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On Finite Subgroups of the Cremona Group*

V. A. Iskovskikh

*Steklov Institute of Mathematics 8 Gubkin Trs.,
Moscow 117966, Russia*

Abstract. This is a survey paper of recent results on the classification of finite subgroups in the Cremona group of birational automorphisms of the plane.

1. Introduction

1.1. The Cremona group $\mathrm{Cr}_2(k)$ over a field k is the group of birational automorphisms of the plane $\mathbb{P}^2_k$ or, equivalently, the group of k-automorphisms of the field $k(x, y)$ of rational functions in two variables x, y. There is a vast classical literature on Cremona transformations (see [18], and also a modern exposition [1] in the classical style), in particular, on classification of finite subgroups of $\mathrm{Cr}_2(k)$ up to conjugation. We shall restrict ourselves to the case of the field $k = \mathbb{C}$ of complex numbers, and we will denote $\mathrm{Cr}_2(\mathbb{C})$ simply by Cr_2 throughout the paper.

Apparently the problem was first raised by E. Bertini in his paper on birational involutions [6]. Kantor wrote a book [24] on the classification of finite subgroups of Cr_2. His classification was improved by Wiman in [32]. The same problem was studied in [2] and [15]. Along with other problems Segre [31] considered the automorphism groups of cubic surfaces that are certainly realised as subgroups of Cr_2 since a cubic surface is rational.

It was already clear at that time that any finite subgroup of G in the Cremona group Cr_2 acts by automorphisms on a suitable nonsingular projective rational surface S (see Sec. 3 below). Such a pair (S, G) is called a *rational G-surface*: any birational map $\chi : S \dashrightarrow \mathbb{P}^2$ reembeds $G \subset \mathrm{Cr}_2$ on subgroup $\chi G \chi^{-1}$, and while χ varies in the class of all such birational maps these subgroups form

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a conjugacy class. Moreover if G acts on two different G-surfaces (S, G) and (S', G) such that there is a G-equivariant birational map $\varphi : S \dashrightarrow S'$, then S and S' determine the same conjugacy class of G in Cr_2. So the problem of classification of finite subgroups $G \subset \mathrm{Cr}_2$ up to conjugation is equivalent to the problem of classification of rational G-surfaces up to G-equivariant birational equivalence.

This problem is very close (let's say, parallel) to the classification problem of geometrically rational (i. e. those that become rational over some extension of the ground field) nonsingular projective surfaces S over a perfect field k. As for G here one takes Galois group $Gal(K/k)$ of some finite normal extension K of the field k, over which the surface $S \otimes_k K$ is birationally equivalent to a plane $\mathbb{P}^2_K$ over K. So two problems are merged and studied together in the context of rational G-surfaces, still there are two cases one has to distinguish: the *geometric* case where G is a subgroup of Cr_2, and the *algebraic* case where G is a Galois group. In this setting the problem of G-birational equivalence was studied by Manin [27, 28], the author of this paper [19 – 21], Gizatullin [14] (see also [16, 17]). For solving the problem of equivariant birational classification of rational G-surfaces it is necessary to, first of all, classify minimal G-surfaces. Recall that a rational G-surface (S, G) is called *minimal* if any G-equivariant birational morphism $f : (S, G) \to (S', G)$ on another G-surface (S', G) is a G-isomorphism. The classification of minimal G-surfaces was begun by Manin in [27] and completed by the author in [20] (see Sec. 3).

1.2. A conceptually new approach to the former classification problem was provided by Mori's theory [29]. Although it was developed as a new theory of minimal models of algebraic varieties in higher dimensions, it appeared to be very useful also in dimension 2, especially in its G-equivariant version. As an illustration of the advantages of this new method in comparison with the classical the theorem on minimal models of surfaces and G-surfaces was reproved in [29]. In this context a minimal G-surface is a G-equivariant Mori fibration of dimension 2, i. e. either a G-equivariant morphism $\varphi : S \to B$ onto a nonsingular curve with fibers - projective conics and G-equivariant Picard number $\rho^G = 2$, or a morphism $\varphi : S \to pt$, where S is a Del Pezzo surface (i. e. its anticanonical class $-K_S$ is ample) with G-equivariant Picard number $\rho^G = 1$ (see Sec. 3).

Since the appearance of equivariant Mori's theory the interest to birational theory of rational G-surfaces was springing up again in both algebraic and geometric cases. So the author (see [21]) classified all elementary birational G-maps (=links) between relatively minimal rational G-surfaces (rational Mori G-fibrations of dimension 2). Following Corti [10] the factorization theorem, i. e. on factorizing any G-birational map between relatively minimal rational G-surfaces according to the algorithm named now the Sarkisov program was reproved. In particular a new proof of the classical theorem of Nöether on the generators of the Cremona group $\mathrm{Cr}_2(k)$ over arbitrary perfect field k (in the classical case $k = \mathbb{C}$) was given. The basic relations that imply all other relations in a standard way were described; in particular, Gizatullin's theorem on relations in $\mathrm{Cr}_2(\mathbb{C})$ was reproved, and moreover, the relations in $\mathrm{Cr}_2(k)$ over arbitrary perfect field k can be described.

Though [21] deals only with the algebraic case, according to the general concept of rational G-surfaces all results have an adequate interpretation in the geometric case. It was shown in the author's notes [22, 23] in the proof of nonconjugacy of two embeddings of a finite subgroup $S_3 \times \mathbb{Z}_2$, where S_3 is the three elements permutation group, into Cr_2 (see Sec. 6 for details).

Recently several authors, starting with Bayle and Beauville [3], have paid attention again to the classical classification problem of finite subgroups in Cr_2, certainly in the context of G-equivariant Mori theory and rational G-surfaces. The emphasis is on an explicit representation of classical results and modernising the methods of the proofs. In particular in [3] the authors gave a new modernised proof of the classification theorem of birational involutions (i. e. elements of order 2 in the Cremona group) up to conjugation. The algebraic analog of this problem is a classical result of Comessatti concerning the classification of real geometrically rational surfaces up to birational $\mathbb{R}$-equivalence ($\mathbb{R}$ - the field of real numbers). A modernised variant of the proof of this result is contained in the paper [30] by Polyakova.

The classification of Cremona transformations of finite order in Cr_2 (in the classical context) is equivalent to the classification of normal multiple planes up to birational equivalence. Several classical research works of Castelnuovo and Enriques (see [9] for example) and modern works of Calabri [7, 8] on the classification of double and triple rational planes were devoted to this problem. Koitabashi [25] studied automorphism groups of general rational surfaces (in particular, Del Pezzo surfaces). Hosoh [16, 17] calculated the automorphism groups of cubic surfaces and Del Pezzo surfaces of degree 4.

The classification of finite subgroups G_p of prime order p in Cr_2 and an explicit description of their actions on Del Pezzo surfaces was given by Zhang in [33] (with appendix by Dolgachev). Another approach to the same problem, leading to a more explicit description, even up to writing down equations, was proposed by De Fernex [12]. An interesting corollary of the results of Zhang and De Fernex is as follows: if $p > 5$, then any subgroup $G_p \subset \mathrm{Cr}_2$ is conjugate to a cyclic subgroup of order p in the projective linear group PGL_3 of automorphisms of $\mathbb{P}^2$. In this connection the following question posed by J.-P. Serre is of interest: whether the order of a cyclic subgroup $G_p \subset \mathrm{Cr}_2(\mathbb{Q})$ in the Cremona group over the field of rational numbers $\mathbb{Q}$ is bounded?

In a recent preprint [5] Beauville studied p-elementary (i. e. of the form $G = (\mathbb{Z}_p)^r$ for some positive integer $r > 0$) subgroups of Cr_2 up to conjugation and and gave their classification for maximal possible r. It turned out that there are very few of them (see Sec. 5): if $p \geqslant 5$, then $r \leqslant 2$, and if $r = 2$, then G is conjugate to a subgroup of a group of diagonal matrices in PGL_3. The maximal value $r = 4$ is possible only for $p = 2$; all the corresponding cases are classified explicitly.

In the last Sec. 6 of these notes we sketch a proof of nonconjugacy of two explicitly given subgroups in Cr_2, each isomorphic to $S_3 \times \mathbb{Z}_2$, using the factorization algorithm of an equivariant birational map between minimal rational

G-surfaces into a sequence of links.

