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Introduction to Branched Galois Covers

Hiroo Tokunanga

Department of Mathematics, Tokyo Metropolitan University, Tokyo, Japan

Abstract. This paper is an introduction to theory and application of D_{2n}-covers. We start with some elementary examples of Galois covers and survey how we treat D_{2n}-covers and apply them through examples.

1. Introduction

In this paper, everything is considered over C. Let us start with the following question:

What are branched Galois covers?

A branched cover is intuitively a space of solutions for an algebraic equation of one variable with coefficients containing parameters. Let us start with elementary examples as below:

Example 0.1. Let us consider a curve $C^{(n)}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equation

$$C^{(n)} : s_0 t_1^n + s_1 t_0^n = 0,$$

where $([s_0, s_1], [t_0, t_1])$ denote bi-homogeneous coordinates of $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathrm{pr}_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection to the first factor:

$$([s_0, s_1], [t_0, t_1]) \mapsto [s_0, s_1].$$

Let $U_{00} = \{p = ([s_0, s_1], [t_0, t_1]) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid s_0 t_0 \neq 0\}$, $V_0 = \{q = [s_0, s_1] \in \mathbb{P}^1 \mid s_0 \neq 0\}$ and $x = s_1/s_0$, $y = t_1/t_0$. In U_{00}, $C^{(n)}$ is given by

$$y^n - x = 0,$$

and put $\pi_n := \mathrm{pr}_1|_{C^{(n)}} : (x, y) \mapsto x$. Given a value x, then we have n points (by taking the multiplicity into account) over x. Moreover, we have a $\mathbb{Z}/n\mathbb{Z}$-action on $C^{(n)}$ given by

$$\sigma : (x, y) \mapsto (x, \zeta_n y), \quad (\zeta_n = \exp(2\pi\sqrt{-1}/n)),$$

and one can easily check this action preserves a fiber of $\pi_n : C^{(n)} \to \mathbb{P}^1$.

We say that C_n is a Galois cover of $\mathbb{P}^1$ with Galois group $\mathbb{Z}/n\mathbb{Z}$, or simply a $\mathbb{Z}/n\mathbb{Z}$-cover. π_n is called the covering of morphism. Note that there is only one point over $[1, 0]$ or $[0, 1]$. These points are called the branch points of π_n.

Example 0.2. We keep the notation as before. Let $D^{(n)}$ be a curve in $\mathbb{P}^1 \times \mathbb{P}^1$ given by the equation

$$s_0 t_1^{2n} - 2s_1 t_1^n t_0^n + s_1 t_0^{2n} = 0.$$

Put $\pi_n := \mathrm{pr}_1|_{D^{(n)}} : D^{(n)} \to \mathbb{P}^1$. One can easily see that, given a point on $\mathbb{P}^1$, there are $2n$ points (counted with multiplicities) over it. Moreover, the dihedral group D_{2n} of order $2n$, $D_{2n} := \langle \sigma, \tau \mid \sigma^2 = \tau^n = (\sigma\tau)^2 = 1\rangle$, acts on $D^{(n)}$ as follows:

$$y^\sigma = \frac{1}{y}$$
$$y^\tau = \zeta_n y.$$

One can easily see that this action is fiber preserving. We call $D^{(n)}$ a Galois cover of $\mathbb{P}^1$ with Galois group D_{2n} or simply D_{2n}-covers. π_n is called the covering morphism. π_n is given by $x = (t^n + 1/t^n)$. The set of the branch points is $\{[1, 1], [1, -1], [0, 1]\}$.

We give one more example.

Example 0.3. Let S_n be a surface in $\mathbb{P}^3$ given by the equation

$$z_0^n + z_1^n + z_2^n + z_3^n = 0,$$

where $[z_0, z_1, z_2, z_3]$ is a homogeneous coordinate. Let

$$\tilde{\pi} : [z_0, z_1, z_2, z_3] \in \mathbb{P}^3 \setminus [0, 0, 0, 1] \mapsto [z_0, z_1, z_2] \in \mathbb{P}^2$$

be the projection centered at $[0, 0, 0, 1]$. $\pi = \tilde{\pi}_{S_n} : S_n \to \mathbb{P}^2$ gives an n-to-1 morphism, and $\mathbb{Z}/n\mathbb{Z}$ acts S_n by

$$\sigma : [z_0, z_1, z_2, z_3] \mapsto [z_0, z_1, z_2, \zeta_n z_3], \quad \zeta_n = \exp(2\pi\sqrt{-1}/n).$$

σ clearly preserves a fiber of π. We call S_n a $\mathbb{Z}/n\mathbb{Z}$-cover of $\mathbb{P}^2$.

