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Zariski Pairs on Sextics I

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Abstract. We study Zariski pairs of sextics which are distinguished by the Alexander polynomials. For this purpose, we present two constructive methods to produce explicit sextics of non-torus type with given configuration of simple singularities.

1. Introduction

A plane curve of degree d is called a (p,q)-torus curve $(p,q \geq 2)$ or a curve of (p,q)-torus type if p,q divide d and C can be defined by a polynomial of the form $f_{d/p}(x,y)^p + f_{d/q}(x,y)^q$ where the degree of $f_{d/p}$, $f_{d/q}$ are d/p and d/q respectively. In this paper, we consider sextics of (2,3)-torus type. Their defining polynomial $f(x,y)$ are written as $f = f_2^3 + f_3^2$. A curve of (p,q)-torus type has $d^2/(pq)$ cusp singularities of the type $y^p + x^q = 0$ at the intersection $f_{d/p} = f_{d/q} = 0$ if two curves $f_{d/p} = 0$ and $f_{d/q} = 0$ intersect transversely. Then the global Alexander polynomial of C is the same with the local Alexander polynomial of the torus link in S^3 defined by $y^p + x^q = 0$ [10]. A curve C of degree d is called *of non-torus type* if the defining polynomial has no such expression for any $p,q \geq 2$ with $p|d$ and $q|d$. For a given torus curve C and a singularity $P \in C$, we say that P is an *inner* singularity if it is on the intersection $f_{d/p} = f_{d/q} = 0$. A torus curve C is called *tame* if it has only inner singularities.

Two plane curves C, C' of the same degree d is called a *Zariski pair* if (i) they have the same configuration of the singularities and there exist topologically equivalent tubular neighborhood, and (ii) the topology of the pair $(\mathbf{P}^2, C)$ and $(\mathbf{P}^2, C')$ are not equivalent.

This notion is introduced by Zariski [16] and formulated as above by Artal Bartolo [16]. The first such example is given by Zariski on sextics: (Z_6, Z_6') where two curves Z_6, Z_6' are sextics with 6 A_2-cusps, Z_6 is a sextic of (2,3)-torus type but Z_6' is not of torus type [16]. The corresponding Alexander polynomials are $t^2 - t + 1$ and 1 respectively. The purpose of this paper is to give further Zariski pairs in sextics which can be distinguished by Alexander polynomials. Pho determined all possible configurations of the singularities of irreducible tame sextics of (2,3)-torus type $C:\ f_2^3+f_3^2=0$ with simple singularities [12]. Hereafter, we simply say a "configuration" instead of a "configuration of the singularities". In fact, they are given by

(1) (1,1,1,1,1,1): $[6A2]$
(2) (2,1,1,1,1): $[A_5, 4A_2]$, $[E_6, 4A_2]$
(3) (2,2,1,1): $[2A_5, 2A_2]$, $[A_5, E_6, 2A_2]$, $[2E_6, 2A_2]$
(4) (2,2,2): $[3A_5]$, $[2A_5, E_6]$, $[A_5, 2E_6]$, $[3E_6]$
(5) (3,1,1,1): $[A_8, 3A_2]$
(6) (3,2,1): $[A_8, A_5, A_2]$, $[A_8, E_6, A_2]$
(7) (3,3): $[2A_8]$
(8) (4,1,1): $[A_{11}, 2A_2]$
(9) (4,2): $[A_{11}, A_5]$, $[A_{11}, E_6]$
(10) (5,1): $[A_{14}, A_2]$
(11) (6): $[A_{17}]$

The numbering corresponds to the partition types of 6 (=the total intersection number of the conic $f_2=0$ and the cubic $f_3=0$). In this paper, we will show that

Theorem 1. *For any configuration* Σ *in* $(1)\sim(11)$*, there is a Zariski partner irreducible sextic, say* C'*, of non-torus type with configuration* Σ*. So choosing an irreducible sextic of torus type* C *with configuration* Σ*, we have a Zariski pair* (C, C') *and they are distinguished by the Alexander polynomial, either* t^2-t+1 *or* 1 *respectively.*

Most of them has been studied by several poeple. We have studied the fundamental group of the complement of the sextics with configuration $[6A_2]$, $[3E_6]$ in [7, 8]. Tokunaga studied the cases with E_6 singularity in [13, 15]. In [3], Eyral and the author have proved the fundamental groups of the complements of the sextics of non-torus type with the following configurations are abelian.

$$[2A_8],\ [A_{17}],\ [A_{11}, E_6],\ [A_{11}, A_5],\ [A_{14}, A_2],\ [A_8, A_5, A_2],\ [A_8, E_6, A_2]$$

Sextics with A_{17} is discussed in [1]. See also [4]. So new results in this paper are only those assertions for the remaining cases. Especially for the cases $[A_5, 4A_2]$, $[E_6, 4A_2]$, the construction is a new result. We also construct 4 dimensional subspace of sextics of non-torus type with $6A_2$ and the fundamental group of the complements are abelian.

In the second part of this paper, we will study Zariski pairs in non-irreducible sextics [11].