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2. Examples of Birational Involutions

The Cremona group Cr_2 contains the projective linear group PGL_3. The simplest example of a nonlinear Cremona transformation is a birational involution $(x, y) \mapsto (1/x, 1/y)$ (where (x, y) are affine coordinates in $\mathbb{A}^2$). It is called a *standard quadratic transformation*; in homogeneous coordinates (x_0, x_1, x_2) with $x = x_1/x_0$, $y = x_2/x_0$ it is given by quadratic formulas

$$\tau : (x_0, x_1, x_2) \mapsto (x_1x_2, x_0x_2, x_0x_1). \tag{1}$$

One can easily see from (1) that the indeterminacy points of τ are exactly the vertices of the coordinate triangle $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, that are blown-up into the opposite coordinate axes: $(1, 0, 0) \leftrightarrow (x_0 = 0)$, $(0, 1, 0) \leftrightarrow (x_1 = 0)$, $(0, 0, 1) \leftrightarrow (x_2 = 0)$. We use double-sided arrows because $\tau = \tau^{-1}$ hence the coordinate lines contract to the opposite vertices.

Question: this involution is conjugate, or not to the linear involution $(x_0, x_1, x_2) \mapsto (x_0, -x_1, -x_2)$? The answer: Yes. Indeed τ acts biregulary on the surface $\mathbb{P}^1 \times \mathbb{P}^1$ with bihomogeneous coordinates $(x_0, x_1) \times (y_0, y_1)$, where $x = x_1/x_0$, $y = y_1/y_0$ by the following formula

$$\tau : (x_0, x_1) \times (y_0, y_1) \mapsto (x_1, x_0) \times (y_1, y_0). \tag{2}$$

From (2) clearly on $\mathbb{P}^1\times\mathbb{P}^1$ there are four fixed points $(1, 1)\times(1, 1)$, $(1, 1)\times(1, -1)$, $(1, -1) \times (1, 1)$ and $(1, -1) \times (1, -1)$. The stereographic projection from one of them (recall that $\mathbb{P}^1\times\mathbb{P}^1$ is embedded into $\mathbb{P}^3$ as quadratic surface $z_0z_1-z_2z_3 = 0$) is a $\langle\tau\rangle$-equivariant map to the plane:

$$x = \frac{x_0 - x_1}{x_0 + x_1}, y = \frac{y_0 - y_1}{y_0 + y_1}.$$

On the plane with coordinates (x, y) the action of τ is given by the formula $(x, y) \mapsto (-x, -y)$, or $(x_0, x_1, x_2) \mapsto (x_0, -x_1, -x_2)$ in homogeneous coordinates.

2.2. Let us give examples of birational involutions that are not conjugate to the linear ones. These are so called *De Jonquieres involutions*

$$\delta : (x, y) \mapsto \Big(x, \frac{P(x)}{y}\Big), \tag{3}$$

[1] Recently (when this survey was already written) I. Dolgachev has provided the author with the abstract of his lectures "Finite subgroups of the plane Cremona group: Lectures in Torino", where the classification of S. Kantor and A. Wiman was reestablished.

where $P(x) = (x - a_1)(x - a_2) \dots (x - a_{2n})$ is a polynomial without multiple roots. We will show below that if $n > 1$ then it is not conjugate to a linear involution. The classical representation of these involutions is as follows. Let $C \subset \mathbb{P}^2$ be a curve of degree $2n$ with an ordinary singular point $P \in C$ of multiplicity $2n-2$ and no other singularities (recall that "ordinary" means that all $2n-2$ local branches are different, or that after blowing-up $\sigma : \mathbb{F}_1 \to \mathbb{P}^2$ at P the exceptional (-1)-curve $E \subset \mathbb{F}_1$ has no multiple intersection points with the strict transform $C' \subset \mathbb{F}_1$ of the curve C). It is clear that C is a hyperelliptic curve and that the pencil of lines in $\mathbb{P}^2$ passing through P gives a double cover $C \to \mathbb{P}^1$; the genus of C equals $g(C) = n - 1$. The involution δ maps a general point $x \in \mathbb{P}^2$ to its harmonic conjugate point x' on the line $\langle P, x\rangle$ with respect to two residual intersection points of the line $\langle P, x\rangle$ with the curve C. By definition this involution retains the pencil of lines passing through P and leaves the curve C fixed. Its indeterminacy points are exactly $2g+2 = 2n$ points where the lines $\langle P, x\rangle$ are tangent to the curve C. Blowing up these points and P on $\mathbb{P}^2$ we get a surface defined by the following affine equation

$$S^0 : yz - P(x) = 0, \tag{4}$$

on which δ acts biregulary by $(x, y, z) \mapsto (x, z, y)$ with fixed hyperelliptic curve $y^2 - P(x) = 0$.

The action of δ has a biregular extension to its nonsingular compactification $S = \overline{S^0}$; the surface S is glued from S^0 and $S^\infty : yz - P\Big(\dfrac{1}{x}\Big)$ by the formula $(x, y, z) \leftrightarrow \Big(\dfrac{1}{x}, \dfrac{y}{x^n}, \dfrac{z}{x^n}\Big)$.

The projection $(x, y, z) \mapsto (x)$ gives S a conic bundle structure $\varphi : S \to \mathbb{P}^1$ with degenerate fibers over the points $x = a_1, \dots, a_{2n}$; each degenerate fiber is a bouquet of lines $\mathbb{P}^1 \vee \mathbb{P}^1$ that are transposed by the involution δ, so that this is a $\langle\delta\rangle$-equivariant Mori fibration, or in the former terminology of [20] a relatively minimal rational $\langle\delta\rangle$-surface with a pencil of rational curves.

Lemma 2.3. [3, 11] *Let $G \subset Cr_2$ be a finite subgroup acting on $\mathbb{P}^2$, and let $\overline{C_1}, \dots, \overline{C_r}$ be normalizations of the fixed curves $C_1, \dots, C_r$ under the action of G. Then each $\overline{C_i}$ is a birational invariant provided its genus $g(\overline{C_i})$ is greater than zero.*

Indeed, any G-equivariant birational map either contracts a fixed curve, that is possible only if the genus of its normalization is zero, or induces a birational map on a fixed curve, if the genus of its normalization is greater than zero. But a birational self-map of a normal curve is an isomorphism.

From this lemma it follows immediately that De Jonquieres involution δ is not conjugate to any linear involution if $n \geqslant 1$, and moreover, up to conjugation all such involutions are parametrized by the isomorphism classes of hyperelliptic curves of genus $g \geqslant 1$ (elliptic curves if $g = 1$). Indeed, the hyperelliptic (elliptic if $n = 2$) curve $C : y^2 - P(x) = 0$ (see (4)) is fixed by δ.

2.4. Geiser Involutions

These birational involutions were discovered yet in the middle of the 19th century. In the classical terminology, each involution $\gamma : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ of this kind is defined by a linear system of curves of degree 8, passing through 7 points $P_i \in \mathbb{P}^2$ in a general position and with multiplicities 3. In our modern interpretation γ acts biregularly on a Del Pezzo surface S_2 of degree 2. The surface S_2 is a blowing-up of $\mathbb{P}^2$ at seven points P_i. The anticanonical linear system $|-K_{S_2}|$ on S_2 consists of strict transforms of the plane cubic curves passing through $P_1, \dots, P_7$. This linear system defines a double covering $\varphi_{-K} : S_2 \to \mathbb{P}^2$ ramified along a nonsingular quartic curve $C_4 \subset \mathbb{P}^2$. The involution γ is the involution of this double covering (i.e. it transposes points in the fibers). Since C_4 is fixed by γ and has genus 3, Geiser involutions are not conjugate to linear ones by Lemma 2.3 and they are parametrized (up to conjugacy) by the isomorphism classes of nonhyperelliptic curves of genus 3.

