x_{i+j-(n+1)} q & \text{if} \quad n+1 \leqslant i+j \leqslant 2n. \end{cases}$$

In particular $QH^*\mathbf{C}P^n \cong \mathbf{Z}[x_1, q]/\langle x_1^{n+1} - q \rangle$.

Before leaving this computation, let us mention two special cases. First, if $i + j \leqslant n$, the term involving q cannot occur, because $|q| > 2n \geq |x_i| + |x_j|$, so we must have $x_i \circ x_j = x_i x_j = x_{i+j}$ (just the cup product). Next, consider the simplest case where $i+j = n+1$, namely $x_1 \circ x_n$. Since $|(x_1 \circ x_n)_0| \leqslant 2n$, this term must be zero, so $x_1 \circ x_n = \lambda q$ where $\lambda = \langle X_1 | X_n | X_n \rangle_1$. Now, $\langle X_1 | X_n | X_n \rangle_1$ is the number of linear maps $\mathbf{C}P^1 \to \mathbf{C}P^n$ which "hit" generic representatives of X_1 (a hyperplane), X_n (represented by a point not on the hyperplane), X_n (represented by another point not on the hyperplane) at three prescribed points of $\mathbf{C}P^1$. There is precisely one such map, namely the (complex, projective) line through the two points (which automatically hits the hyperplane in one point), so $\lambda = 1$ and $x_1 \circ x_n = q$. From these two cases, it turns out all other quantum products $x_i \circ x_j$ may be deduced, e.g. $x_2 \circ x_n = x_1^2 \circ x_n = (x_1 \circ x_1) \circ x_n = x_1 \circ (x_1 \circ x_n) = x_1 q$. Thus, essentially the only Gromov-Witten invariant which has to be computed "by hand" is $\langle X_1 | X_n | X_n \rangle_1$, and this is 1 because there exists one line through two distinct points.

Example 8.2. Let

$$F_3 = F_{1,2}(\mathbf{C}^3) = \{(L,V) \in \ \mathrm{Gr}_1(\mathbf{C}^3) \times \ \mathrm{Gr}_2(\mathbf{C}^3) \ \text{ such that } \ L \subseteq V\}$$
$$\cong U_3/U_1 \times U_1 \times U_1$$

be the (full) flag manifold of the unitary group U_3. The "Borel description" of $H^*(F_3;\mathbf{Z})$ is

$$H^*(F_3;\mathbf{Z}) \cong \mathbf{Z}[x_1,x_2,x_3]/\langle \sigma_1,\sigma_2,\sigma_3\rangle$$

where $\sigma_1,\sigma_2,\sigma_3$ are the elementary symmetric functions of x_1,x_2,x_3. Geometrically, $x_i = -c_1(\mathcal{L}_i)$, where $\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_3$ are the complex line bundles on F_3 whose fibres over (L,V) are L, $L^\perp \cap V$, $V^\perp$ respectively. On the other hand the "Schubert description" of $H_*(F_3;\mathbf{Z})$ involves specific cycles, the Schubert varieties, which generalize the subvarieties $\mathbf{C}P^i$ of $\mathbf{C}P^n$. With respect to a fixed reference flag $E_1 \subseteq E_2 \subseteq \mathbf{C}^3$ there are six Schubert varieties. To compute the intersection of two Schubert homology classes (hence the products of the Poincaré dual cohomology classes) one chooses two reference flags so that the (set-theoretic) intersection of the two Schubert varieties is either empty or another Schubert variety.

Excluding the trivial cases "F_3" and "a point", we list the Schubert varieties below.

(1) $\{L \subseteq E_2\} \ = \ \mathrm{PD}\,(a) \ = \ A \ \in \ H_4(F_3;\mathbf{Z})$
(2) $\{E_1 \subseteq V\} \ = \ \mathrm{PD}\,(b) \ = \ B \ \in \ H_4(F_3;\mathbf{Z})$
(3) $\{L = E_1\} \ = \ \mathrm{PD}\,(a^2) \ = \ A^2 \ \in \ H_2(F_3;\mathbf{Z})$
(4) $\{V = E_2\} \ = \ \mathrm{PD}\,(b^2) \ = \ B^2 \ \in \ H_2(F_3;\mathbf{Z}).$

Here, $\{L \subseteq E_2\}$ is an abbreviation for $\{(L,V) \in F_3 | L \subseteq E_2\}$, etc. The notation A^2 is simply an abbreviation for $PD(a^2)$, etc. With this notation, the Schubert variety "a point" would be $\{L = E_1, V = E_2\} = PD(a^2b) = A^2B$ or $PD(ab^2) = AB^2$.

Let us choose the basis $A_0,\ldots,A_5$ as follows:

$H_0(F_3;\mathbf{Z})$	$H_2(F_3;\mathbf{Z})$	$H_4(F_3;\mathbf{Z})$	$H_6(F_3;\mathbf{Z})$
$A_0 = \text{point}$	$A_1 = B^2$	$A_3 = B$	$A_5 = F_3$
	$A_2 = A^2$	$A_4 = A$	

The dual basis $b_0,\ldots,b_5$ is:

$H^0(F_3;\mathbf{Z})$	$H^2(F_3;\mathbf{Z})$	$H^4(F_3;\mathbf{Z})$	$H^6(F_3;\mathbf{Z})$
$b_0 = 1$	$b_1 = a$	$b_3 = a^2$	$b_5 = a^2b = ab^2$
	$b_2 = b$	$b_4 = b^2$	

The remaining cup products are determined by $ab = a^2 + b^2$ (which implies $a^3 = 0, b^3 = 0$); this equation can also be verified by considering Schubert varieties. The relation with the Borel description is that $a = x_1$, $b = x_1 + x_2$.

Geometrically, $a = -c_1(\mathcal{L})$ and $b = -c_1(\mathcal{V})$, where $\mathcal{L}$, $\mathcal{V}$ are the holomorphic "tautological" bundles whose fibres over (L, V) are L, V respectively.

In particular we have:

$$H_2(F_3; \mathbf{Z}) = \mathbf{Z}B^2 \oplus \mathbf{Z}A^2 = \mathbf{Z}\{V = E_2\} \oplus \mathbf{Z}\{L = E_1\}.$$

If $[f] = D_1A_1 + D_2A_2$ then it follows that

$$D_1 = f^*a = -c_1 f^*\mathcal{L}, \quad D_2 = f^*b = -c_1 f^*\mathcal{V}$$

It is easily verified that there exists a holomorphic map f such that $[f] = D_1A_1 + D_2A_2 = (D_1, D_2)$ if and only if either (a) $D_2 \geq D_1 \geq 0$, or (b) $D_2 = 0, D_1 \geq 0$.