For Examples 0.1, 0.2 and 0.3, one can regard $C^{(n)}$, $D^{(n)}$ and S_n as "spaces" of solutions for an algebraic equation of one variable over parameter space. In fact, they correspond to equations as below:

$$C^{(n)} : y^n - x = 0,$$
$$D^{(n)} : y^{2n} - 2xy^n + 1 = 0,$$
$$S_n : z^n + x^n + y^n + 1 = 0,$$

where x is a parameter for $C^{(n)}$ and $D^{(n)}$ and x, y for S_n.

As we have seen as above, a branched Galois cover is always considered as a space of solutions of an algebraic equation over the base space.

In the next section, we give a precise definition of Galois covers. In Sec. 2 and Sec. 3, we survey some results and applications for dihedral covers, Galois covers having the dihedral groups as their Galois groups.

One can see that, although dihedral groups are considered most elementary finite non-abelian groups, such Galois covers give some non-trivial results.

1. Definitions and Terminologies

Let X and Y be normal projective varieties. We denote the rational function field of X and Y by $\mathbb{C}(X)$ and $\mathbb{C}(Y)$, respectively. Let $\pi : X \to Y$ be a finite surjective morphism. By considering the pull-back of functions, $\mathbb{C}(Y)$ is regarded as a subfield of $\mathbb{C}(X)$ and it is known that $\deg \pi = [\mathbb{C}(X) : \mathbb{C}(Y)]$ (see [9]).

The following fact is important in the study of Galois covers.

Fact. Let K be a finite extension of $\mathbb{C}(Y)$, let X_K be the K-normalization of Y and let $\pi_K : X_K \to Y$ be the normalization morphism. Then we have (i) $\mathbb{C}(X_K) = K$ and (ii) X_K is unique in the following sense:

Let X be a normal projective variety with a finite morphism $\pi : X \to Y$. If $\mathbb{C}(X) \cong K$ over $\mathbb{C}(Y)$, then there exists an isomorphism $\alpha : X \to X_K$ such that $\pi = \pi_K \circ \alpha$.

See [8], Theorem 2.2.4 for the normalization of varieties.

Definition 1.1.

(i) *We call $\pi : X \to Y$ a (branched) Galois cover if $\mathbb{C}(X)/\mathbb{C}(Y)$ is a Galois extension.*

(ii) *Let G be a finite group. We call $\pi : X \to Y$ a G-cover if it is Galois and $Gal(\mathbb{C}(X)/\mathbb{C}(Y)) \cong G$.*

Remark 1.1. Under the above setting (ii), G acts on X and $Y = X/G$ (see [16], for example).

We next define the branch locus of π. The branch locus, denoted by Δ_π or $\Delta(X/Y)$, is a subset of Y given by

$$\Delta_\pi = \{y \in Y \mid \ \pi \text{ is not (analytically) locally isomorphic over } y.\}.$$

It is known that Δ_π is an algebraic subset of codimension 1 if Y is smooth (see [23]).

Assume that Y is smooth. Let G be a finite group and let $\pi : X \to Y$ be a G-cover with $\Delta_\pi = B_1 + \cdot + B_r$, B_i being irreducible component of the divisor Δ_π. The ramification index along B_i is the one along the smooth part of B_i. More precisely, if we let $(w_1, \ldots, w_k)$ be a local coordinate of Y such that B_i

is given by $w_1 = 0$ and $(z_1, \dots, z_k)$ be a local coordinate of X such that π is given by $(z_1, \dots, z_k) \mapsto (w_1, \dots, w_k) = (z_1^{e_i}, z_2, \dots, z_k)$, then the ramification index along B_i is e_i.

We say that a G-cover $\pi : X \to Y$ is branched at $e_1B_1 + \cdots + e_rB_r$ if (i) $\Delta_\pi = B_1 + \cdots + B_r$ and (ii) the ramification index along B_i is e_i $(i = 1, \dots, r)$.