2.5. Bertini Involutions

They were discovered almost at the same time as Geiser involutions. A birational map $\beta : \mathbb{P}^2 \to \mathbb{P}^2$ is defined by a linear system of curves of degree 17 passing through 8 points $P_1, \dots, P_8 \in \mathbb{P}^2$ in a general position and with multiplicities 6. In the modern interpretation it is a biregular involution on a Del Pezzo surface S_1 of degree 1. The surface S_1 is a blowing-up of $\mathbb{P}^2$ at 8 points $P_1, \dots, P_8$. The anticanonical linear system $|-K_{S_1}|$ consists of strict transforms of plane cubics passing through P_i, $i = 1, \dots, 8$. The ninth intersection point of these cubics P_0 is the only base point of the pencil $|-K_{S_1}|$. The linear system $|-2K_{S_1}|$ defines a double cover $\varphi_{-2K} : S_1 \to Q^* \subset \mathbb{P}^3$ of a quadratic cone Q^* in $\mathbb{P}^3$ branched in its vertex P_0 and a nonsingular curve $C_6 \subset Q^*$ that is cut out on Q^* by a cubic surface in $\mathbb{P}^3$ not passing through P_0. The involution of this double cover is the Bertini involution β. The fixed curve C_6 is a nonhyperelliptic curve of genus 4, so by Lemma 2.3 β is conjugate neither to a linear involution, nor to a De Jonquieres or Geiser involution. Up to conjugation Bertini involutions are parametrized by isomorphism classes of nonhyperelliptic curves of genus 4, such that their canonical models are contained in quadratic cones.

We state a classical result which was reproved by modern methods.

Theorem 2.6. [3] *Each birational involution in* Cr_2 *is conjugate to one of the following:*

(i) *linear involution;*
(ii) *De Jonquieres involution of genus* $g \geqslant 1$*;*
(iii) *Geiser involution;*
(iv) *Bertini involution.*

3. Some General Results

3.1. In view of previous examples one sees that birational involutions acting on $\mathbb{P}^2$ with indeterminacy points are lifted up to biregular actions on the corresponding nonsingular projective rational surfaces. It turns out that this is a general principle of regularization.

Let $\chi : X' \dashrightarrow X$ be a birational map of projective varieties. If $G \subset \mathrm{Bir}(X)$ is a subgroup of birational automorphisms of X and $G' = \chi^{-1}G\chi$, then the map χ is called G-*equivariant*, and the pairs (X, G), (X', G') - G-*birationally equivalent*. If moreover X' is nonsingular and $G' \subset \mathrm{Aut}(X')$, then the pair (X', G') is called a *resolution of indeterminacies* (or *regularization*) of the pair (X, G).

The following result is well known in characteristic 0.

Theorem 3.2. *Let (X, G) be as above and G be a finite subgroup of $Bir(X)$. Then there exists a regularization (X', G') of the pair (X, G).*

Sketch of a proof. Let $K = k(X)$ be the field of rational functions on X. Group G acts on K in a natural way. Consider the subfield of invariants $K^G \subset K$. Since group G is finite, there exists a normal projective model Y with such field of rational functions $k(Y) = K^G$. Let $X^N \to Y$ be a finite morphism of normalization of Y in K normal projective variety on which G acts biregularly and Y is the factor X^N/G. It remains to use Hironaka's result on the existence of G-equivariant resolution of singularities $(X', G') \to (X, G)$.

In the case of surfaces the proof is simplified, since the indeterminacy locus consists of isolated points, so G-equivariant resolution is rather algorithmic (see eg. [12]). ∎

Hence, if a finite subgroup $G \subset \mathrm{Cr}_2$ acts birationally on $\mathbb{P}^2$, then by 3.2 there exists a regularization (S, G) with S a nonsingular projective surface which is rational by definition, and $G \subset \mathrm{Aut}(X)$. Generally speaking, there are quite a lot of such G-surfaces. For example, G-action lifts on blowup $\sigma : S' \to S$ of 0-dimensional G-orbit in S. The exceptional locus of σ is a 1-dimensional G-orbit.

This is a clue on how to make a minimal rational G-surface from an arbitrary one - one has to contract G-orbits of mutually disjoint (-1)-curves, and explicitly describe the situation when there are no such G-orbits.

Although classification of minimal rational G-surfaces was first obtained in [20] before the appearance of Mori theory, but namely the equivariant Mori theory gives a conceptually transparent proof.

3.3. We recall some elements of this theory in the case of rational surfaces S (see [29]). Let $\mathrm{NE}(S)$ be the cone of effective divisor classes in $\mathrm{Pic}(S) \otimes \mathbb{R}$, $\overline{\mathrm{NE}}(S)$ – its closure in the real topology, $\overline{\mathrm{NE}}(S)_{K \geqslant 0} = \{Z \in \overline{\mathrm{NE}}(S) | K_S \cdot Z \geqslant 0\}$. *Mori's cone theorem* in this special case states that:

$$\overline{\mathrm{NE}}(S) = \overline{\mathrm{NE}}(S)_{K \geqslant 0} + \sum_C \mathbb{R}_+[C],$$

where $[C]$ is a class of smooth rational curve C with $C^2 = -1$, 0 or 1. $\mathbb{R}_+[C]$ are called extremal rays and form a finite, or countable set. Each extremal ray

$R = \mathbb{R}_+[C]$ gives rise to a contraction morphism $\phi_R : S \to T$. The classification of these contractions for nonsingular projective rational surfaces is very simple:

(i) contraction of (-1)-curve, if $R = \mathbb{R}_+[C]$ and $C^2 = -1$;

(ii) the morphism $\phi_R : S \to \mathbb{P}^1$ is a locally trivial bundle with fibers isomorphic to $\mathbb{P}^1$, if $C^2 = 0$ (then $S \simeq \mathbb{F}_N$ is a geometrically ruled surface, also known as Hirtzebruch surface, with $-N = E^2$, $E \subset \mathbb{F}_N$ is a unique section with negative selfintersection index if $N > 0$; if $N = 0$ then $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$);

(iii) $\phi_R : S = \mathbb{P}^2 \to pt$ is a contraction of $\mathbb{P}^2$ to the point, if $C^2 = 1$.

Now let's have a look at the equivariant theory under the action of a finite group G on S. The cone theorem has the form:

$$\overline{\mathrm{NE}}^G(S) = \overline{\mathrm{NE}}^G(S)_{K \geqslant 0} + \sum \mathbb{R}_+ F^G,$$

here F^G denote G-invariant faces, formed by G-orbits of extremal rays $\mathbb{R}_+[C]$. Each face F^G gives rise to an extremal contraction – G-equivariant morphism $\phi_F : S \to T$, moreover classification $(i)-(iii)$ is modified as follows:

(i)^G $\phi_{F^G} : S \to T$ – a contraction of an orbit of mutually disjoint (-1)-curves;

(ii)^G $\phi_{F^G} : S \to \mathbb{P}^1$-a G-equivariant conic bundle with two possible cases:

(a) morphism ϕ_{F^G} is smooth – it is the former (ii) with $\phi : \mathbb{F}_N \to \mathbb{P}^1$ and the action of G;

(b) there are degenerate fibers which are bouquets of two lines $\mathbb{P}^1 \vee \mathbb{P}^1$ transposed by G-action;

In both cases $\mathrm{Pic}^G(S) \simeq \mathbb{Z} \oplus \mathbb{Z}$ generated by the class of a fiber $[f]$ and the class of invariant section $[E]$ in case (a), and anticanonical class $-K_S$ in case (b).

(iii)^G besides case (iii) above, there is a series of Del Pezzo surfaces S with $\mathrm{Pic}^G(S) = \mathbb{Z}$ and contraction morphism $\phi_{F^G} : S \to pt$ to the point; with possible cases $d = K_S^2 = 1, 2, \ldots, 8$, if $d = 8$: $S \simeq \mathbb{P}^1 \times \mathbb{P}^1$ with $\mathrm{Pic}^G(S) = \frac{1}{2}\mathbb{Z}(-K_S) \simeq \mathbb{Z}$, for the other d it is a blowing-up at $r = 9 - d$ points on $\mathbb{P}^2$ in general position (see 6.2 a) below); for such S one has $\mathrm{Pic}^G(S) = \mathbb{Z}(-K_S)$.

Now let's formulate our classification result.

Theorem 3.4. [20, 29] *Up to G-isomorphism minimal rational G-surfaces (G-equivariant rational Mori fibrations of dimension* 2) *are exhaustive by two families*(ii)^G *or* (iii)^G *above.*

Indeed, by contracting faces F^G G-equivariantly as in (i)^G, after a finite number of steps (since each contraction decreases the rank of Picard group) we come to one of the cases (ii)^G or (iii)^G. ∎

Let us mention some properties of minimal rational G-surfaces.