It is known that $c_1(TF_3) = 2a + 2b$. Therefore, for $q_1 = q^{A_1}$ and $q_2 = q^{A_2}$ we have

$$|q_1| = 2\langle 2a + 2b, B^2\rangle = 4, \quad |q_2| = 2\langle 2a + 2b, A^2\rangle = 4.$$

To calculate the quantum product $\circ$, we begin by investigating the numerical condition for $\langle X|Y|Z\rangle_D \neq 0$. We have $\langle c_1(TF_3), D\rangle = 2D_1 + 2D_2$, so the numerical condition is

$$|x| + |y| + |z| = 6 + 4D_1 + 4D_2.$$

Since $\mathrm{Hol}^{F_3,p}_{D_1,D_2}$ is empty when either of D_1 or D_2 is negative, we have

$$x \circ y = \sum_{D_1,D_2 \geq 0} (x \circ y)_{D_1,D_2}\ q_1^{D_1} q_2^{D_2}.$$

Since the degree $|(x \circ y)_{D_1,D_2}|$ is given by $|x| + |y| - 4D_1 - 4D_2$, and this must be 0, 2, 4, or 6, the relevant values of (D_1, D_2) are severely restricted. We shall give one example:

Proposition 8.1. $a \circ a = a^2 + q_1$.

Proof. We have $a \circ a = (a \circ a)_{0,0} + (a \circ a)_{1,0}q_1 + (a \circ a)_{0,1}q_2$. Now, $(a \circ a)_{0,0}$ is necessarily a^2, so it remains to calculate the degree 0 cohomology classes $\lambda = (a \circ a)_{1,0}$ and $\mu = (a \circ a)_{0,1}$.

By definition, $\lambda = \langle A|A|Z\rangle_{1,0}$, where Z denotes the generator of $H_0(F_3; \mathbf{Z})$. Any holomorphic map of degree $(1, 0)$ is of the form $\boldsymbol{P}(H) \to F_3$, $L \mapsto (L, H)$, where H is a fixed two-dimensional subspace of $\mathbf{C}^3$, so we shall attempt to count the number of such maps which hit subvarieties of the form

$\{L \subseteq E_2'\}$ (representing A)
$\{L \subseteq E_2''\}$ (also representing A)
(E_1, E_2) (a single point of F_3)
at three distinct points.

The logic behind this argument is somewhat convoluted so we reiterate it. We are attempting to find a line E_1 and planes H, E_2', E_2'', E_2 such that the holomorphic map determined by H hits each of the three subvarieties defined by E_1, E_2', E_2'', E_2. With E_1, E_2', E_2'', E_2 fixed, we count the number of such H. If this number is zero or finite, then it must be equal to λ (in particular, it is independent of the choice of E_1, E_2', E_2'', E_2). If it is infinite, then we can conclude

nothing — except that this naive method of computing λ is inadequate. In fact, thanks to the special nature of the flag manifold, the naive method always works. We shall come across an example where it fails a little later, in the case of the Hirzebruch surface.

Returning to the calculation, let us choose E_2, E_2', E_2'' in general position, i.e. such that the intersection of any two of them is a line, and the intersection of all three is the origin. Let us choose E_1 to be any line in E_2. Then there is precisely one H with the required property, namely $H = E_2$! The line $\boldsymbol{P}(H)$ intersects the three subvarieties, respectively, at the points

(L', E_2')
(L'', E_2'')
(E_1, E_2).

where $L' = H \cap E_2'$, $L'' = H \cap E_2''$. We conclude that $\lambda = 1$. A similar calculation gives $\mu = \langle A|A|Z\rangle_{0,1} = 0$. ∎

All quantum products of additive generators of H^*F_3 may be computed in a similar way. The results are collected in the table below:

	1	a	b	a^2	b^2	$a^2b = ab^2$
1	1	a	b	a^2	b^2	a^2b
a	a	$a^2 + q_1$	ab	bq_1	ab^2	$b^2q_1 + q_1q_2$
b	b	ba	$b^2 + q_2$	ba^2	aq_2	$a^2q_2 + q_1q_2$
a^2	a^2	bq_1	a^2b	b^2q_1	q_1q_2	aq_1q_2
b^2	b^2	b^2a	aq_2	q_1q_2	a^2q_2	bq_1q_2
a^2b	a^2b	$b^2q_1 + q_1q_2$	$a^2q_2 + q_1q_2$	aq_1q_2	bq_1q_2	abq_1q_2

As in the case of $\mathbf{C}P^n$, it is not really necessary to calculate all quantum products in this table from Gromov-Witten invariants. Once the products of degree at most six have been computed, the rest follow from associativity and commutativity.

By Theorem 2.2 of [4], the relations defining the relations defining the quantum cohomology algebra are "quantum modifications" of the Borel relations

$$x_1 + x_2 + x_3, \quad x_1x_2 + x_2x_3 + x_3x_1, \quad x_1x_2x_3.$$

From the table we obtain:

$$x_1 \circ x_2 + x_2 \circ x_3 + x_3 \circ x_1 = -a \circ a - b \circ b + a \circ b = -q_1 - q_2$$

and

$$x_1 \circ x_2 \circ x_3 = -a \circ b \circ b + a \circ a \circ b = bq_1 - aq_2.$$

The quantum cohomology algebra QH^*F_3 is therefore the quotient of the polynomial algebra $\mathbf{C}[x_1, x_2, x_3, q_1, q_2]$ by the ideal generated by

$$x_1 + x_2 + x_3, \quad x_1x_2 + x_2x_3 + x_3x_1 + q_1 + q_2, \quad x_1x_2x_3 + x_3q_1 + x_1q_2,$$

or, more efficiently,

$$\frac{\mathbf{C}[a,b,q_1,q_2]}{\langle a^2+b^2-ab-q_1-q_2, a^2b-ab^2-bq_1+aq_2\rangle}.$$

Example 8.3. The Hirzebruch surface $\Sigma_k = \boldsymbol{P}(\mathcal{O}(0)\oplus\mathcal{O}(-k))$, where $\mathcal{O}(i)$ denotes the holomorphic line bundle on $\mathbf{C}P^1$ with first Chern class i, may be described explicitly as a smooth projective algebraic variety (see [12]):

$$\Sigma_k = \{([z_0;z_1;z_2],[w_1;w_2]) \in \mathbf{C}P^2\times\mathbf{C}P^1 \text{ such that } z_1w_1^k = z_2w_2^k\}.$$

The subvarieties

$$\begin{aligned} X_1 &= \{z_2 = w_1 = 0\}\\ X_2 &= \{z_1 = z_2 = 0\}\\ X_3 &= \{z_1 = w_2 = 0\}\\ X_4 &= \{z_0 = 0\}. \end{aligned}$$

are all isomorphic to $\mathbf{C}P^1$, and represent two-dimensional homology classes. If Σ_k is regarded as "$\mathcal{O}(-k)\cup\ \infty$-section", then X_1 and X_3 are fibres, X_2 is the 0-section, and X_4 is the ∞-section.