As for the examples in Introduction, we have the following table:

	G	Δ_π with ramification index
Example 0.1	C_n	$n[1,0] + n[0,1]$
Example 0.2	D_{2n}	$2[1,1] + 2[1,-1] + n[0,1]$
Example 0.3	C_n	$n\{z_0^n + z_1^n + z_2^n = 0\}$

One of fundamental problems in the study of branched Galois covers may be formulated in the following way:

Question 1.1. Let G be a finite group, let Y be a smooth projective variety and let $B = B_1 + \cdot + B_r$ be a reduced divisor on Y. Give a necessary and sufficient condition for the existence of a G-cover of Y branched at $e_1B_1 + \cdots e_rB_r$.

We consider Question 1.1 for some special cases such as $Y = \mathbb{P}^2$, B is a reduced plane curve and $G = D_{2n}$, the dihedral group of order $2n$. Even such special case, our question is not so trivial and there are interesting applications in the study of the topology of $\mathbb{P}^2 \setminus B$. We end this section with a basic facts which relates G-covers with the topology of $\mathbb{P}^2 \setminus B$:

Fact. There exists a G-cover $\pi : S \to \mathbb{P}^2$ with $\Delta_\pi = B \Leftrightarrow$ there exists a surjective homomorphism $\pi_1(\mathbb{P}^2 \setminus B, p_o) \to G$.

$\Rightarrow$ is immediate, while $\Leftarrow$ follows from Grauert-Remmert theorem [7].

2. D_{2n}-covers, an Introduction

2.1. An Observation on Cyclic Covers

Let us start with an observation on $\mathbb{Z}/n\mathbb{Z}$-covers. Let Y be a smooth projective variety and let X be a $\mathbb{Z}/n\mathbb{Z}$-cover. $\mathbb{C}(X)$ is a $\mathbb{Z}/n\mathbb{Z}$-extension of $\mathbb{C}(Y)$. Let $\theta \in \mathbb{C}(X)$ be an element such that (i) $\mathbb{C}(X) = \mathbb{C}(Y)(\theta)$ and (ii) the minimal polynomial of θ is of the form $z^n - \varphi$, $\varphi \in \mathbb{C}(Y)$. Under this setting, one can see that X is isomorphic to a $\mathbb{Z}/n\mathbb{Z}$-cover of X_φ of Y obtained by the following way:

$X' :=$ the closure of $\{z^n - \varphi = 0\} \subset Y \times \mathbb{P}^1$,

$\tilde{X}' :=$ the normalization of X', and

$X_\varphi :=$ the Stein factorization of $\tilde{X}' \to X' \xrightarrow{\mathrm{pr}_1} Y$.

Let $\pi_\varphi : X_\varphi \to Y$ is the canonical morphism. Since $\mathbb{C}(X_\varphi) \cong \mathbb{C}(X)$, there is an isomorphism $X_\varphi \cong X$ over Y by the uniqueness of $\mathbb{C}(X)$-normalization.

By the above observation, one can describe Δ_π in terms of the divisor of φ. Put

$$(\varphi)_0 = \sum_i n_i D_i^{(0)},$$
$$(\varphi)_\infty = \sum_j m_j D_j^{(\infty)},$$

then we have

$$\Delta_{\pi_\varphi} = \Delta_\pi = \cup_{i, n \not| n_i} D_i^{(0)} \cup \cup_{j, n \not| m_j} D_j^{(\infty)}.$$

2.2. Preliminaries from Field Theory

We here prepare two lemmas for a D_{2n}-extension of a field. Let k be a field of characteristic 0 containing n-th roots of unity for any n.

Lemma 2.1. *Let L be a D_{2n}-extension (n: odd) of k. Then there exist an element $\theta \in L$ such that*

$$\theta^\sigma = \frac{1}{\theta}$$
$$\theta^\tau = \zeta_n \theta, \quad \zeta_n = \exp(2\pi\sqrt{-1}/n)$$

Proof. Let L^τ be the fixed field by τ. Since L/L^τ is a $\mathbb{Z}/n\mathbb{Z}$-extension, there exists $\theta_o \in L$ such that $L = L^\tau(\theta_o)$ and the minimal polynomial of θ_o is of the form $x^n - a, a \in L^\tau$ and $\theta_o^\tau = \zeta_n \theta_o$. Put $\theta = (\theta_o^\sigma/\theta_o)^k$, where k is an integer satisfying $(n-2)k \bmod n$. As n is odd, one see that θ satisfies the desired properties. ∎

Lemma 2.2. *Let K/k be a quadratic extension and let φ be an element of K satisfying (i) $\varphi^\sigma = 1/\varphi$ and (ii) $z^n - \varphi$ is irreducible in $K[z]$. Then $L := K(\sqrt{[n]\varphi})$ is a D_{2n}-extension.*

Since the statement follows easily from standard field theory, we omit it.