3.5. Conic bundles

Let $\phi : S \to \mathbb{P}^1$ be a conic bundle with the action of G as in (ii)^G. If morphism ϕ is smooth as in (ii)^G a), then $S = \mathbb{F}_N$ is a ruled surface with a unique $(-N)$-section provided $N > 0$ and $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. In view of its uniqueness the $(-N)$-section is G-invariant. The full automorphism group $\mathrm{Aut}(\mathbb{F}_N), N \geqslant 1$, is represented in the extension form

$$0 \to K \to \mathrm{Aut}(\mathbb{F}_N) \to \mathrm{PGL}_2 \xrightarrow{\phi_*} 1 \quad (5)$$

where K is represented also in the extension form

$$0 \to \bigoplus_{i=0}^{N} \mathbb{C} \to K \to \mathbb{C}^* \to 1,$$

where $\mathbb{C}^*$ acts on $\mathbb{C}^{N+1}$ via scalar multiplications. If $G \subset \mathrm{Aut}(\mathbb{F}_N)$ - a finite group, it has the form (since there are no nontrivial finite subgroups in $\mathbb{C}^{N+1}$)

$$1 \to \mu_n \to G \to H \to 1,$$

where $H \subset \mathrm{PGL}_2$ is a finite subgroup of PGL_2, and $\mu_n \in \mathbb{C}^*$ is a cyclic group of n-th roots of unity. In the case $N = 0$, the group $\mathrm{Aut}(\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1)$ is represented in the extension form

$$1 \to \mathrm{PGL}_2 \times \mathrm{PGL}_2 \to \mathrm{Aut}(\mathbb{P}^1 \times \mathbb{P}^1) \to \{1, \alpha\} \to 1 \quad (6)$$

where α is a permutation involution of factors. If $G \subset \mathrm{Aut}(\mathbb{F}_0)$ - a finite subgroup and $\mathrm{rk}\ \mathrm{Pic}^G(\mathbb{F}_0) = 2$, then $G = H_1 \times H_2$, where $H_i \subset \mathrm{PGL}_2$ are finite subgroups, if $\mathrm{rk}\ \mathrm{Pic}^G(\mathbb{F}_0) = 1$, then

$$1 \to H \to G \to \{1, \alpha\} \to 1, \quad H \subset \mathrm{PGL}_2$$ [2]

The classification of finite subgroups of PGL_2 is well known. These are cyclic groups, dihedral groups, tetrahedral group of even permutation A_4, octahedral symmetric group S_4 and icosahedral group $I \simeq A_5$.

If $\phi : S \to \mathbb{P}^1$ has degenerations $\mathbb{P}^1 \vee \mathbb{P}^1$ in $m > 0$ fibers, then $K_S^2 = 8 - m$ and $\mathrm{rk}\ \mathrm{Pic}(S) = 2 + m$. If $G \subset \mathrm{Aut}(S)$ - a finite subgroup and $\mathrm{rk}\ \mathrm{Pic}^G(S) = 2$, then G transposes fiber components, consequently its order $|G|$ is even. The group G is represented in the exact sequence form

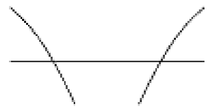
$$1 \to G' \to G \xrightarrow{\phi_*} G'' \to 1$$

where G'' is the projection of G into the group of automorphisms of the base PGL_2, $G' \subset G$ acts fiberwise and does not transpose fibers. For example, in the case of De Jonquieres involutions δ acts fiberwise and $\phi_*\delta = id$.

3.6. Minimal Del Pezzo G-surfaces

For them $d = K_S^2 = 1, \ldots, 6, 9$. The number d is called *the degree* of Del Pezzo surface $S = S_d$. If $d = 7$, S_7 is a blowing-up at two points on $\mathbb{P}^2$ with the following configuration of three (-1)-curves of the form

[2]These are only two examples from the general list of finite subgroups in $\mathrm{Aut}(\mathbb{F}_0)$ from the Dolgachev's lectures mentioned above. The list consists of 22 groups.



In this configuration the horizontal (-1)-curve must be G-invariant, since it intersects two other (-1)-curves, in the same way the pair of vertical (-1)-curves is G-invariant, since they intersect only one (-1)-curve. As a result $\mathrm{Pic}^G(S_7) \simeq \mathbb{Z} \oplus \mathbb{Z}$, so S_7 is not G-minimal. Similarly $S_8 = \mathbb{F}_1$ contains a G-invariant (-1)-curve, and hence it is not G-minimal. In the case $d = 9$ $S_9 \simeq \mathbb{P}^2$ with a linear action of G; in the case $d = 8$ $S_8 \simeq \mathbb{P}^1 \times \mathbb{P}^1$ with $\mathrm{Pic}^G(S_8) = \frac{1}{2}\mathbb{Z}(-K_S)$ – this is a quadric in $\mathbb{P}^3$ with the action of G induced by the linear action of $\mathbb{P}^3$. Since $\mathrm{Pic}^G(S_8) \simeq \mathbb{Z}$, G transposes factors of $\mathbb{P}^1 \times \mathbb{P}^1$ and exact sequence (6) shows that the projection $\phi_* : G \to \{1, \alpha\}$ is surjective, which means that the order $|G|$ is even.

The modern definition of Del Pezzo surfaces – these are nonsingular projective surfaces S with ample anticanonical divisor class (*i.e.* two-dimensional Fano varieties). If the degree $K_S^2 = d \geqslant 3$, then anticanonical linear system $|-K_S|$ is very ample, it gives an embedding of S onto the surface S_d of degree d in $\mathbb{P}^d$. The cases $d = 1$ and 2 were considered in 2.5 and 2.4, respectively under the consideration of Bertini and Geiser involutions. All Del Pezzo surfaces S_d, except $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$, are obtained from $\mathbb{P}^2$ by blowing up $m = 9 - d$ points in a general position, and on every such surface there is a finite number of (-1)-curves (these are lines under the embedding $S_d \subset \mathbb{P}^d$ with $d \geqslant 3$), the number of which is shown in the following table (see *e.g.* [28]):

d	1	2	3	4	5	6	7	8
m	8	7	6	5	4	3	2	1
l	240	56	27	16	10	6	3	1
rk Pic(S)	9	8	7	6	5	4	3	2

(7)

For example, on cubic surface $S_8 \subset \mathbb{P}^3$ there is a famous configuration of 27 lines. For $d \leqslant 7$ the group $\mathrm{Pic}(S_d)$ is generated by (-1)-curves. Now let's have a look at the structure of automorphism groups $\mathrm{Aut}(S_d)$. If $d = 6$ (the case $d = 7$ was excluded above in view of non-minimality of such a G-surface), then S_6 is obtained as a blowing-up at three points in a general position on $\mathbb{P}^2$. Let $N \subset \mathrm{PGL}_3$ be a subgroup of automorphisms of $\mathbb{P}^2$ that leave this triple invariant, then $\mathrm{Aut}(S_6)$ is a semi-direct product

$$1 \to N \to \mathrm{Aut}(S_6) \to \{1, \tau\} \to 1$$

where $\{1, \tau\} \subset \mathrm{Aut}(S_6)$ is a subgroup of order 2 obtained by lifting to S_6 the standard involution τ from 2.1. If the cases $d = 1, 2, \ldots, 5$ group $\mathrm{Aut}(S_d)$ is contained in $\mathrm{Aut}(R)$ of root lattice of classical series $\mathbb{A}, \mathbb{D}, \mathbb{E}$ (automorphism group of graph corresponding to the configuration of (-1)-curves, see *e.g.* [28]). More precisely (see [13]) $\mathrm{Aut}(S_d) \subset W \subset \mathrm{Aut}(R)$, where W is a Weyl group for root system R. We write down the root systems R, Weyl groups W and their orders $|W|$ that appear here:

R	$\mathbb{A}_4$	$\mathbb{D}_5$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
W	S_5	$\mathbb{Z}_2^4 \rtimes S_5$	$\mathbb{E}_6$ type	$\mathbb{E}_7$ type	$\mathbb{E}_8$ type
$\|W\|$	$5!$	$2^4 \cdot 5!$	$2^7 \cdot 3^4 \cdot 5$	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$

(8)

Thus, if $G \subset \mathrm{Cr}_2$ – a finite subgroup acting via automorphisms on $S_d, 1 \leqslant d \leqslant 5$, then it is a subgroup of the known groups W from the table (8). Under the embedding $S_d \subset \mathbb{P}^d$ group G acts by linear automorphisms of the ambient space $\mathbb{P}^d$.