2. Two Computational Methods

In this section, we introduce two computational methods to obtain a sextics with a prescribed configuration of singularities. First we recall basic properties of sextics of torus type. Consider a sextic of torus type

$$T:\ f(x,y)=f_2(x,y)^3+f_3(x,y)^2=0$$

where f_2, f_3 are polynomials of degree 2 and 3 respectively. Hereafter, we mean by a sextic of torus type a sextic of (2, 3)-torus type. Recall that a singularity $P\in T$ is called an inner singularity if P is at the intersection of the conic $C_2:\ f_2(x,y)=0$ and the cubic $C_3:\ f_3(x,y)=0$. If C_3 is non-singular at P, the corresponding singularity of T at P is described by the local intersection number $\iota:=I(C_2,C_3;P)$ and the singularity at P is isomorphic to $A_{3\iota-1}$. If C_3 is singular at P, (T,P) is a simple singularity if and only if when C_2 is reduced (so smooth at P) and $I(C_2,C_3;P)=2$. In this case, we have $(T,P)\cong E_6$.

In [9], we have introduced an local invariant $\rho(P,5)$ for any local singular point $P\in C$ by $\dim_{\mathbf{C}}\mathcal{O}_P/\mathcal{I}_{P,5,6}$ where C is a sextic and the ideal $\mathcal{I}_{P,5,6}$ is defined by the adjuction ideal [5. 6. 2. 1] and put $\rho(C,5):=\sum_P\rho(P,5)$. For a sextic of torus type, we observe [9].

Proposition 2. *Assume that T is a sextic of torus type and P be an inner simple singularity. Then $\rho(P,5)=\iota(C_3,C_2;P)$. Thus $\sum_{P:inner}\rho(P,5)=6$ for a sextics of torus type with only simple singularities where the summation is taken for all inner singularities P.*

Suppose that a sextic $C=\{g(x,y)=0\}$ is given and assume that the singularities of C are simple singularities contained in the list: $\{A_1,\ E_6,\ A_2,\ A_5,\ A_8,\ A_{11},\ A_{14},\ A_{17}\}$. Here C need not to be of torus type. Let us denote the configuration set of singularities of C by $\Sigma(C)$. The following is a convenient criterion by Tokunaga to check if C is of torus type.

Lemma 3. (Tokunaga Criterion [14]) *The sextic C is of torus type if and only if there exists a conic $C_2:\ g_2(x,y)=0$ such that $C_2\cap C\subset\Sigma(C)$ and $I(C,C_2;P)=2\rho(P,5)$ for any $P\in C\cap C_2$.*

This is a useful tool to check if a given sextic C is of torus type or not when C is already given. For our purpose, we are interested in explicit construction of sextics of non-torus type with given configuration of singularities Σ.

Definition 4. *Assume that $O\in C$ is an $A_{3\iota-1}$ singularity. We assume that $y=t_1x$ is the tangent cone of (C,O). Then there exist unique complex numbers $t_2,\dots,t_\tau$, with $\tau=[3\iota/2]$ such that we can write*

$$\tilde g(x,y):=g(x,y+t_1x+t_2x^2+\cdots+t_\tau x^\tau)=ay^2+bx^{3\iota}+\textit{(higher terms)},\ a,b\neq 0$$

Higher terms are linear combinations of monomials $x^\alpha y^\beta$, $2\alpha+3\iota\beta>6\iota$. We call $y=t_1x+t_2x^2+\cdots+t_\tau x^\tau$ **the maximal contact coordinate curve** *of (C,O).*

Let E be the curve defined by $y-(t_1x+t_2x^2+\cdots+t_\tau x^\tau)=0$. Then by definition, we have $I(C,E;O)=\mathrm{order}_x\, g(x,t_2x^2+\cdots+t_\tau x^\tau)=3\iota$.

Method 1. $\sharp(\Sigma) \leqslant 4$. By the action of $\mathrm{PGL}(3, \mathbf{C})$, we may assume that the locations of singularities are on the chosen explicit points. For example, we explain the construction of a sextic of non-torus type with $\Sigma = [A_{14} + A_2 + A_1]$. We start from the generic form of sextic:

$$g(x, y) = \sum_{i+j \leqslant 6} a_{ij} x^i\, y^j$$

We have 27 free coefficients. Let us assume that $C = \{g(x, y) = 0\}$ has A_{14} singularity at $O = (0, 0)$, A_2 at $P = (1, 0)$ and A_1 at $Q = (-1, 1)$. The condition $(C, Q) \cong A_1$ is given by 3 linear equation. The condition $(C, P) \cong A_2$ is given by 3 linear condition and one quadratic equation i.e., the hessian vanishes at P. To avoid this quadratic equation, we may assume that $x = 1$ is the tangent cone at P and $x = 0$ is the tangent cone at the origin. Thus the quadratic equation is replaced by two linear equation $\dfrac{\partial^2 g}{\partial x \partial y}(P) = \dfrac{\partial^2 g}{\partial y^2}(P) = 0$. So up to here, we eliminated 8 coefficients. The most difficult one is the condition for O to be A_{14}. For this purpose, we consider the maximal contact coordinate curve for A_{14}. Namely we suppose that $x = 0$ is the tangent cone of (C, O) and $x = t_2\, y^2 + \cdots + t_7\, y^7$ is the maximal contact coordinate curve at O. Here $t_2, \dots, t_7$ are variables to be determined later. Thus to find a sextic of non-torus type C, we consider the polynomial $h(x, y) := g(x + t_2\, y^2 + \cdots + t_7\, y^7,\, y)$ and put it as $h(x, y) = \sum_{ij} b_{ij} x^i\, y^j$. (This has $19 + 6 = 25$ free variables in their coefficients, where 19 of them are linear.) We use the notation $\mathrm{Coeff}\,(h, x^i\, y^j)$ for b_{ij}. Note that b_{ij} is a linear combination of $a_{k\ell}$ with coefficients in $\mathbf{C}[t_2, \dots, t_7]$. The assumption