It follows from this that the relations between the Poincaré dual cohomology classes x_1,x_2,x_3,x_4 are:

(1) $x_1 = x_3$, $x_4 = x_2 + kx_1$.

(2) $x_1x_3 = x_2x_4 = 0$, $x_1x_2 = x_1x_4 = x_2x_3 = x_3x_4 = z$, where z is a generator of $H^4(\Sigma_k;\mathbf{Z}) \cong \mathbf{Z}$.

(3) $x_1^2 (= x_3^2) = 0$, $x_2^2 = -kz$, $x_4^2 = kz$.

The cohomology algebra of Σ_k is

$$\begin{aligned} H^*(\Sigma_k;\mathbf{Z}) &\cong \mathbf{Z}[x_1,x_2,x_3,x_4]/\langle x_1-x_3, x_4-x_2-kx_1, x_1x_3, x_2x_4\rangle\\ &\cong \mathbf{Z}[x_1,x_4]/\langle x_1^2, x_4^2-kz\rangle. \end{aligned}$$

Let H_1, H_2 be the restrictions to Σ_k of the tautologous line bundles on $\mathbf{C}P^1$, $\mathbf{C}P^2$. It may be verified that $c_1(H_1) = -x_1$ and $c_1(H_2) = -x_4$.

We shall choose the following homology basis:

$$A_0 = \text{point}, \quad A_1 = X_2, \quad A_3 = \text{fibre}, \quad A_4 = \Sigma_k.$$

The dual cohomology basis is

$$b_0 = 1, \quad b_1 = x_1, \quad b_2 = x_4, \quad b_3 = x_1x_4 = z.$$

Thanks to the explicit embedding in $\mathbf{C}P^2\times\mathbf{C}P^1$, we have an explicit description of holomorphic maps $f : \mathbf{C}P^1 \to \Sigma_k$ in terms of polynomials. Namely, f is of the form

$$f = ([p_4; p_2p_3^k; p_2p_1^k],[p_1;p_3])$$

where p_1, p_2, p_3, p_4 are arbitrary complex polynomials such that p_1, p_3 have no common factor, and p_2, p_4 have no common factor. (The notation is chosen so that $p_i \equiv 0$ if and only if $f(\mathbf{C}P^1) \subseteq X_i$.) If $[f] = D_1A_1 + D_2A_2 = (D_1, D_2)$, we have

$$D_1 = -c_1(f^*H_1) = \max\{\deg p_1, \deg p_3\}$$
$$D_2 = -c_1(f^*H_2) = \max\{\deg p_4, \deg p_2 p_3^k, \deg p_2 p_1^k\}.$$

It follows that there exists a holomorphic map f such that $[f] = D_1A_1 + D_2A_2$ if and only if either

(a) $D_2 \geq kD_1 \geq 0$, or
(b) $D_2 = 0$, $D_1 > 0$.

It is known that $c_1(T\Sigma_k) = 2x_4 - (k-2)x_1$. Therefore, for $q_1 = q^{A_1} = q^{X_2}$ and $q_2 = q^{A_2} = q^{X_1}$ we have

$$|q_1| = 2\langle 2x_4 - (k-2)x_1, X_2\rangle = 2(2-k), \quad |q_2| = 2\langle 2x_4 - (k-2)x_1, X_1\rangle = 4.$$

Here we have the first intimation of trouble: Σ_k is Fano only for $k = 0, 1$. Moreover, it is convex only for $k = 0$, which is the trivial case $\mathbf{C}P^1 \times \mathbf{C}P^1$ (whose quantum cohomology is $QH^*\mathbf{C}P^1 \otimes QH^*\mathbf{C}P^1$). Therefore, there is *not a single interesting case* where we can be confident that the naive geometrical method can be used to compute Gromov-Witten invariants of Σ_k. In fact the situation is even worse, as (unlike in the flag manifold example) we cannot be sure that the above cycles X_i are a sufficiently flexible collection — and indeed they are not, as, even for ordinary cohomology, we cannot compute x_2^2 because we have identified only one cycle of type X_2.

However, purely by good luck, we will be able to compute enough Gromov-Witten invariants in the case $k = 1$ to arrive at the correct answer. *From now on, therefore, we assume that* $k = 1$. We have $|q_1| = 2$ and $|q_2| = 4$.

Before starting, we note that (from the polynomial representation):

(1) the holomorphic maps of degree $(1, 0)$ are given by the "0-section", X_2, and
(2) the holomorphic maps of degree $(0, 1)$ are given by the "fibres".

Proposition 8.2. $x_1 \circ x_1 = x_2 q_1$.

Proof. Let us write $x_1 \circ x_1 = x_1^2 + \lambda q_2 + \mu q_1$ where $\lambda \in H^0(\Sigma_1; \mathbf{Z})$ and $\mu \in H^2(\Sigma_1; \mathbf{Z})$. We have $\lambda = \langle X_1 | X_1 | Z\rangle_{0,1}$, where Z is the generator of $H_0(\Sigma_1; \mathbf{Z})$. Let us choose X_1, X_3, and any point of Σ_1 as representatives of the three homology classes. Then each holomorphic map of degree $(0, 1)$ (i.e. each "fibre") fails to intersect all three representatives. Hence $\lambda = 0$.

To calculate μ, we must calculate $\langle \mu, Y\rangle = \langle X_1 | X_1 | Y\rangle_{1,0}$, for two independent homology classes $Y \in H_2(\Sigma_1; \mathbf{Z})$.