4. Subgroups of Prime Order $p > 2$ in Cr_2

4.1. We expose here a part of results by Zhang [33] and De Fernex [11] (see also the note by Beauville and Blanc [4]) on classification of finite order p subgroups in Cremona group. The case $p = 2$ was examined in 2, so we shall assume $p \geqslant 3$. As remarked in the Introduction, classification results in [33] and [11] essentially are same, just diffrent by the method of proof. Here we follow [11] where a more directt and constructive approach is used.

First of all let's mention a result of [4] on the conjugacy of linear automorphisms.

Proposition 4.2. [4] *Linear projective automorphisms of the same order n are conjugate in Cr_2 (we make a remark that in the linear projective group PGL_3 they are not conjugate if they have non-proportional collections of eigenvalues).*

Sketch of a proof. First of all any automorphism of finite order is diagonalizable. Let $T = \mathbb{C}^* \times \mathbb{C}^*$ be the standard maximal torus in PGL_3. We shall consider it as an affine subset in $\mathbb{P}^2$. Then its automorphisms can be extended to Cremona transformations of $\mathbb{P}^2$ and the whole group of its automorphisms $\mathrm{GL}_2(\mathbb{Z})$ is a subgroup in Cr_2. Element $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ acts by the formula: $(x, y) \to (x^a y^b, x^c y^d)$. We have to show that any two elements of order n in T can be transformed each to other via conjugation by some element of $\mathrm{GL}_2(\mathbb{Z}) \subset \mathrm{Cr}_2$.

The kernel of multiplication by n in T is isomorphic to $\mathbb{Z}_n \oplus \mathbb{Z}_n$. Considering it as a $\mathbb{Z}_n$-module, any element of order n in $\mathbb{Z}_n \oplus \mathbb{Z}_n$ can be assumed as a basis element. Group $\mathrm{SL}_2(\mathbb{Z}_n)$ acts transitively on such elements. Any transformation from $\mathrm{SL}_2(\mathbb{Z}_n)$ sending one element of order n to another, can be lifted to an element in $\mathrm{SL}_2(\mathbb{Z})$, since the map $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}_n)$ is surjective. ∎

Thus let $\sigma \in \mathrm{Cr}_2$ be an element of prime order $p \geqslant 3$, then by results of 3 it acts on a minimal Del Pezzo $\langle\sigma\rangle$-surface $S_d, d = 1, 2, \ldots, 6, 9$. The case $d = 8$ and conic bundles are excluded since $p \neq q2$.

Theorem 4.3. [11] *Let σ be an element of prime order $p \geqslant 3$ in Cr_2. Then* *up to conjugacy any action of σ acts on a minimal Del Pezzo $<\sigma>$-surface S_d either of the following types:*

(i) $d = 9$*:* $\sigma \in PGL_3$ *– a linear automorphism,* $p \geqslant 3$*;*

(ii) $d = 3, p = 3$*:* σ *acts on a cubic surface* $S_3 \subset \mathbb{P}^3$ *with equation* $x^3 = F(y, z, w)$ *by formula* $(x, y, z, w) \to (\lambda x, y, z, w), \lambda^3 = 1, \lambda \neq q1$*, where* (x, y, z, w) *are homogeneous coordinates in* $\mathbb{P}^3$*;*

(iii) $d = 1, p = 3$*:* S_1*- a Del Pezzo surface of degree* 1 *embedded into the weighted projective space* $\mathbb{P}(1, 1, 2, 3)$ *with coordinates* x, y, z, w *and weights* $1, 1, 2, 3$ *respectively, as a surface of degree* 6 $z^3 = F(x, y, w)$*, and* σ *acts by formula* $(x, y, z, w) \mapsto (x, y, \lambda z, w), \lambda^3 = 1, \lambda \neq q1$*;*

(iv) $d = 1, p = 5$*:* S_1*- a Del Pezzo surface of degree* 1*, embedded into* $\mathbb{P}(1, 1, 2, 3)$ *as above, just with equation* $xy^5 = F(x, z, w)$*, and the action of* σ *is* $(x, y, z, w) \mapsto (x, \mu y, z, w)$*, where* $\mu \in \mathbb{C}, \mu^5 = 1, \mu \neq 1$*.*

Idea of the proof. If σ acts on a surface $\mathbb{F}_N$, then by (5) and (6) it is a subgroup of PGL_2, so this is case (i).

Now consider actions of σ on Del Pezzo surfaces $S_d, d = 9, 6, 5, \ldots, 1$. If $d = 9$, then $S_d \simeq \mathbb{P}^2$ and the action is linear. In the remaining cases consider any $\langle\sigma\rangle$-orbit of the (-1)-curve E, say $D = E + \sigma E + \ldots + \sigma^{p-1}E$, then, since $\mathrm{Pic}^\sigma(S_d) = \mathbb{Z}(-K_S)$, $D \in |-aK_{S_d}|$ for some integer $a > 0$. We have

$$p = -K_S \cdot D = aK_S^2 = ad.$$

Consequently $d = 1$, or p, because p is prime. All (-1)-curves on S_d splits into $\langle\sigma\rangle$-orbits of length p, so p divides the number of lines on S_d, which can be found in the table (7). It follows that there remain only two possibilities: $p = 3$ and $p = 5$. Let us consider these cases:

Case $d = p = 3$. Here $S_3 \subset \mathbb{P}^3$ and σ acts by a linear automorphism of the space $\mathbb{P}^3$. It means that one can choose a coordinate system so that the action becomes diagonalized with cubic roots of unity on the diagonal. But this is still not sufficient. One has to sudy the locus of fixed points of σ-action on S_3. It can be a disjoint union of points, lines, and possibly a linear section of S_3- a smooth plane cubic curve (see eg. [33], 1.2).

Let us show that in this case the fixed locus is only the smooth plane cubic curve. Indeed, $\langle\sigma\rangle$ acts on $\mathrm{Pic}^{(S_3)} \otimes \mathbb{C}$ whose dimension is 7 (see table (7)) with 1-dimensional invariant subspace $\mathrm{Pic}^G(S_3) \otimes \mathbb{C} = \mathbb{C}(-K_S)$. Then eigenvalues are $1, \lambda, \lambda, \lambda, \lambda^2, \lambda^2, \lambda^2$, where $\lambda = e^{\frac{2\pi i}{3}}$ (see *e.g.* [33], 1.6). So trace $\mathrm{tr}(\sigma) = -2$ and since $H^1 = H^3 = 0$ on S_3 the alternated sum of traces of σ-action on cohomologies is equal to 0. By Lefschetz' theorem, the latter sum is equal to the sum of Euler characteristics of the fixed locus components. But Euler characteristics of a point and a line are positive, it remains only possibility- a smooth plane cubic.

Let $x = 0$ be an equation of the plane cutting out this cubic and $F(y, z, w) = 0$ - equation of the fixed cubic on this plane as the ramification divisor for the factor-map $S_3 \to S_3/\langle\sigma\rangle$. It follows that the equation for S_3 is of the form $x^3 - F(y, z, w) = 0$ with the action of σ as in statement (ii) of the theorem.

Case $d = 1, p = 3$. Group $\langle\sigma\rangle$ acts on S_1 leaving the pencil $|-K_{S_1}|$ invariant. The action of σ on this pencil is fiberwise, i.e. it is trivial on the base. Indeed, if the action of σ on the base is not trivial, then the fixed locus consists of points, fibers and fiber components. Their Euler characteristics are nonnegative, but the trace of σ acting on $\oplus_{i=0}^4 H^i(S_1, \mathbb{Z})$ equals to -1.