$$h(x, y) = a\, x^2 + c\, y^{15} + (\text{higher terms}), \quad a, c \neq 0$$

implies that

$$b_{0j} = 0,\ j \leqslant 14 \quad \text{and} \quad b_{1s} = 0,\ s \leqslant 7.$$

which is comparatively easy to be solved. We have 23 equations. So at the end of calculation, we have to solve some non-linear equations. Now we explain how we can solve the equations explicitly (preferably over $\mathbf{Q}$) so that the resulting sextics are of non-torus type. Assume that C is a sextic of torus type i.e., there exists a conic $C_2 := \{h_2(x, y) = 0\}$ and a cubic $C_3 := \{h_3(x, y) = 0\}$ such that $g(x, y) = h_2(x, y)^3 + h_3(x, y)^2$. Then C_3 is smooth at O and assume that the equation $h_3(x, y) = 0$ can be solved locally as $x = \sum_{i \geq 2} s_2\, y^i$. Put $x_1 = x - \sum_{i \geq 2} s_2\, y^i$. We see that the assumption $I(C_2, C_3; O) = 5$ implies that $\mathrm{ord}_y\, h_2(\sum_{i \geq 2} s_i\, y^i, y) = 5$. This implies that $h_3 = x_1 U$ with U being a unit, $h_2(x_1 + \sum_{i \geq 2} s_i\, y^i, y) = \alpha x_1 + \beta\, y^5 + (\text{higher terms})$, $\alpha,\ \beta \neq 0$ and

$$h_2(x_1 + \sum_{i \geq 2} s_i\, y^i, y)^3 + h_3(x_1 + \sum_{i \geq 2} s_i\, y^i, y)^2 = U(0) x_1^2 + \beta^3\, y^{15} + (\text{higher terms})$$

Thus by the uniqueness of the maximal contact coordinate curve, we have $t_i = s_i$, $i \leqslant 5$. Thus *The maximal contact coordinate of inner $A_{3\iota-1}$ singularity of* *a sextic of torus type is given by solving* $f_3(x,y) = 0$ *in* x. Solving the five equalities

$$h_2(1,0) = 0, \quad \text{and } \mathrm{Coeff}(h_2(t_2y^2 + \cdots + t_5y^5,\, y),\, y^j) = 0, \quad j = 0,\ldots,4$$

and thus we see that $h_2(x,y)$ is written as $h_2(x,y) = c\,(t_2^2y^2 + t_3xy + x^2t_2 - xt_2)$, $c \neq 0$ and the last equality $\mathrm{Coeff}(h_2(t_2y^2 + \cdots + t_5y^5, y), y^5) = 0$ implies the equality

$$J := t_3t_4 + 2t_2^2t_3 - t_5t_2 = 0.$$

Note that there are 5 free coefficient in h_2 and 6 equations can not satisfied without the condition $J = 0$. This implies that the maximal contact coordinate for a sextic of a torus type must satisfy $J(t_2,\ldots,t_5) = 0$, under the above assumption on the location of the singularities. Thus in the construction of sextics with configuration $[A_{14} + A_2 + A_1]$, we choose the coefficients $t_2,\ldots,t_5$ such that $J(t_2,\ldots,t_5) \neq 0$ which is enough to guarantee that the sextic to be obtained will be of non-torus type. An example of sextic with $[A_{14} + A_2 + A_1]$ is give by:

$$\begin{aligned} g(x,y) = {} & 208716480\,x^5 - 74370240\,x^6 - 38384640\,y^2\,x + 331067520\,y^4\,x \\ & +19192320\,x^2 - 660935520\,y^2\,x^4 + 19192320\,y^4 - 175129920\,x^4 \\ & -701719200\,y^4\,x^2 - 9596160\,y\,x^2 + 28788480\,y^5 + 760495680\,y^3\,x^2 \\ & -226709280\,y^2\,x^2 - 106757280\,y^6 - 319072320\,y\,x^3 - 374250240\,y^5\,x \\ & +926029440\,y^2\,x^3 - 338264640\,y\,x^5 + 666933120\,y\,x^4 \\ & -19192320\,y^3\,x - 858856320\,x^3\,y^3 + 21591360\,x^3 \end{aligned} \tag{1}$$