The triple product $\langle X_1 | X_1 | X_1\rangle_{1,0}$ is equal to 1, as we may choose three distinct "fibres" representing X_1, and then there is a unique configuration (P_1, P_2, P_3, X_2) which intersects these fibres (respectively) in the points P_1, P_2, P_3.

The triple product $\langle X_1 | X_1 | X_4\rangle_{1,0}$ is equal to zero, because $X_2 \cap X_4 = \emptyset$.

We conclude that $\langle \mu, X_1 \rangle = 1$ and $\langle \mu, X_4 \rangle = 0$, and hence $\mu = x_2$. ∎

Proposition 8.3. $x_1 \circ x_4 = x_1 x_4$.

Proof. Let us write $x_1 \circ x_4 = x_1 x_4 + \lambda q_2 + \mu q_1$. We have $\lambda = \langle X_1 | X_4 | Z \rangle_{0,1}$. By taking Z as any point in the complement of $X_1 \cup X_4$, we see that $\lambda = 0$. A holomorphic map of degree $(1,0)$ cannot intersect X_4, so $\mu = 0$. ∎

Proposition 8.4. $x_2 \circ x_4 = q_2$.

Proof. Let us write $x_2 \circ x_4 = x_2 x_4 + \lambda q_2 + \mu q_1$. We have $\lambda = \langle X_2 | X_4 | Z \rangle_{0,1}$, and this is equal to 1, as there is a unique "fibre" which intersects X_2, X_4 and a point in the complement of $X_2 \cup X_4$. The triple products $\langle X_2 | X_4 | Y \rangle_{1,0}$ are necessarily zero, so we have $\mu = 0$. ∎

The three remaining quantum products are not amenable to direct computation in our present framework, but we can obtain them indirectly from the relation $x_4 = x_2 + x_1$:

$$\begin{aligned} x_4 \circ x_4 &= z + q_2 \\ x_1 \circ x_2 &= z - x_2 q_1 \\ x_2 \circ x_2 &= -z + q_2 + x_2 q_1. \end{aligned}$$

The quantum cohomology algebra is therefore

$$QH^*\Sigma_1 \cong \frac{\mathbf{C}[b_1, b_2, q_1, q_2]}{\langle b_1^2 - (b_2 - b_1)q_1, b_2^2 - b_1 b_2 - q_2 \rangle}.$$

Examples like this suggest already that quantum cohomology is not functorial in any obvious sense, in contrast to ordinary cohomology. The reason for this lack of functoriality is that the quantum cohomology algebra QH^*M is only the "tip of the iceberg"; underlying it there is a deeper structure.

9. Homological Geometry

The first significant steps in uncovering the deeper structure of quantum cohomology were taken by Givental in [6] and [7]. He observed that the additional structure on the cohomology vector space provided by quantum cohomology has a differential geometric or symplectic aspect, and he introduced the term "homological geometry" to describe it.

In this section we shall describe very briefly this point of view, primarily as historical motivation. In addition to the articles just mentioned, we recommend [8] (which introduced an intriguing relation between QH^*F_n and the Toda lattice, a completely integrable Hamiltonian system), and [2, 3].

Let M be a complex manifold whose quantum cohomology is defined (as in Secs. 7 and 8). The construction of this section requires an additional hypothesis:

Assumption. $H^*(M; \mathbf{Z})$ is generated as an algebra by the subgroup $H^2(M; \mathbf{Z})$.

When this condition does not hold, one may replace $H^*(M;\mathbf{Z})$ by the subalgebra $H^\sharp(M;\mathbf{Z})$ generated by $H^2(M;\mathbf{Z})$ and work with that.

As in sections 2.2 and 2.3 we choose bases

$$H_*M = \bigoplus_{i=0}^{s} \mathbf{Z}A_i, \quad H^*M = \bigoplus_{i=0}^{s} \mathbf{Z}b_i$$

such that $\langle b_i, A_j\rangle = \delta_{ij}$. In anticipation of a symplectic interpretation, we denote a general element of $H_2(M;\mathbf{C})$ by $\sum_{i=1}^r p_i A_i$ (modifying our earlier notation $\sum_{i=1}^r D_i A_i \in H_2(M;\mathbf{Z})$). A general element of $H^2(M;\mathbf{C})$ will be denoted by $\sum_{i=1}^r t_i b_i$ as usual.

Thus, we regard p_i, t_j as the standard coordinate functions:

$$p_i = \langle b_i, \ \rangle : H_2(M;\mathbf{C}) \cong \mathbf{C}^r \to \mathbf{C}$$
$$t_j = \langle \ , A_j\rangle : H^2(M;\mathbf{C}) \cong \mathbf{C}^r \to \mathbf{C}.$$

Similarly, we regard $q_j = e^{t_j}$ as a map

$$q_j = e^{\langle \ , A_j\rangle} : H^2(M;\mathbf{C}) \cong \mathbf{C}^r \to \mathbf{C}^* = \mathbf{C} - \{0\}.$$

The exponential map $\exp : \mathbf{C} \to \mathbf{C}^*$ induces an identification $\mathbf{C}/2\pi\sqrt{-1}\mathbf{Z} \cong \mathbf{C}^*$. Using this, we have an identification

$$B = H^2(M;\mathbf{C}/2\pi\sqrt{-1}\mathbf{Z}) \cong H^2(M;\mathbf{C}^*) \cong (\mathbf{C}^*)^r$$

Thus there is an induced map

$$q_j = e^{\langle \ \ , A_j\rangle} : B \to \mathbf{C}^* = \mathbf{C} - \{0\}.$$

for which we shall use the same notation.

By the above assumption, $H^*(M;\mathbf{C})$ has the form

$$H^*(M;\mathbf{C}) \cong \mathbf{C}[p_1,\dots,p_r]/\langle R_1, R_2, \dots\rangle$$

where $\langle R_1, R_2, \dots\rangle$ denotes the ideal generated by some relations $R_1, R_2, \dots$. For certain M (such as the examples in the previous section) it then follows that

$$QH^*M \cong \mathbf{C}[p_1,\dots,p_r,q_1,\dots,q_r]/\langle \mathcal{R}_1, \mathcal{R}_2, \dots\rangle$$

for some relations $\mathcal{R}_1, \mathcal{R}_2, \dots$ which are "quantum versions" of $R_1, R_2, \dots$.