Let's now consider the action of σ on fibers of this pencil which are elliptic curves. At least one fixed point is known- it is a unique base point of the pencil $|-K_{S_1}|$. Thus the factor of the elliptic curve by the action of σ is isomorphic to $\mathbb{P}^1$ and by Hurwitz' formula there must be two more branch points, i.e. there exists a fixed curve $R \subset S_1$ intersecting fibers of the pencil $|-K_{S_1}|$ in two points, so $R \in |-2K_{S_1}|$. Let's choose coordinates on $\mathbb{P}(1,1,2,3)$ as eigenfunctions of σ-action, then equation for R is $z = 0$. Now it is clear that the equation for S_1 and the action of σ as in the theorem, the case (iii).

The remaining case $d = 1, p = 5$ is considered analogously. ∎

We did not concern here the possiblity $d = 5, p = 5$ examined also in [11], 4.6, where a Cremona transformation of order 5 on $\mathbb{P}^2$ was written down: $(x, y, z) \to (xz, x(y-z), z(x-y))$ which is lifted to automorphism σ on Del Pezzo surface of degree 5 with $\mathrm{Pic}^{<\sigma>}(S_5) = \mathbb{Z}(-K_S)$. However it was shown in [4] that this transformation is conjugate to a linear. Indeed, the action of σ on S_5 has fixed points, since contrariwise Euler characteristic $\chi(S_5) = 7$ were divisible by 5. The projection from a tangent to $S_5 \subset \mathbb{P}^5$ plane at any σ-fixed point is a σ-equivariant birational map $S_5 \dashrightarrow \mathbb{P}^2$ transferring σ to a linear automorphism from PGL_3. In [4] even one of matrices for such an automorphism was written down.

In connection with this remark the following question arises: whether transformations σ from (ii)-(iv) of Theorem 4.3 are conjugate to linear ones, or betweeen themselves ? The answer is given in the following:

Theorem 4.4. ([11], Th.F) *Let $\tau \in Cr_2$ be an element of prime order p (here including the case $p = 2$) and $G_\tau \subset Cr_2$- cyclic subgroup $\langle\tau\rangle$ generated by τ, $[G_\tau]$ - the conjugacy class of subgroup G_τ in Cr_2. Then there is a bijective correspondence between the set of conjugacy classes $\{[G_\tau]\}$ and the set of isomorphism classes of the following curves:*

- (i) *hyperelliptic curves (elliptic if $g = 1$) of genus $g \geqslant 1$, if τ - a De Jonquieres involution of genus $g \geqslant 1$;*
- (ii) *nonhyperelliptic curves of genus 3, if τ - a Geiser involution;*
- (iii) *nonhyperelliptic curves of genus 4, with canonical model on a quadratic cone, if τ - a Bertini involution;*
- (iv) *elliptic curves, if τ is lifted to the action of σ as in* 4.3 (ii);
- (v) *smooth curves of genus 2, if τ is lifted to the action of σ as in* 4.3 (iii);
- (vi) *elliptic curves, if τ is lifted to the action of σ as in* 4.3 (iv);

Sketch of a proof. Cases (i)-(iii) are already examined in 2. By Lemma 2.3 normalizations of fixed curves of genera > 0 are the basic invariant. In cases (i)-(vi) the fixed curve is irreducible, nonsingular and of genus > 0, so it determines the corresponding G_τ-surface uniquely, up to equivariant birational G_τ-map. ∎

Remark 4.5. A more general approach to G-equivariant birational classification of rational G-surfaces based on Nöether's inequality was proposed in [19–23, 27, 28]. For example, in [27] (see also [21]) it was shown that a minimal Del Pezzo G-surface of degree 1 is G-birationally rigid, i.e. it has no equivariant birational maps to minimal rational G-surfaces, except G-automorphisms to itself. The work [21] describes all possible equivariant birational maps between minimal rational G-surfaces (in the algebraic case, but as we already remarked, all results have adequate geometric interpretation). In the most cases these equivariant birational maps are birational G-automorphisms generated by various involutions.

5. On Beauville's Classification of p-Elementary Subgroups in Cr_2

5.1. The next step after the classification of the subgroups of the prime order p is the classification of the finite p-elementary subgroups $(\mathbb{Z}_p)^r, r \in \mathbb{Z}_+$, in Cremona group Cr_2. This problem was studied by A. Beauville in [5]. As usual, group $G \simeq (\mathbb{Z}_p)^r$ acts on the minimal G-surface and, if $p \neq 2$, then only on $\mathbb{P}^2, \mathbb{F}_N$ and del Pezzo surfaces $S_d, d = 1, \dots, 6$. The conjugacy class $[G]$ in Cr_2 depends only on the equivariant birational class of this G-surface.

Theorem 5.2. [5] *Let $G \simeq (\mathbb{Z}_p)^r \subset Cr_2$ - a subgroup where p is a prime number. Then*

a) *if $p \geqslant 5$, then $r \leqslant 2$; and if $r = 2$, then G is conjugate to the p-torsions subgroup in the maximal torus $(\mathbb{C}^*)^2 \subset PGL_3$;*

b) *if $p = 3$, then $r \leqslant 3$; and if $r = 3$, then G is conjugate to the 3-torsion subgroup in the diagonal torus $(\mathbb{C}^*)^3 \subset PGL_4$ which acts on Fermat's cubic $x^3 + y^3 + z^3 + w^3 = 0$;*

c) *if $p = 2$, then $r \leqslant 4$; and if $r = 4$, then G is conjugate to one of the following subgroups*

c$_1$) *the 2-torsion subgroup of the diagonal torus $(\mathbb{C}^*)^4 \subset PGL_5$ which acts on del Pezzo surface $S_4 \subset \mathbb{P}^4$ defined by the following equations:*

$$\begin{cases} x^4 + y^4 + z^4 + w^4 + u^4 = 0, \\ ax^4 + by^4 + cz^4 + dw^4 + eu^4 = 0 \end{cases} \tag{10}$$

for some mutually different $a, b, c, d, e \in \mathbb{C}^*$;

c$_2$) *the subgroup in Cr_2 generated by the involutions*

$$(x, y) \mapsto (-x, y), (x, y) \mapsto (1/x, y), (x, y) \mapsto \Big(x, \frac{\alpha(u)}{y}\Big),$$

$(x, y) \mapsto \big(x, \frac{\beta(u)y-\alpha(u)}{y-\beta(u)}\big)$ for rational functions $\alpha(u), \beta(u) \in \mathbb{C}(u), u = x^2 + x^{-2}$ and $\alpha(u) \neq 0,\ \beta^2(u)$.

Idea of the proof. First of all let's clarify which p-elementary subgroups are contained in $\mathrm{PGL}_2(k)$ where k is either $\mathbb{C}$, or $\mathbb{C}(t)$ - the field of rational functions. Let $C_p \subset \mathrm{PGL}_2$ be the subgroup of dilations $z \mapsto \lambda z, \lambda^p = 1$ and $\delta : \mathrm{PGL}_2(k) \to k^*/k^{*2}$ be the matrix determinant homomorphism. ∎

Lemma 5.3. ([5], 2.1) *Let $G \simeq (\mathbb{Z}_p)^r$ be a nontrivial subgroup in $PGL_2(k)$ where p is a prime number. Then*

(i) *if $p \neq 2$, then $r = 1$, if $p = 2$, then $r \leqslant 2$;*

(ii) *if $p \neq 2$, then G is conjugate to C_p;*

(iii) *if $p = 2$ and $r = 1$, then up to conjugation G is generated by involution $z \mapsto -z$ if $k = \mathbb{C}$; and by one of the involutions $z \mapsto -z$ or $z \mapsto \frac{\alpha(t)}{z}$ if $k = \mathbb{C}(t)$ where $\alpha(t) \in k^*/k^{*2}$; if $p = 2$ and $r = 2$, then G is conjugate to the subgroup $V \subset PGL_2(\mathbb{C})$ generated by $z \mapsto -z$ and $z \mapsto 1/z$ in the case of $k = \mathbb{C}$ and to the subgroup $V_{\alpha,\beta} \subset PGL_2(k)$ generated by $z \mapsto \frac{\alpha(t)}{z}$ and $z \mapsto \frac{\beta(t)z-\alpha(t)}{z-\beta(t)}$ in the case of $k = \mathbb{C}(t), \alpha(t) \neq 0, \beta^2(t)$;*