Method 2: Semi-torus method. This method is quite useful for the case $\sharp(\Sigma) \geq 5$ but it is also convenient for the case $\sharp(\Sigma) \leqslant 4$. In the case $\sharp(\Sigma) \geq 5$, the method 1 does not work in general as the locations of all singularities can not be arbitrary, though we can fix the locations of four singularities. This is the case when we consider the configurations for example $4A_2 + A_5$, $4A_2 + E_6$, $6A_2$. To consider these cases systematically, we consider *a sextic of semi-torus type* which is defined by

$$(ST) \quad C:\, g(x,y) \;=\; f_2(x,y)^3 + g_2(x,y)^2\,h_2(x,y) = 0$$

where f_2, g_2 (respectively h_2) are polynomials of degree 2 (resp. of degree $\leqslant 2$). A singularity $P \in C$ is called *wild* if P is on the intersection of three conics: $\{f_2(x,y) = g_2(x,y) = h_2(x,y) = 0\}$. We say C is *nice* if C does not have any wild singularities. As in the case of (2,3)-torus curve, we call a singular point $P \in C$ *inner* (respectively *outer*) if $f_2(P) = g_2(P) = 0$ (resp. $f_2(P) \neq 0$) and P is not wild. By the same argument as in [12], it is easy to see that the possible configurations of inner singularities of a nice sextic C of semi-torus type are

$$(\sharp):\ [4A_2],\ [A_5 + 2A_2],\ [E_6 + 2A_2],\ [2E_6],\ [A_8, A_2],\ [A_{11}].$$

We can check by a direct computation that *the possible outer singularities are* A_j, $j \leqslant 5$ *and* E_6 *is not possible as an outer singularity.* (We do not consider other singularities.) So we can construct sextics with configuration $\Sigma = \Sigma_{inner} \cup \Sigma_{outer}$ where Σ_{outer} can be either $[2A_2]$ or $[A_5]$. Just like the case of sextic of torus type, we can show easily that

Proposition 5. *An inner singularity $P \in \{f_2(x,y) = g_2(x,y) = 0,\, h_2(x,y) \neq 0)\}$ is an E_6 if only if the conic $g_2(x,y) = 0$ is singular at P and $I(f_2, g_2; P) = 2$. This implies that $g_2(x,y) = 0$ is either a union of two lines meeting at P or a line of mutiplicity 2 passing through P.*

Thus $[2E_6]$ is obtained only if $g_2(x,y) = 0$ is a double line $\ell(x,y)^2 = 0$ and $\ell(x,y) = 0$ intersects the conic $f_2(x,y) = 0$ at two distinct points. We assume that a sextic C of semi-torus type is given in the form (ST) and its inner configuration Σ_{inner} is given from the list $(\sharp)$. To explain a little more in detail, we construct sextics of semi-torus type with $\Sigma_{inner} \cup \{2A_2\}$, $\Sigma_{inner} \cup \{A_5\}$ systematically. Let P, Q be two outer A_2 singularities. In the case $P = Q$, we assume that $(C, P) \cong A_5$. First we put $t_1 := f_2(P)$, $s_1 := g_2(P)$, $t_2 := f(Q)$, $s_2 := g_2(Q)$. We assume that $t_1,\, t_2,\, s_1,\, s_2 \neq 0$ and

$$(C,P) \cong A_2,\, (C,Q) \cong A_2, \quad \text{if } P \neq Q \tag{2}$$

$$(C,P) \cong A_5, \quad \text{if } P = Q. \tag{3}$$

Then we first observe that:

Proposition 6. *The linear system of conics $j_2 = 0$ satisfying $I(j_2, g; S) = 2I(f_2, g_2; S)$ for any inner singularity S is given by*

$$C_2(t,s): \quad j_2(x,y) := t\, f_2(x,y) + s\, g_2(x,y) = 0, \quad s, t \in \mathbf{C}.$$

Proof. Note that the space of conics is 5-dimensional and the condition is 4-dimensional. So the space is a linear system. Assume that S is an inner singular point. For a generic t, s, the conic $j_2 = 0$ is smooth and $I(j_2, g_2; S) = I(j_2, f_2; S)$, and therefore we have $I(j_2, f_2^3 + g_2^2 h_2; S) = 2I(j_2, g_2; S)$. ∎

Now we assert

Theorem 7. *Assume that C is a nice sextic and let Σ_{inner} be the inner configuration of C. Then C is a sextics of non-torus type which contains*

(a) *$\Sigma_{inner} \cup \{2A_2\}$, if $P \neq Q$ and $t_1, s_1, t_2, s_2 \neq 0$ and $t_1 s_2 - t_2 s_1 \neq 0$, or*

(b) *$\Sigma_{inner} \cup \{A_5\}$, if $P = Q$ and $t_1, s_1 \neq 0$ and the tangent cone of C at P is not the tangent line of the conic $C_2(s_1, -t_1)$ at P.*

Proof. Suppose that there exists a conic j_2 passing through the singular points of C with the prescribed intersection number. Then as it passes through P, Q, we must have the equality: $t_j\, t + s_j\, s = 0$ for $j = 1, 2$. This has a solution $(t, s) \neq (0, 0)$ if and only if $t_1 s_2 - t_2 s_1 = 0$, contradicting to the assumption. If $P = Q$, the condition that $j_2 = 0$ passes through P implies that $(t, s) \sim (s_1, -t_1)$.

If $(C, P) \cong A_5$ and if C is of torus type defines as $j_2^3 + j_3^2 = 0$ for some cubic form j_3, the tangent cone is given by tangent line of $j_2 = 0$ at P. ■

Remark.
1. The equation $f(P) = 0$ is usually difficult to solve (i.e., the elimination of a parameter involves a big denominator). This difficulty is resolved by introducing new parameter t_1, s_1 so that $f_2(P) = t_1$ and $g_2(P) = s_1$ as the latter are easy to be solved in a coefficient of f_2 or g_2 respectively. Then the equation $f(P) = 0$ is a linear equation in the coefficients of h_2.
2. For a wild singularity P of a semi-torus sextic C, so $f_2(P) = g_2(P) = h_2(P) = 0$ with $\iota = I(f_2, g_2; P)$, the corresponding singularity is $D_{3\iota+1}$ for $\iota = 1, 2, 3, 4$.