2.3. D_{2n}-Covers

We here introduce some notations. Given a D_{2n}-cover $\pi : X \to Y$, we canonically obtain the double cover, $D(X/Y)$, of Y by taking the $\mathbb{C}(X)^\tau$-normalization of Y, where $\mathbb{C}(X)^\tau$ is the fixed field of $\langle\tau\rangle$. X is an n-cyclic cover of $D(X/Y)$ by its definition. We denote these covering morphisms by $\beta_1(\pi) : D(X/Y) \to Y$ and $\beta_2(\pi) : X \to D(X/Y)$, respectively. Note that $\beta_1(\pi) \circ \beta_2(\pi) = \pi$. In this section, we only consider very special D_{2n}-covers of $\mathbb{P}^2$ as below. As for general theory for D_{2n}-covers, see [14, 18].

Let B be a sextic curve such that $B = C + Q$, where (i) C is a smooth conic, (ii) Q is a quartic with 3-nodes and (iii) C is tangent to Q at 4 distinct smooth points.

As for a D_{2n}-cover of $\mathbb{P}^2$ branched at $2C+nQ$, we have the following result:

Theorem 2.1. *Let B be the sextic curve as above. Then:*

Either (i) *there exists a D_{2n}-cover $\pi_n : S_n \to \mathbb{P}^2$ branched at $2C+nQ$ for any n,*

or (ii) *there exists no D_{2n}-cover $\pi_n : S_n \to \mathbb{P}^2$ branched at $2C+nQ$ for any odd integer $n \geq 3$.*

Lemma 2.3. *If there exists a D_{2n}-cover $\pi_n : S \to \mathbb{P}^2$ branched at $2C+nQ$ for odd n, then $D(S/\mathbb{P}^2)$ is a double cover branched at $2C$.*

Proof. Since $\mathbb{P}^2$ is simply connected, $\Delta_{\beta_1(\pi_n)} \neq \emptyset$. Hence $\Delta_{\beta_1(\pi_n)} = C, Q$ or $C+Q$. By the assumption on the ramification index, we infer that $\Delta_{\beta_1(\pi_n)} = C$. ■

Let $f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ be a double cover given by

$$(x, y) \mapsto (u, v) = (x + y, xy),$$

where (x, y) and (u, v) denote inhomogeneous coordinates. f is a double cover branched along a conic given by the equation $u^2 - 4v = 0$. By taking a suitable change of coordinate, we may assume that C is given by the equation $u^2 - 4v = 0$. By the third condition on B, f^*Q consists of 2 irreducible components and we write it as in the form $f^*Q = Q^+ + Q^-$. There are two possibilities:

Either

Case (i) both Q^+ and Q^- are curves given by bi-homogeneous polynomials of bi-degree $(2, 2)$,

or Case (ii) Q^+ is given by a bi-homogeneous polynomial of bi-degree $(3, 1)$, while Q^- is given by a bi-homogeneous polynomial of bi-degree $(1, 3)$ (or vice versa).

Lemma 2.4. *If Case* (i) *occurs, then there exists a D_{2n}-cover $\pi_n : S_n \to \mathbb{P}^2$ branched at $2C + nQ$ for any n.*

Proof. Let F^+ and F^- be bi-homogeneous polynomials of bi-degree $(2, 2)$ which give Q^+ and Q^-, respectively. Put $\varphi := F^+/F^-$. By Lemma 2.2, a field extension $L := \mathbb{C}(\mathbb{P}^1 \times \mathbb{P}^1)(\sqrt{[n]\varphi})$ gives a D_{2n}-extension of $\mathbb{P}^2$. Let S_n be the L-normalization of $\mathbb{P}^1 \times \mathbb{P}^1$ and let $g_n : S_n \to \mathbb{P}^1 \times \mathbb{P}^1$ be the induced $\mathbb{Z}/n\mathbb{Z}$-cover. By the observation at the beginning of this section, $\Delta_g = Q^+ + Q^-$. Hence $\pi_n := f \circ g_n : S_n \to \mathbb{P}^2$ gives rise to a D_{2n}-covers of $\mathbb{P}^2$ branched at $2C + nQ$. ■