(iv) *in the case $p = 2, r = 2$ and $k = \mathbb{C}(t)$ the homomorphism $\delta : PGL_2(k) \to k^*/k^{*2}$ induces a one-to-one correspondence between conjugacy classes of subgroups $G \simeq (\mathbb{Z}_2)^2$ and subgroups of the 4th order in k^*/k^{*2}.*

All these statements can be verified directly, let's make it just for statement (ii). If $p \neq 2$, then σ is represented by matrix $A \in \mathrm{GL}_2(k)$ satisfying condition $A^p = aI$ where $a \in k^*$. The passage of taking determinant gives condition $(detA)^p = a^2$, so $a^2 \equiv 1(mod\ k^{*p})$. Since p is odd it follows from the last condition that $a \in k^{*p}$; so matrix A can be diagonalized and σ is conjugate to an element from C_p. ■

Let's continue our discussion of Theorem 5.2. If $G \simeq (\mathbb{Z}_p)^r$ acts on smooth conic bundles i.e. on the surfaces $\mathbb{F}_N, N = 1, 2, \ldots$, then it follows from (5) that G is a subgroup of $\mathrm{PGL}_2(\mathbb{C})$ and Lemma 5.3 can be applied. If G acts on $\mathbb{P}^1 \times \mathbb{P}^1$ and $p \neq 2$, then $G \subset \mathrm{PGL}_2 \times \mathrm{PGL}_2$ (see (6)) and the result is again clear from lemma. If $p = 2$, there are two cases: $G \subset \mathrm{PGL}_2 \times \mathrm{PGL}_2$, then $r \leqslant 4$ and if $r = 4$, then this is a statement c_2); if G transposes factors of $\mathbb{P}^1 \times \mathbb{P}^1$, then from exact sequence (6) one has

$$1 \to G' \to G \to \{1, \alpha\} \to 1$$

where $G' \subset V \times V$ as a diagonal subgroup (see the definition of V in Lemma 5.3, (iii)). So $r \leqslant 3$.

If G acts on conic bundles with the degenerate fibers and transposes components of these fibers, then $p = 2$ in this situation. Group G is decomposed into a direct sum of $G_1 \simeq (\mathbb{Z}_2)^{r_1}$ and $G_2 \simeq (\mathbb{Z}_2)^{r_2}$ where $r = r_1 + r_2$. Group G_1 acts on a general fiber of morphism $\varphi : S \to \mathbb{P}^1$ so it is contained in $\mathrm{PGL}_2(K)$ where $K = \mathbb{C}(t)$ and $G_2 \subset \mathrm{PGL}_2(\mathbb{C})$. Hence by lemma $r \leqslant 4$.

Now, let $r = 4$, then G_1 and G_2 - maximally possible 2-subgroups. One can choose coordinates on the base $\mathbb{P}^1$ such that $G_2 = V$, then G_1 is a subgroup of the form $V_{\alpha,\beta}$ from Lemma 5.3 (iii). But in order for group $G = G_1 \oplus G_2$ to act on $\varphi : S \to \mathbb{P}^1$ coherently, $V_{\alpha,\beta}$ needs to commute with V, in other words, functions $\alpha(t)$ and $\beta(t)$ must be invariant under $t \mapsto -t$ and $t \mapsto 1/t$. Invariant subfield $\mathcal{L} \subset \mathbb{C}(t)$ is generated by the function $u = t^2 + t^{-2}$. Thus α and β must be the functions in u as formulated in Theorem 5.2, c_1 with x instead of t and y instead of z.

Now assume that $G_2 \simeq (\mathbb{Z}_2)^{r_2}$ acts on del Pezzo surface $S = S_d$ and $Pic^G(S) \simeq \mathbb{Z}$. Since linear system $|-aK_S|$, $a \in \mathbb{Z}_+$ is invariant, G acts on S via restriction of the linear actions of the ambient space where surface S is embedded by this linear system. Therefore for us the following fact is important.

Lemma 5.4. ([5], 3.1) *Let $G \subset PGL_n$ be a subgroup isomorphic to $(\mathbb{Z}_p)^r$ where p – a prime number. Suppose that p is not a divisor of n, then $r \leqslant n-1$ and if $r = n-1$, then G is conjugate to the p-torsion subgroup of the diagonal torus $(\mathbb{C}^*)^{n-1} \subset PGL_n$.*

Proof. Let's consider the extension

$$1 \to \mu_n \to \mathrm{SL}_n \to \mathrm{PGL}_n \to 1,$$

where μ_n is the group of n-th roots of unity. The lifts of G to SL_n are parameterized by the cohomology group $H^2(G, \mu_n)$ vanishing by n and p. Consequently the extension of G by μ_n is trivial and G is lifted to subgroups in SL_n which are isomorphic to $(\mathbb{Z}_p)^r$. All such subgroups are contained in the maximal torus which are conjugate. ∎

In the case of $S = \mathbb{P}^2$, according to 5.4 if $p \neq 3$, then $r \leqslant 2$; and if $r = 2$, then G is the p-torsion subgroup of the torus $(\mathbb{C}^*)^2$. If $p = 3$ it is knows that $r \leqslant 2$ and if $r = 2$, besides the diagonal subgroup, there is also a subgroup generated by the automorphisms $(x, y, z) \mapsto (y, z, x)$ and $(x, y, z) \mapsto (x, \lambda y, \lambda^2 z), \lambda = e^{2\pi i/3}$. In the case of del Pezzo surfaces $S = S_d$ of degree $d = 6, 5, \ldots, 1$ such a surface is a blowing-up at $9 - d$ points on $\mathbb{P}^2$ in a general position and if $d \leqslant 5$, G is contained in the Weyl group W as in the table (9). When $d = 6$ from the exact sequence (8) we find that if $p \neq 2$, then $G \subset N$ and by Lemma 5.4 $r \leqslant 2$. If $p = 2$, then by the same lemma $r \leqslant 3$.

Let us study the remaining cases $d = 1, 2, \ldots, 5$. If $d = 1$ or 2, then Bertini and Geiser involutions determine double coverings with ramification in the curve C of genus $g = 4$ and 3 respectively. This induces an exact sequence

$$0 \to \mathbb{Z}_2 \to Aut(S) \to Aut(C).$$

From the Hurvitz genus formula for the coverings of curves it is easy to deduce (see [5], 3.7) that if $G \simeq (\mathbb{Z}_p)^r$ acts effectively on C, then p^{r-1} divides $2g - 2$.

From the table (9) one can see that if $p \geqslant 7$, then the order $|W|$ is not divided by p^2, consequently in W there are no subgroups $G \simeq (\mathbb{Z}_p)^r$ with $r \geqslant 2$. When $p = 5$ the order $|W|$ is divided by p^2 only if $d = 1$, but then 5 divides $2g - 2 = 6$ that is impossible. Hence there remain only two possibilities: $p = 3$ or 2.

The above exposed arguments are sufficent for getting the statement *b*) when $p = 3$ and the statement c_1) when $p = 2$ in 5.2. ∎

Let's make an addition that from the explicitly indicated actions on Fermat's cubic in the case *b*) and on the intersection of two quadrics (10) in the case c_1) one sees that there is no G-invariant 0-dimesional orbit of length $l < d = degS$ on the correspondent surface. By Noether's inequality ([21], 2.4) it follows immediately that these G-surfaces are birationally rigid, i.e. they have no birational G-mappings on the other minimal G-surfaces except G-isomorphisms on itself (see [21], 1.6 (ii)). (The proof of this fact in [5] involves the fixed curves as in Lemma 2.3). Hence the conclusion - the conjugacy class of $[G]$ in Cr_2 for these two cases is uniquely determined by the corresponding G-surface.

6. Appendix

6.1. In this appendix let's consider the conjugacy question for two specific embeddings of the group $S_3 \times \mathbb{Z}_2$ in Cr_2 where S_3 is the permutation group of 3 elements (see [22, 23]). This question arose in [26] in connection with the study of one property of linear algebraic groups, namely the property to be a Cayley group.