Let $\mathcal{M}_{ST}(6A_2; 6)$ be the configuration space of sextics with semi-torus type with $6A_2$ and the non-torus condition $t_1 s_2 - t_2 s_1 \neq 0$. This is a subspace of the configuration space of irreducible sextics with non-conical $6A_2$ (let us denote this space by $\mathcal{M}_{nt}(6A_2; 6)$ and we claim that $\mathcal{M}_{ST}(6A_2; 6)/PGL(3, \mathbf{C})$ *is connected of dimension* 4. To see the irreducibility, we start from the generic forms of f_2, g_2, h_2 and consider the slice condition:

($\sharp$): 2 outer A_2 are at $P_1 := (0, 1)$, $P_2 := (0, -1)$ with the respective tangent cones $y \mp 1 = 0$.

The corresponding conditions can be written as

$$f_2(P_i) = t_i,\ g_2(P_i) = s_i,\ f(P_i) = 0,\ i = 1, 2$$
$$\frac{\partial f}{\partial x}(P_i) = \frac{\partial f}{\partial y}(P_i) = \frac{\partial^2 f}{\partial x \partial y}(P_i) = \frac{\partial^2 f}{(\partial x)^2}(P_i) = 0,\ i = 1, 2$$

Using these equations, we can eliminate 10 coefficients and the number of the remaining *free coefficients* are $17 - 10 - 1 = 6$. (One dimension comes from the change $(g_2, h_2) \mapsto (\alpha\, g_2, h_2/\alpha)$). This slice is clearly connected (or irreducible) as it is a Zariski open subset of $\mathbf{C}^6$. As we have two dimensional isotopy group which fix P_1, P_2 and the tangent cone $y \mp 1 = 0$. So the quotient is 4 dimensional.

Proposition 9. *For any sextic* $C \in \mathcal{M}_{ST}(6A_2; 6)$*, the fundamental group* $\pi_1(\mathbf{P}^2 - C)$ *is isomorphic to the cyclic group* $\mathbf{Z}/6\mathbf{Z}$*.*

In fact by the connectivitity of the configuration slice, it is enough to show the assertion for one explicit $C \in \mathcal{M}_{ST}(6A_2; 6)$. Recall that we have shown shown that $\pi_1(\mathbf{P}^2 - C) \cong \mathbf{Z}/6\mathbf{Z}$ in [7] where C is defined by $f_6(x, y) = 0$,

$$f_6(x, y) := x^2\,(x-1)^2\,(x^2 + 2\,x) - x^2\,(x-1)^2\,(y^2 - 1)$$
$$+ \frac{1}{3}\,(x^2 - 1)\,(y^2 - 1)^2 - \frac{1}{27}\,(y^2 - 1)^3$$

On the other hand, f_6 has a semi-torus decomposition as

$$f_2(x, y) = -\frac{1}{3}\,y^2 + x^2 - \frac{2}{3}, \quad g_2(x, y) = -(x - 1)\,y, \quad h_2(x, y) = 2\,x + 1$$

Remark 10. We have a conjecture that the space $\mathcal{M}_{nt}(6A_2;6)$ is connected and the fundamental group of the complement $\pi_1(\mathbf{P}^2 - C)$ is abelian for any C in $\mathcal{M}_{nt}(6A_2;6)$. The above Proposition is a partial answer to this. We know that the Alexander polynomial is trivial for any C in $\mathcal{M}_{nt}(6A_2;6)$. We expect that the dimension of $\mathcal{M}_{nt}(6A_2;6)/\mathrm{PGL}(3,\mathbf{C})$ is $19 - 6 \times 2 = 7$ dimensional. We can check directly by computation that the dimension of the quotient space $\mathcal{M}(\Sigma;6)/\mathrm{PGL}(3,\mathbf{C})$ with a given configuration of simple singularities Σ is 19 minus the total Milnor number, if the number of singularities are less than 4.

3. Construction of Irreducible Sextics of Non-torus Type

In this section, we construct irreducible non-torus sextics with configurations (1), (2), (3), (4), (5) and (8).

1. First we consider the configuration $\Sigma = [A_{11}, 2A_2]$. This can be constructed either by Method 1 or also Method 2. Usually the computation by Method 2 is less heavy. For example, we look for a sextics of semi-torus type:

$$C:\ g(x,y) = f_2(x,y)^3 + g_2(x,y)^2\, h_2(x,y) = 0$$

where A_{11} is at (1,0) as an inner singularity and we put two $2A_2$ as outer singularities at $(0,1)$, $(0,-1)$. For example, we can take the equations:

$$\begin{aligned}
f_2 :=\ & 16962 + 43740\,y - 33924\,x - 9816\sqrt{3} - 9816\,x^2\sqrt{3} - 25286\,y\sqrt{3}\\
& - 15792\,y^2\sqrt{3} + 19632\,x\sqrt{3} + 27445\,y^2 + 16962\,x^2\\
g_2 :=\ & 6 + 6\,x^2 - 38\,y^2 + 9\sqrt{3} + 9\,x^2\sqrt{3} - 11\,y^2\sqrt{3}\\
& + 16\,y\sqrt{3} - 18\,x\sqrt{3} - 12\,x + 3\,y\\
h_2 :=\ & -(-136995863187\,x^2 - 45018774416\,y^2 + 24984902205\sqrt{3}\\
& + 80265133314\,x^2\sqrt{3} + 23477770096\,y^2\sqrt{3} + 49893109990\,y\sqrt{3}\\
& - 23083475220\,x\sqrt{3} + 36411797484\,x\,y + 35951630964\,x - 83905166058\,y\\
& - 41386138293 - 20817696276\,x\,y\sqrt{3})\big/\big((-3539 + 1787\sqrt{3})\,(7 + 4\sqrt{3})^4\big)
\end{aligned}$$

The conics $f_2(x,y) = 0$ and $g_2(x,y) = 0$ are tangent at $(1,0)$ with intersection multiplicity 4 so that C has A_{11} singularity at $(1,0)$.

2. Now we construct a sextics of non-torus type with $\Sigma = [A_8 + 3A_2]$ as its configuration. We can use both methods in Case 1 or 2. Here we use the second method. So we consider sextic of semi-torus type whose inner singularities are A_8 at (1,1) and two outer A_2 at $(1,0)$, $(0,1)$. As an example, we obtain:

$$\begin{aligned}
g(x,y) &= f_2(x,y)^3\ +\ g_2(x,y)^2\, h_2(x,y)\\
f_2 &:=\ -y^2 - \frac{1}{5}\,y\,x\sqrt{59} - \frac{13}{5}\,y\,x + \frac{1}{5}\,y\sqrt{59} + \frac{13}{5}\,y + 3\,x^2\\
&\quad + \frac{1}{5}\,x\sqrt{59} - \frac{12}{5}\,x - \frac{1}{5}\sqrt{59} + \frac{2}{5}
\end{aligned}$$

$$g_2 := -\frac{1}{219010}(-155+6\sqrt{59})(1810\,y^2+173\sqrt{59}+1724 \\ +1724\,y\,x-5084\,y+1506\,x+173\,y\,x\sqrt{59}-233\,y\sqrt{59} \\ -1680\,x^2-30\,x^2\sqrt{59}-83\,x\sqrt{59})$$

$$h_2 := \frac{1}{1876463765}(-26177+2292\sqrt{59})(2996295\,x^2-256330\,y^2 \\ +24928\,y\,x\sqrt{59}+14372\,y\sqrt{59}-371968\,x\sqrt{59} \\ +306120\,x^2\sqrt{59}+956794+183424\,y\,x \\ +77308\sqrt{59}+346616\,y-3822204\,x)$$

Notice that two conics $f_2(x,y)=0$ and $g_2(x,y)=0$ intersect at (1,1) with intersection multiplicity 3 to make A_8 with one more transverse intersection which makes an inner A_2.

3. We consider $\Sigma_{4,1} := [3A_5]$, $\Sigma_{4,2} := [2A_5+E_6]$, $\Sigma_{4,3} := [A_5+2E_6]$. For $[2A_5+E_6]$, we use Method 2. For $[3A_5]$, $[2E_6+A_5]$, we can use either Method 1 or 2. For the configuration $[3E_6]$, we have already constructed in [8].

3-1. A sextic with configuration $[3A_5]$ is given by Method 2 as:

$$g(x,y) = f_2(x,y)^3 + g_2(x,y)^2 h_2(x,y)$$

$$f_2 := 1+2\,x^2-\frac{25}{1407}\,I\,x^2\sqrt{1407}-y^2+x\,y+\frac{25}{1407}\,I\,x\sqrt{1407}-2\,x$$

$$g_2 := \frac{5}{4}\,y^2-\frac{5}{4}\,x\,y+\frac{125}{5628}\,I\,x^2\sqrt{1407}-\frac{1}{4}\,x^2-\frac{125}{5628}\,I\,x\sqrt{1407}+\frac{5}{2}\,x-\frac{5}{4}$$

$$h_2 := -\frac{155}{16}\,y^2+\frac{2025}{3752}\,I\,x\,y\sqrt{1407}-\frac{11}{2}\,x\,y-\frac{2025}{3752}\,I\,y\sqrt{1407}+\frac{131449}{7504}\,x^2 \\ +\frac{275}{2814}\,I\,x^2\sqrt{1407}-\frac{275}{2814}\,I\,x\sqrt{1407}-\frac{138953}{3752}\,x+\frac{138953}{7504}$$

Another example is obtained using Method 1:

$$f := 27800+7600\,y^5-15200\,y^3+86200\,x^3\,y^3+1650\,y^4\,x^2+15050\,x^6 \\ -3900\,y^2\,x^4-86200\,x^3\,y+58000\,y\,x^4+58000\,y\,x^2+59650\,x^2 \\ -58600\,x+7600\,y-58600\,y^4\,x+117200\,y^2\,x-58000\,y^3\,x^2-40600\,y\,x^5 \\ +40600\,y^4-62000\,y^2+61900\,x^4-6400\,y^6-61300\,y^2\,x^2-65200\,x^3 \\ -6400\,y^3\,x+3200\,y\,x+65200\,y^2\,x^3+3200\,y^5\,x-40600\,x^5$$