Homological geometry begins with the manifold T^*B, the cotangent bundle of the complex algebraic torus B. As B is a group, we have canonical isomorphisms

$$TB \cong B \times H^2(M;\mathbf{C}), \quad T^*B \cong B \times H^2(M;\mathbf{C})^*.$$

Via this identification, the natural (complex) symplectic form of T^*B is $d\lambda$, where

$$\lambda = \sum_{i=1}^{r} \frac{dq_i}{q_i} \wedge p_i.$$

With respect to this symplectic structure, the Poisson bracket of two functions $f, g : T^*B \to \mathbf{C}$ is given by

$$\{f,g\} = \sum_{i=1}^{r} q_i \Big(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial g}{\partial p_i}\frac{\partial f}{\partial q_i} \Big).$$

Let us consider next the algebraic variety V_M defined by

$$V_M = \{(q_1,\dots,q_r,p_1,\dots,p_r) \in T^*B \text{ such that } \mathcal{R}_1 = \mathcal{R}_2 = \cdots = 0\}.$$

It is shown in [8] and [3] that, under certain conditions, V_M is a Lagrangian subvariety of T^*B. This means that V_M is maximal isotropic with respect to the symplectic form $d\lambda$, i.e. $d\lambda$ is identically zero on V_M, and V_M is maximal with respect to this property. For the moment we just remark that the Lagrangian property is related to the ideal $\langle \mathcal{R}_1, \mathcal{R}_2, \dots \rangle$ being closed under Poisson bracket.

Let us identify the variety V_M and verify the closure property for the examples of the previous section.

Example 9.1. For $M = \mathbf{C}P^n$ (see Example 8.1) we have $V_M = \{(q,p) \in \mathbf{C}^* \times \mathbf{C}$ such that $p^{n+1} = q\}$. It is one-dimensional and therefore automatically Lagrangian.

Example 9.2. For the flag manifold $M = F_3$ (see Example 8.2), V_M is the subvariety of

$$T^*B = \{(q_1,q_2,p_1,p_2) \in (\mathbf{C}^*)^2 \times \mathbf{C}^2\}$$

defined by the equations $\mathcal{R}_1 = 0, \mathcal{R}_2 = 0$ where

$$\mathcal{R}_1 = p_1^2 + p_2^2 - p_1p_2 - q_1 - q_2,\ \mathcal{R}_2 = p_1^2p_2 - p_1p_2^2 - p_2q_1 + p_1q_2.$$

A simple computation gives $\{\mathcal{R}_1, \mathcal{R}_2\} = 0$, so the closure property certainly holds in this case.

Example 9.3. For $M = \Sigma_1 = \boldsymbol{P}(\mathcal{O}(0) \oplus \mathcal{O}(-1))$, V_M is the subvariety of

$$T^*B = \{(q_1,q_2,p_1,p_2) \in (\mathbf{C}^*)^2 \times \mathbf{C}^2\}$$

defined by the equations $\mathcal{R}_1 = 0, \mathcal{R}_2 = 0$ where

$$\mathcal{R}_1 = p_1^2 - (p_2 - p_1)q_1,\ \mathcal{R}_2 = p_2^2 - p_1p_2 - q_2.$$

This time we obtain $\{\mathcal{R}_1, \mathcal{R}_2\} = q_1\mathcal{R}_2$, so again the closure property holds.

In the case $M = F_3$, the stronger than expected property $\{\mathcal{R}_1, \mathcal{R}_2\} = 0$ suggests that this example is special. In fact the same property (Poisson brackets of relations of QH^*M are all zero) holds for more generally for $M = F_n$. This fact has the interpretation that V_M is the phase space for a completely integrable Hamiltonian system in the sense of classical mechanics. Each relation $\mathcal{R}_i$ can be regarded as a Hamiltonian function on the symplectic manifold T^*B, and the integral curves of the Hamiltonian vector field corresponding to $\mathcal{R}_i$ are the solution curves of an ordinary differential equation on T^*B. Each such equation is

said to be completely integrable because T^*B has dimension $2(n-1)$ and there are $n-1$ Poisson-commuting "conserved quantities" (or "integrals of motion"), namely $\mathcal{R}_1, \ldots, \mathcal{R}_{n-1}$. Remarkably, the completely integrable Hamiltonian system arising in this way from QH^*F_n was already well known; it is the so-called Toda lattice. Although the case $M = F_n$ is clearly very special, the connection with the Toda lattice came as a surprise when it was discovered by Givental and Kim [8], and was an early stimulus for research on quantum cohomology and integrable systems.

10. The Quantum Differential Equations

The quantum differential equations are the system of differential equations

$$h\partial_i \Psi = b_i \circ_t \Psi, \quad i = 1, \ldots, r$$

for $\Psi : H^2(M; \mathbf{C}) \to H^*(M; \mathbf{C})$, where h is a complex parameter. We use $\circ_t$ rather than $\circ$ here and we use cohomology with complex coefficients, because these are convenient from the point of view of differential equations. However, we shall usually abbreviate $H^*(M; \mathbf{C})$ by H^*M, as this will not cause confusion.

A deeper understanding of relations defining the quantum cohomology can be obtained by studying these equations, and that is the first reason for doing so. Perhaps surprisingly, it turns out that geometrical/topological aspects of relations defining the quantum cohomology are reflected in algebraic/analytic properties of the quantum differential equationse. For example, the coefficients of power series solutions of the quantum differential equations turn out to be related to Gromov-Witten invariants. Singularities of solutions can be traced back to properties of the manifold M. The very existence of solutions (i.e., the compatibility of the equations, in the p.d.e. case) turns out to be related to the associativity of the quantum product.

We begin by looking at the quantum differential equations in two somewhat different ways, one algebraic and one differential geometric.

First version: matrix of structure constants.

The quantum differential equations can be written as a matrix system

$$h\partial_i \begin{pmatrix} \psi_0 \\ \vdots \\ \psi_s \end{pmatrix} = \begin{pmatrix} & & \\ & C_i & \\ & & \end{pmatrix} \begin{pmatrix} \psi_0 \\ \vdots \\ \psi_s \end{pmatrix}$$

where we write $\Psi = \sum_0^s \psi_j b_j$ and consider b_j to be the column vector with 1 in position j and zeros elsewhere ($0 \leqslant j \leqslant s$). The matrix C_i is the matrix of "quantum multiplication by b_i" with respect to the basis $b_0, \ldots, b_s$, where $1 \leqslant i \leqslant r$. That is,

$$b_i \circ_t b_j = \sum_0^s (C_i)_{kj} b_k,$$

i.e. the coefficients $(C_i)_{kj}$ of the differential equations are the "structure constants" of the relations defining the quantum cohomology algebra.