Now let G denote a linear algebraic group and $\mathfrak{g}$ - its Lie algebra. Group G acts on itself by conjugacy $x \mapsto g{\cdot}x{\cdot}g^{-1}$, $g \in G$, $x \in X = G$ as an affine algebraic variety. On $Y = \mathfrak{g}$ group G acts as adjoint representation $Ad_G g(y) = g \cdot y \cdot g^{-1}$, $g \in G$, $y \in Y$. The authors of [26] call a group G 'Cayley' if there exists a G-equivariant birational map $\lambda : X - - \to Y$, *i.e.* $\lambda(gxg^{-1}) = Ad_G(\lambda(x))$. The question of being a Cayley group for simple algebraic groups in [26] is reduced to the question on the existence of W-equivariant birational map of the maximal torus T on its Lie algebra where W is the Weyl group.

In particular for a simple algebraic group as $\mathbb{G}_2$ the maximal torus T : $xyz = 1$ has a dimension 2 and the Weyl group $W = S_3 \times \mathbb{Z}_2$. The action of W on T is given by

(I) $T : (xyz = 1)$. S_3 acts by the permutations on the coordinates and $\mathbb{Z}_2$ acts via the involution $(x, y, z) \mapsto (x^{-1}, y^{-1}, z^{-1})$;
On the Lie algebra $P : x + y + z = 0$ W acts in the following way

(II) $P : (x + y + z = 0)$. S_3 acts by the permutations on the coordinates and $\mathbb{Z}_2$ acts by the sign inversion $(x, y, z) \mapsto (-x, -y, -z)$.

Each of these actions gives an embedding of W in Cr_2. The question is whether the images under these embeddings are conjugate or not? The negative answer obtained in [22, 23] means that the group $\mathbb{G}_2$ is not a Cayley group. It is interesting that in [26] it is shown that the group $\mathbb{G}_2 \times G_m^2$, where G_m is the multiplicative group of the field with trivial action on W, is already a Cayley group. This means the stable Cayley property of the group $\mathbb{G}_2$, i.e. although T and P are not W-equivariantly birational they are stably W-birational: $T \times (\mathbb{C}^*)^2 \underset{\sim}{W} P \times (\mathbb{C}^*)^2$ with trivial action of W on the second factors.

6.2. A sketch of the proof. Consider the smooth projective completions X and Y of the affine rational surfaces T and P respectively, so that X and Y are minimal rational W-surfaces. Then $X = X_6 \subset \mathbb{P}^6$ is a del Pezzo surface of degree 6 with $Pic^W(X) = \mathbb{Z}(-K_X)$ and Y – the plane $\mathbb{P}^2$ with linear W-action.

Suppose two indicated embeddings of $W \subset \mathrm{Cr}_2$ are conjugate, then there exists W-equivariant birational map $\chi : X - \to Y = \mathbb{P}^2$. According to the general theory (see [21]) there is an algorithm, called equivariant Sarkisov's program, decomposing map χ into a sequence of elementary W-equivariant birational maps called links . These links are constructed according to the equivariant Mori theory (see [21]) and all of them are classified. In our situation links are constructed from the 0-dimensional W-orbits, of length $l < degX$ on X and of lengths $l_i < degX_i$ on W-surfaces X_i that appear as a result of applying links $\Phi_i : X_i - \to X_{i+1}$. A necessary condition for the existence of a link - the orbit points must be in a general position in the following sense

a) if X is a del Pezzo surface and $\sigma : X' \to X$ is a blowing-up at all the orbit points, then X' is also a del Pezzo surface;

b) if $\phi : X \to \mathbb{P}^1$ is W-equivariant conic bundle, then the points $x_1, \dots x_l$ are in a general position if no point lies on a degenerate fibre and any two of them do not lie on the same fibre. This is a necessary and sufficient condition for the existence of an elementary transformation $\Phi = \mathcal{E}_{x_1,\dots x_l}$

$$\begin{array}{ccc} X & \overset{\Phi}{--\to} & X' \\ \downarrow & & \downarrow \\ \mathbb{P}_1 & \simeq & \mathbb{P}_1 \end{array}$$

blowing-up points $x_1, ...x_l$ and contracting the strict transforms of fibres where these points lie.

The algorithm if decomposing χ is constructed as follows. On $Y = \mathbb{P}^2$ fix a very ample W-invariant linear system, say $|-K_{\mathbb{P}^2}|$, and consider its strict transform with respect to χ on X:

$$\mathcal{H} = \chi_*^{-1}(|-K_{\mathbb{P}^2}|).$$

By definition of the strict transform $\mathcal{H}$ has no fixed components, it is W-invariant and $\mathcal{H} \sim -aK_X$ for some $a \in \mathbb{Z}_+$ since $Pic^W(X) = \mathbb{Z}(-K_X)$.

In view of Noether's inequality ([21], 2.4) linear system $\mathcal{H}$ must have *a maximal singularity* (since χ is not an isomorphism), *i.e.* a 0-dimensional W-orbit $x \subset X$ of multiplicity $r = r(x) > a$ in a base set of $\mathcal{H}$. The length $l = l(x)$ of an orbit must be less then K_X^2 because of the condition on multiplicity, in our case $l < 6$.

Now if the orbit $x \subset X$ is a maximal singularity, then one can check easily that its points are in a general position and there is link $\Phi_x : X \dashrightarrow X_1$ with center in x transferring linear system $\mathcal{H}$ to $\mathcal{H}_1 \sim -a_1 K_{X_1}$ with $a_1 < a$. Hence one needs to classify W-orbits of length $l < 6$ on X and to find out for which of them the points are in a general position. From the equation $xyz = 1$ one sees that there are only three such orbits:

(1) $P = (1, 1, 1)$ – a W-fixed point;
(2) $\{P_1, P_{-1}\} = \{(\lambda, \lambda, \lambda), (\lambda^2, \lambda^2, \lambda^2)\}, \lambda = e^{2\pi i/3}$;
(3) $\{Q_1, Q_2, Q_3\} = \{(1, -1, -1), (-1, 1, -1), (-1, -1, 1)\}$.

Orbit (3) can not be a maximal singularity; its points are not in a general position (the curves $E_1 : (z = -1), E_2 : (y = -1), E_3 : (x = -1)$ become (-2)-

curves after blowing-up points Q_1, Q_2, Q_3 and the anticanonical class fails to be very ample).

With orbit (2) there is an "untwisting" link – a birational involution $\Phi_{P_1,P_{-1}} : X \dashrightarrow X$ which decreases a coefficient a at $-K_X$. The link $\Phi_P : X \dashrightarrow X_2 \subset \mathbb{P}^3$ – projection from a plane tangent to $X = X_6 \subset \mathbb{P}^6$ at P, is associated with the fixed point $P \in X$. This projection blows-up the point P and contracts three conics (with equations $x = 1, y = 1, z = 1$ in $xyz = 1$) passing through P, into a W-invariant triple of points on $C_0 \subset X_2$ – the image of the blown-up line.

The equation of X_2 in $\mathbb{P}^3$ has the form

$$X_2 : xy + yz + zx = 3w^2$$

where (x, y, z, w) are the homogenous coordinates on $\mathbb{P}^3$ and the equation of the conic $C_0 :$ $(w = 0$ on $X_2)$. Group S_3 acts by transposing coordinates x, y, z and fixing w, $\mathbb{Z}_2$ acts via sign inversion $(x, y, z, w) \mapsto (-x, -y, -z, w)$ fixing w and $Pic^W(X_2) \simeq \frac{1}{2}\mathbb{Z}(K_X) = \mathbb{Z}[C_0]$.

Let $\mathcal{H}_2 \sim -a_2 K_{X_2}$ be the strict transform of linear system $\mathcal{H}$ on X_2. Then $\chi_2 = \chi \circ \Phi_P : X_2 \dashrightarrow Y = \mathbb{P}^2$ is not an isomorphism and again by Noether lemma there must be a maximal singularity $x_2 \subset X_2$ of multiplicity $r(x_2) > a_2$ and of length $l(x_2) < K^2_{X_2} = 8$. There are two W-orbits on X_2 of length 2, two of length 3 and few of length 6. The associated with them "untwisting links" are either birational involutions and X_2 is mapped nowhere, or the associated with the orbits of length 3 links bringing us back to X.

Thus the analysis shows that it is impossible to decompose χ into a sequence of links, that contradicts theorem 2.5 in [21]. Consequently there is no W-equivariant map $\chi : X \dashrightarrow Y = \mathbb{P}^2$, or no affine surface T on P. Hence two indicated embeddings $W \subset \mathrm{Cr}_2$ lie in different conjugacy classes. ■

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