3-2. A sextics with $[2A_5+E_6]$ is given by the following. We have used Method 1 and $2A_5$ are $(0,-1)$, $(0,1)$ with respective tangent cones $y\mp 1=0$ and E_6 is at $(1,0)$.

$$\frac{1664357}{25425}\,x+2\,y+\frac{203063}{25425}\,y^2-\frac{135871}{2825}\,y\,x+y^6-4\,y^3+\frac{18148}{565}\,y\,x^5-\frac{34736}{2825}\,y\,x^2$$
$$-\frac{73177}{25425}\,x^6-\frac{88819}{25425}+2\,y^5-\frac{5222126}{25425}\,x^2-\frac{135871}{2825}\,y^5\,x+\frac{1664357}{25425}\,y^4\,x$$
$$+\frac{6641603}{25425}\,x^3-\frac{139669}{25425}\,y^4-\frac{3903338}{25425}\,x^4-\frac{4882462}{25425}\,y^4\,x^2+\frac{39260}{1017}\,x^5$$
$$+\frac{271742}{2825}\,y^3\,x-193\,y^3\,x^3-\frac{3328714}{25425}\,y^2\,x-\frac{471008}{2825}\,y\,x^4+193\,x^3\,y$$
$$-\frac{6641603}{25425}\,y^2\,x^3+\frac{1122732}{2825}\,y^2\,x^2+\frac{34736}{2825}\,y^3\,x^2-\frac{337334}{25425}\,y^2\,x^4=0$$

3-3. A sextic with $[2E_6+A_5]$. We put $2E_6$ at $(0,\pm 1)$ with respective tangent cones $y=\pm 1$ and we put A_5 at $(1,0)$). Take $g=f_2^3+g_2h_2$ where

$$f_2 := -\frac{667}{676}\,y^2-\frac{135}{676}\,I\,y^2\,\sqrt{3}-\frac{53}{26}\,x\,y+\frac{15}{26}\,I\,y\,x\,\sqrt{3}+y-x^2+x,\quad g_2:=y^2$$
$$h_2 := \frac{409455}{228488}\,x^2+\frac{915381}{742586}\,x\,y+\frac{2175255}{1485172}\,I\,y\,x\,\sqrt{3}+\frac{16605}{28561}\,I\,\sqrt{3}-\frac{25029}{57122}$$
$$+\frac{20915415}{38614472}\,I\,y^2\,\sqrt{3}-\frac{291195}{228488}\,I\,x\,\sqrt{3}+\frac{18225}{228488}\,I\,x^2\,\sqrt{3}-\frac{262035}{228488}\,I\,y\,\sqrt{3}$$
$$-\frac{373005}{228488}\,x+\frac{282123}{228488}\,y-\frac{15287373}{19307236}\,y^2$$

4. Sextics with $[2A_5+2A_2]$, $[A_5+E_6+2A_2]$, $[2E_6+2A_2]$.

4-1. A sextic with $[2A_5+2A_2]$ with $2A_5$ as inner singularities. We use Method 2. We put one inner A_2 at $(1,0)$ and an outer A_5 at O. Take $g=f_2^3+g_2h_2$ where

$$f_2 := 1-\frac{1}{3}\,y\,x+\frac{8203}{31752}\,y^2+\frac{1}{3}\,y-\frac{5444}{3969}\,x+\frac{1475}{3969}\,x^2$$
$$g_2 := 1-y\,x+\frac{21139}{31752}\,y^2+y-\frac{14857}{7938}\,x+\frac{6919}{7938}\,x^2$$
$$h_2 := -1+y\,x-\frac{24667}{31752}\,y^2+y+\frac{1475}{3969}\,x-\frac{36115}{47628}\,x^2$$

4-2. A sextic with $[A_5+E_6+2A_2]$ with E_6+2A_2 as inner singularities. We put E_6 at $(1,0)$ as an inner singularity and an A_5 as an outer singularity at O. Take $g=f_2^3+g_2h_2$ where

$$f_2 := -\frac{1}{24}\,y^2-y\,x+y+\frac{13}{12}\,x^2-\frac{25}{12}\,x+1$$
$$g_2 := \frac{3}{8}\,y^2-\frac{3}{2}\,y\,x+\frac{3}{2}\,y+x^2-2\,x+1$$
$$h_2 := \frac{1}{8}\,y^2-\frac{1}{4}\,y\,x-\frac{37}{32}\,x^2+\frac{9}{4}\,x-1.$$

Note that $g_2(x,y)=-1/16\left(4\,x-3\,y+y\sqrt{3}-4\right)\left(-4\,x+4+3\,y+y\sqrt{3}\right)$.