Second version: a connection.

This is more interesting, and is the origin of the link to integrable systems. Let us re-write the system as

$$(\partial_i - \tfrac{1}{\hbar} C_i) \begin{pmatrix} \psi_0 \\ \vdots \\ \psi_s \end{pmatrix} = 0.$$

These equations say that the covariant derivative of Ψ is zero, with respect to the connection $\nabla = d - \frac{1}{\hbar} \sum_1^r C_i dt_i$.

Let us recall briefly some terminology from the theory of connections. We regard $\Psi : H^2M \to H^*M$ as a section of the trivial vector bundle

$$H^2M \times H^*M \to H^2M.$$

A connection (or covariant derivative operator) on this vector bundle is a differential operator ∇ on sections of the bundle. More precisely, for each tangent vector field X on the base manifold $H^2M \cong \mathbf{C}^r$, we have a first order differential operator ∇_X, and $\nabla_X \Psi$ may be interpreted as the derivative of Ψ in the direction of X. For example, in the case of the "trivial" connection $\nabla = d$, $\nabla_X \Psi$ is, by definition, the usual directional derivative of Ψ in the direction of X, sometimes denoted $X\Psi$. Any other connection may be written in the form $\nabla = d + \omega$, where ω is a matrix of 1-forms (matrix-valued 1-form). Thus, our connection $\nabla = d - \frac{1}{\hbar} \sum_1^r C_i dt_i$ corresponds to the particular matrix of 1-forms $-\frac{1}{\hbar} \sum_1^r C_i dt_i$.

For $r = 1$ the connection is not particularly helpful, but for $r \geq 2$ it is fundamental, because the condition for local existence of solutions — near a regular point — is precisely that the curvature of the connection is zero. (For $r = 1$, the curvature is always zero, and, as is well known, an ordinary differential equation always admits local solutions near a regular point.) We shall see shortly that the properties of relations defining the quantum cohomology guarantee that the curvature of the above connection is zero. This is the starting point of the entire theory. The connection is often referred to as the Dubrovin connection, or the Givental connection.

The condition for local existence of solutions (often called the "compatibility condition" or "consistency condition") is an elementary result, based on Frobenius' Theorem, but it is so important that we shall review the statement (and various reformulations) carefully, before we proceed any further.

Basic Lemma. *Let $\nabla = d - \beta$ be a connection in the trivial bundle $\mathbf{C}^a \times \mathbf{C}^b \to \mathbf{C}^a$, with $\beta = \sum_1^a \beta_i dt_i$. Each β_i is a $b \times b$ matrix function of $t \in \mathbf{C}^a$, holomorphic in a neighbourhood N of the origin. Then the dimension of the space of holomorphic maps $\Psi : N \to \mathbf{C}^b$ which satisfy the system $(\partial_i - \beta_i)\Psi = 0$, $1 \leqslant i \leqslant a$ is either 0 or b. The dimension is b if and only if any of the following equivalent conditions holds:*

(1) *$d - \beta$ is a flat connection, i.e. has zero curvature.*

(2) *$d\beta - \beta \wedge \beta = 0$.*

(3) $\beta = dGG^{-1}$ *for some holomorphic map* $G : N \to GL_b\mathbf{C}$.

In (3), the map G is simply a "fundamental solution matrix" for the linear system, i.e.

$$G = \begin{pmatrix} | & & | \\ \Psi^{(1)} & \dots & \Psi^{(b)} \\ | & & | \end{pmatrix}$$

where $\Psi^{(1)}, \dots, \Psi^{(b)}$ are any linearly independent solutions. It is unique up to multiplication on the right by an invertible $b \times b$ matrix, i.e. by an element of $\mathrm{GL}_b\mathbf{C}$.

Theorem 10.1. *The quantum differential equations are consistent, i.e.* $d - \frac{1}{h}C$ *has zero curvature, for any nonzero value of the parameter* h. *That is,* $dC = C \wedge C = 0$.

Proof. By condition (2) above, the zero curvature condition says that $d\Big(\frac{1}{h}C\Big) - \Big(\frac{1}{h}C\Big) \wedge \Big(\frac{1}{h}C\Big) = 0$ for all h. Comparing coefficients of powers of h, we see that this condition is equivalent to $dC = C \wedge C = 0$. Now, the fact that $C \wedge C = 0$ is an immediate consequence of the commutativity and associativity of the quantum product. The fact that $dC = 0$ is less obvious, and we omit the proof. ∎

As in the Basic Lemma, we write

$$\frac{1}{h}C = dGG^{-1}$$

but now G is an $(s+1) \times (s+1)$ matrix-valued function of $t \in H^2M$ and $h \in \mathbf{C}^* = \mathbf{C} - \{0\}$. It is defined in a neighbourhood of any point where the matrices C_i are holomorphic functions of t; we shall not make any assumptions about such neighbourhoods yet, nor about a specific choice of G (which is unique only up to multiplication on the right by an $(s+1)\times(s+1)$ matrix-valued function of h).

Next we discuss a well known feature of first order linear differential equations for vector functions, which will be very important for relations defining the quantum cohomology: such systems may sometimes be re-written as higher order systems of differential equations for *scalar* functions. cohomologyc

In the case of the quantum differential equations for $\Psi = (\psi_0, \dots, \psi_s)$, it may be possible to introduce an equivalent system of p.d.e.

$$D_1^h\psi = 0, \ \dots, \ D_u^h\psi = 0$$

for a scalar function ψ, the equivalence being given through expressions of the form

$$\psi_i = \text{ differential polynomial in } \psi$$

(The superscript h in the differential operator D_i^h indicates dependence on the parameter h.) The o.d.e. case is familiar: passage from an n-th order equation

for ψ to a system of n first order equations for $\psi_0, \dots, \psi_s$ may be achieved by writing $\psi_i = \partial^i \psi$. But for p.d.e., it is not immediately obvious how, or whether, such a procedure can be carried out (and the same can be said of the converse procedure of passing from a vector system to a scalar system — even in the o.d.e. case).