4-3. A sextic with $[2E_6+2A_2]$ with $2E_6$ as inner singularities at O and $(1,0)$. $2\,A_2$ are at $(0,1)$, $(1,1)$. Take $g=f_2^3+g_2h_2$ where

$$f_2 := \frac{14}{9}\,y^2 + \frac{2}{27}\,I\,y^2\,\sqrt{42} + y\,x - \frac{5}{9}\,y - \frac{2}{27}\,I\,y\,\sqrt{42} - \frac{1}{9}\,x^2 + \frac{1}{9}\,x, \quad g_2 := y^2$$

$$h_2 := -\frac{23}{12}\,y^2 - \frac{5}{9}\,I\,y^2\,\sqrt{42} - 7\,y\,x - \frac{2}{3}\,I\,y\,x\,\sqrt{42} + \frac{1}{6}\,y + \frac{8}{9}\,I\,y\,\sqrt{42}$$
$$-\frac{11}{3}\,x^2 + \frac{11}{3}\,x + \frac{2}{3}\,I\,x\,\sqrt{42} + \frac{3}{4} - \frac{1}{3}\,I\,\sqrt{42}$$

5. A sextic with $[A_5 + 4A_2]$, $[E_6 + 4A_2]$.

5-1. A sextic with $[A_5 + 4A_2]$ is given by Method 2. We put $4\,A_2$ as inner singularities and an A_5 at O as an outer singularity. Take $g = f_2^3 + g_2 h_2$ where

$$f_2(x,y) := 1 + \frac{33863}{1650}\,x + y - \frac{33863}{1650}\,x^2 + \frac{23404966}{226875}\,x\,y - \frac{12222527}{907500}\,y^2$$
$$g_2(x,y) := 2 + \frac{13727}{550}\,x + 7\,y - \frac{14277}{550}\,x^2 + \frac{43781281}{302500}\,x\,y - \frac{1217817}{75625}\,y^2$$
$$h_2(x,y) := -\frac{1}{4} - \frac{2517}{275}\,x + y + \frac{9243}{1100}\,x^2 - \frac{1667401}{151250}\,x\,y + \frac{209923}{151250}\,y^2$$

5-2. A sextic with $[E_6 + 4A_2]$ is obtained by Method 2. We put $2\,A_2$ at $(0,1)$, $(1,0)$ as outer singularities and an inner E_6 at $(1,1)$. Take $g = f_2^3 + g_2 h_2$ where

$$f_2 := \Big(\frac{163}{171} - \frac{1}{171}\,\sqrt{854}\Big)\,y^2 + \Big(\Big(-\frac{221}{152} - \frac{1}{342}\,\sqrt{854}\Big)\,x - \frac{683}{1368} + \frac{1}{114}\,\sqrt{854}\Big)\,y$$
$$+\Big(\frac{31}{152} + \frac{1}{342}\,\sqrt{854}\Big)\,x^2 + x - \frac{31}{152} - \frac{1}{342}\,\sqrt{854}$$
$$g_2 := y^2 + \Big(-\frac{7}{8}\,x - \frac{9}{8}\Big)\,y + \frac{1}{8}\,x^2 + \frac{5}{8}\,x + \frac{1}{4}$$
$$h_2 := \Big(-\frac{2179}{4332} + \frac{4}{1083}\,\sqrt{854}\Big)\,y^2 + \Big(\Big(\frac{1350}{361} - \frac{49}{1083}\,\sqrt{854}\Big)\,x - \frac{2036}{1083} + \frac{10}{361}\,\sqrt{854}\Big)\,y$$
$$+\Big(-\frac{53}{1083}\,\sqrt{854} - \frac{131}{1444}\Big)\,x^2 + \Big(\frac{29}{361}\,\sqrt{854} - \frac{1655}{722}\Big)\,x - \frac{34}{1083}\,\sqrt{854} + \frac{1997}{1444}$$

4. Further Application

The semi-torus method is also very useful to construct other plane curves. We give an example. We can construct a curve C of degree 10 with 10 A_4 singularities where 8 of them are inner and two A_4 are outer and we put them at $(0, \pm 1)$.

$$f(x,y) = f_2(x,y)^5 \;+\; g_4(x,y)^2\,h_2(x,y)$$
$$f_2(x,y) = -\frac{9}{4}\,y^2 + \frac{19}{4}\,x^2 + x + \frac{13}{4}$$
$$g_4(x,y) := -\frac{105}{32}\,y^4 - \frac{199}{16}\,x^4 - 8\,x^3 + \frac{165}{16}\,y^2 - \frac{21}{4}\,x$$
$$-\frac{1}{24}\,x^3\,\sqrt{30} - \frac{661}{32}\,x^2 + \frac{1}{12}\,x^2\,\sqrt{30}$$
$$+\frac{405}{32}\,y^2\,x^2 + \frac{15}{4}\,y^2\,x - \frac{1}{24}\,x^4\,\sqrt{30} - \frac{257}{32}$$
$$h_2(x,y) = \frac{15}{4}\,y^2 - \frac{19}{4} - \frac{813}{16}\,x^2 - 2\,(-\frac{661}{32} + \frac{1}{12}\,\sqrt{30})\,x^2 - 2\,x.$$

Consider the normal form of A_4 singularity $P \in P$: $v^2 + u^5 = 0$. The ideal of adjuction $\mathcal{I}_{P,k,10}$ (see [1, 9] for definition) is defined as

$$\mathcal{I}_{P,10,9} = \langle u^2, v\rangle, \quad \mathcal{I}_{P,k,10} = \langle u, v\rangle \; K = 7, 8$$

and the explicit computation of $\sigma_k : \mathcal{O}(k-3) \to \oplus \mathcal{O}_P/\mathcal{I}_{P,k,10}$ shows that the Alexander polynomial of C is trivial. Thus together with a generic curve of $(5,2)$-torus C', we have a Zariski pair (C, C') which are distinguished by their Alexander polynomials.

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