However, if they exist, the higher order operators D_i^h have a very interesting interpretation, in terms of *relations* of the relations defining the quantum cohomology algebra QH^*M. If R is a polynomial such that $R(b_1, \dots, b_r) = 0$ in QH^*M, then the form of the quantum differential equations suggests that $R(h\partial_1, \dots, h\partial_r)\psi$ might also be zero — but this turns out not to be correct. If $R(h\partial_1, \dots, h\partial_r)\psi = 0$ then in fact $R(b_1, \dots, b_r) = 0$, but in general the "differential relation" corresponding to $R(b_1, \dots, b_r) = 0$ contains additional terms involving h. The following observation of [8] elucidates this point:

Theorem 10.2. *Let $P(X_0, \dots, X_{2r})$ be a polynomial in $2r+1$ variables, written so that, in each monomial term, X_i precedes X_j if $1 \leqslant i \leqslant r$ and $r+1 \leqslant j \leqslant 2r$. If*

$$P(h, e^{t_1}, \dots, e^{t_r}, h\partial_1, \dots, h\partial_r)(\Psi^{(u)}, 1) = 0,$$

for $u = 0, \dots, s$, then the relation $P(0, e^{t_1}, \dots, e^{t_r}, b_1, \dots, b_r) = 0$ holds in the quantum cohomology algebra QH^*M.

Proof. For any $f : H^2M \to H^*M$,

$$\begin{aligned} h\partial_i(\Psi^{(u)}, f) &= (h\partial_i \Psi^{(u)}, f) + (\Psi^{(u)}, h\partial_i f) \\ &= (b_i \circ_t \Psi^{(u)}, f) + (\Psi^{(u)}, h\partial_i f) \\ &= (\Psi^{(u)}, b_i \circ_t f) + (\Psi^{(u)}, h\partial_i f) \\ &= (\Psi^{(u)}, b_i \circ_t f + h\partial_i f) \\ &= (\Psi^{(u)}, B_i f), \text{ say.} \end{aligned}$$

It follows that

$$h\partial_{i_j} \dots h\partial_{i_1}(\Psi^{(u)}, f) = (\Psi^{(u)}, B_{i_j} \dots B_{i_1} f)$$

and similarly for any polynomial in $h\partial_1, \dots, h\partial_r$.

Let us take f to be the constant function 1 in these formulae. We have

$$\begin{aligned} B_i 1 &= b_i \\ B_j B_i 1 &= b_j \circ_t b_i \\ B_k B_j B_i 1 &= b_k \circ_t b_j \circ_t b_i + h\partial_k(b_j \circ_t b_i) \end{aligned}$$

and more generally $B_{i_j} \dots B_{i_1} 1 = b_{i_j} \circ_t \dots \circ_t b_{i_1} + O(h)$. Hence

$$P(h, e^{t_1}, \dots, e^{t_r}, h\partial_1, \dots, h\partial_r)(\Psi^{(u)}, 1) = (\Psi^{(u)}, P(0, e^{t_1}, \dots, e^{t_r}, b_1, \dots, b_r)) + O(h).$$

By hypothesis the left hand side is zero for all u. Setting $h = 0$ and using the fact that $\Psi^{(0)}(t), \dots, \Psi^{(s)}(t)$ are a basis of H^*M, we obtain the required result $P(0, e^{t_1}, \dots, e^{t_r}, b_1, \dots, b_r) = 0$.

11. Examples of Quantum Differential Equations

To gain some experience, let us convert some concrete examples of quantum differential equations to higher order scalar equations. For more detailed versions of some of these calculations, see [9].

Example 11.1. We begin with $\mathbf{C}P^n$. With respect to the usual basis $1, x_1, \ldots, x_n$ of $H^*\mathbf{C}P^n$, the quantum differential equations are the system

$$h\partial \begin{pmatrix} \psi_0 \\ \vdots \\ \vdots \\ \psi_n \end{pmatrix} = \begin{pmatrix} 0 & & & q \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_0 \\ \vdots \\ \vdots \\ \psi_n \end{pmatrix}$$

for the vector function $\Psi(q) = \sum_0^n \psi_j(q)x_j$, with $q = e^t$.

Let us write $\psi = \psi_n$. Then the quantum differential equations give

$$\psi_{n-1} = h\partial\psi,\ \psi_{n-2} = (h\partial)^2\psi,\ \ldots,\ \psi_0 = (h\partial)^n\psi$$

(which express $\psi_0, \ldots, \psi_{n-1}$ in terms of ψ) together with

$$(h\partial)^{n+1}\psi = q\psi.$$

Conversely, if we start with the o.d.e. $(h\partial)^{n+1}\psi = q\psi$, and if we define $\psi_0, \ldots, \psi_{n-1}$ in terms of $\psi = \psi_n$ as above, we obtain the original quantum differential equations.

Example 11.2. Next we consider the flag manifold F_3. With $t = t_1a + t_2b$, we have $C = C_1dt_1 + C_2dt_2$ where the matrices C_1, C_2 are the matrices of the quantum multiplication operators $a\circ_t$, $b\circ_t$ on H^*F_3. From the multiplication table given earlier we have

$$C_1 = \left(\begin{array}{c|cc|cc|c} 0 & q_1 & 0 & 0 & 0 & q_1q_2 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q_1 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & q_1 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right), \quad C_2 = \left(\begin{array}{c|cc|cc|c} 0 & 0 & q_2 & 0 & 0 & q_1q_2 \\ \hline 0 & 0 & 0 & 0 & q_2 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & q_2 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right).$$

The quantum differential equations are

$$h\partial_1\Psi = C_1\Psi$$
$$h\partial_2\Psi = C_2\Psi$$

namely

$$h\partial_1\psi_0 = q_1\psi_1 + q_1q_2\psi_5$$
$$h\partial_1\psi_1 = \psi_0$$
$$h\partial_1\psi_2 = q_1\psi_3$$

$$h\partial_1\psi_3 = \psi_1 + \psi_2$$
$$h\partial_1\psi_4 = \psi_2 + q_1\psi_5$$
$$h\partial_1\psi_5 = \psi_4$$

and

$$h\partial_2\psi_0 = q_2\psi_2 + q_1q_2\psi_5$$
$$h\partial_2\psi_1 = q_2\psi_4$$
$$h\partial_2\psi_2 = \psi_0$$
$$h\partial_2\psi_3 = \psi_1 + q_2\psi_5$$
$$h\partial_2\psi_4 = \psi_1 + \psi_2$$
$$h\partial_2\psi_5 = \psi_3.$$

Let us choose $\psi = \psi_5$.

Then five of the above twelve equations may be used to express $\psi_0, \psi_1, \psi_2, \psi_3, \psi_4$ in terms of ψ as follows:

$$\psi_0 = h^3\partial_1^2\partial_2\psi - q_1h\partial_2\psi$$
$$\psi_1 = h^2\partial_1\partial_2\psi - h^2\partial_1^2\psi + q_1\psi$$
$$\psi_2 = h^2\partial_1^2\psi - q_1\psi$$
$$\psi_3 = h\partial_2\psi$$
$$\psi_4 = h\partial_1\psi$$

(this may be done in various ways; we have made a particular choice). The remaining seven equations reduce to the following system of equations for ψ:

$$(h^2\partial_1^2 + h^2\partial_2^2 - h^2\partial_1\partial_2 - q_1 - q_2)\psi = 0$$
$$(h^3\partial_1\partial_2^2 - h^3\partial_1^2\partial_2 - q_2h\partial_1 + q_1h\partial_2)\psi = 0$$
$$(h^3\partial_1^3 - q_1h\partial_1 - q_1h\partial_2)\psi = q_1h\psi$$
$$(h^4\partial_1^3\partial_2 - 2q_1h^2\partial_1\partial_2 + q_1h^2\partial_1^2 - q_1^2 - q_1q_2)\psi = q_1h^2\partial_2\psi$$
$$h^4\partial_1^2\partial_2^2 - q_1h^2\partial_2^2 - q_2h^2\partial_1^2)\psi = 0.$$

The last three equations, in this group of five, follow from the first two. We conclude that the system

$$h\partial_1\Psi = C_1\Psi$$
$$h\partial_2\Psi = C_2\Psi$$

is equivalent to the scalar system

$$D_1^h\psi = 0, D_2^h\psi = 0,$$

where

$$D_1^h = h^2\partial_1^2 + h^2\partial_2^2 - h^2\partial_1\partial_2 - q_1 - q_2$$
$$D_2^h = h^3\partial_1\partial_2^2 - h^3\partial_1^2\partial_2 - q_2 h\partial_1 + q_1 h\partial_2,$$

the correspondence being given by

$$\Psi = \begin{pmatrix} (h^3\partial_1^2\partial_2 - q_1 h\partial_2)\psi \\ (h^2\partial_1\partial_2 - h^2\partial_1^2 + q_1)\psi \\ (h^2\partial_1^2 - q_1)\psi \\ (h\partial_2)\psi \\ (h\partial_1)\psi \\ \psi \end{pmatrix}.$$

Example 11.3. We shall consider the quantum differential equations for the Hirzebruch surface $\Sigma_1 = \boldsymbol{P}(\mathcal{O}(0) \oplus \mathcal{O}(-1))$. With $t = t_1x_1 + t_2x_4$, we have $C = C_1dt_1 + C_2dt_2$ where the matrices C_1, C_2 are the matrices of the quantum multiplication operators $x_1\circ_t$, $x_4\circ_t$ on $H^*\sum_1$. From our computations of the quantum products,

$$C_1 = \left(\begin{array}{c|ccc} 0 & 0 & 0 & q_1q_2 \\ \hline 1 & -q_1 & 0 & 0 \\ 0 & q_1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \end{array}\right), \quad C_2 = \left(\begin{array}{c|ccc} 0 & 0 & q_2 & q_1q_2 \\ \hline 0 & 0 & 0 & q_2 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \end{array}\right)$$

The quantum differential equations are

$$h\partial_1\psi_0 = q_1q_2\psi_3$$
$$h\partial_1\psi_1 = \psi_0 - q_1\psi_1$$
$$h\partial_1\psi_2 = q_1\psi_1$$
$$h\partial_1\psi_3 = \psi_2$$

and

$$h\partial_2\psi_0 = q_2\psi_2 + q_1q_2\psi_3$$
$$h\partial_2\psi_1 = q_2\psi_3$$
$$h\partial_2\psi_2 = \psi_0$$
$$h\partial_2\psi_3 = \psi_1 + \psi_2.$$

Let us choose $\psi = \psi_3$. Then three of the above eight equations may be used to express ψ_0, ψ_1, ψ_2 in terms of ψ as follows:

$$\psi_0 = h^2\partial_1\partial_2\psi$$
$$\psi_1 = h\partial_2\psi - h\partial_1\psi$$
$$\psi_2 = h\partial_1\psi$$

The remaining five equations reduce to the following system of equations for ψ:

$$
\begin{aligned}
(h^2\partial_2^2 - h^2\partial_1\partial_2 - q_2)\psi &= 0\\
(h^2\partial_1^2 - q_1 h\partial_2 + q_1 h\partial_1)\psi &= 0\\
(h^3\partial_1^2\partial_2 - q_1 q_2)\psi &= 0\\
(h^3\partial_1\partial_2^2 - q_2 h\partial_1 - q_1 q_2)\psi &= 0
\end{aligned}
$$

The last two equations, in this group of four, follow from the first two. Hence the original system

$$
\begin{aligned}
h\partial_1\Psi &= C_1\Psi\\
h\partial_2\Psi &= C_2\Psi
\end{aligned}
$$

is equivalent to the system

$$D_1^h\psi = 0, D_2^h\psi = 0$$

where

$$
\begin{aligned}
D_1^h &= h^2\partial_2^2 - h^2\partial_1\partial_2 - q_2\\
D_2^h &= h^2\partial_1^2 - q_1 h\partial_2 + q_1 h\partial_1,
\end{aligned}
$$

the correspondence being given by

$$\Psi = \begin{pmatrix} (h^2\partial_1\partial_2)\psi \\ (h\partial_2 - h\partial_1)\psi \\ (h\partial_1)\psi \\ \psi \end{pmatrix}.$$

12. Final Comments

Our discussion of the quantum differential equations in Sec. 9-11 suggests (see [10, 1]) that the deeper structure underlying the quantum cohomology algebra QH^*M is a D-module of the form

$$D/(D_1^h, \ldots, D_u^h)$$

where D is a ring of differential operators and $(D_1^h, \ldots, D_u^h)$ denotes the (left) ideal generated by differential operators $D_1^h, \ldots, D_u^h$. The variety V_M is then the so called characteristic variety of this D-module. The concept of D-module (see [13]) is at the heart of the relation betwen quantum cohomology and integrable systems, as we shall discuss at length in the forthcoming book [11].

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