Lemma 2.5. *If Case* (ii) *occurs, then there exists no D_{2n}-cover $\pi_n : S_n \to \mathbb{P}^2$ branched at $2C + nQ$ for any odd integer $n \geq 3$.*

Proof. Suppose that such D_{2n}-cover $\pi_n : S_n \to \mathbb{P}^2$ exists. We may assume that $D(S_n/\mathbb{P}^2) = \mathbb{P}^1 \times \mathbb{P}^1$ and $\beta_1(\pi_n) = f$. By Lemma 2.1, there exists $\theta \in \mathbb{C}(S_n)$

such that $\theta^\sigma = 1/\theta$ and $\theta^\tau = \zeta_n \tau$. Put $\varphi := \theta^n$. As $\Delta_{\beta_2(\pi_n)} = Q^+ + Q^-$, we may assume that

$$(\varphi) = (kQ^+ + nD) - (kQ^- + n\sigma^* D),$$

where k is an natural number with gcd $(n, k) = 1$ and D is some effective divisor given by a bi-homogeneous equation of bi-degree (a, b). Considering this relation in terms of bi-degree of the defining equation, we have $(3k + na, k + nb) = (k + nb, 3k + na)$. Hence we have $2k = n(b - a)$ but this is impossible. ∎

Theorem 2.1 is now immediate from Lemmas 2.4 and 2.5.

Remark 2.1. The argument in our proof of Theorem 2.1 is generalized, and we can apply it to find examples of Zariski k-plet (see [3]).

We end this section with giving explicit example:

Example 2.1.
(i) Let Q_1 be a quartic curve given by the equation

$$4 - 12u + 4u^3v + 5u^2 + 6u^2v - 22uv + 8u^2 - 4u^4 - 2uv^3 - 12uv^2 + v^4 + 6v^3 13v^2 + 12v = 0.$$

The defining equation of f^*Q_1 is

$$(-2 + xy + 5x + y - 2x^2 + 2y^2 - 2xy^2 + x^2y^2) \times (-2 + xy + 5y + x - 2y^2 + 2x^2 - 2x^2y + x^2y^2) = 0.$$

This gives an example for Case (i).

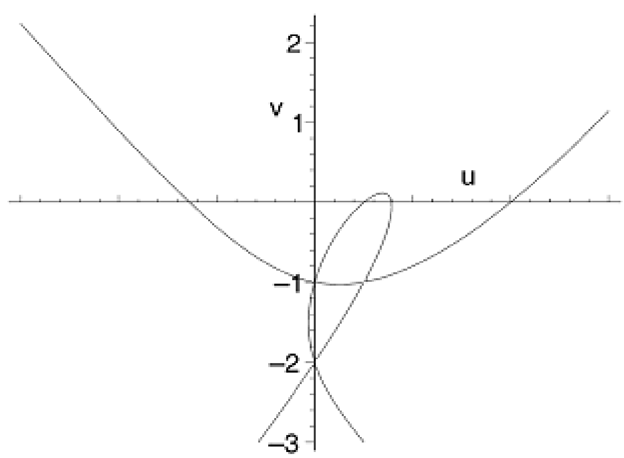

Fig. 1. The curve Q_1

(ii) Let Q_2 be a quartic curve given by the equation

$$1 - u^2v + v^4 + 2v^2 = 0.$$

The defining equation of f^*Q_2 is

$$(x^3y - 1)(xy^3 - 1) = 0.$$

This gives an example for Case (ii).

Corollary 2.1. *Let Q_1 and Q_2 be the quartic curves as above. There is no homeomorphism between the pairs $(\mathbb{P}^2, C \cup Q_1)$ and $(\mathbb{P}^2, C \cup Q_2)$. In particular, the pair $(C \cup Q_1, C \cup Q_2)$ is a Zariski pair (see [***artal1***] for a Zariski pair).*

3. D_{2n}-Covers and Non-Commutativity of the Fundamental Group

In this section, we survey some results obtained as applications of the study of D_{2n}-covers without proof. For details, see [16 - 18].

Throughout this section, our basic question is the following:

Question 3.1. What one can say about the topology of $\mathbb{P}^2 \setminus B$ just from the data of local types of singularities of B?

Hereafter we simply say *the configuration of singularities* in the place of *the data of local topological types of singularities*. Since Question 3.1 seems to be rather vague, we propose a little more specific problem:

Question 3.2. Under what condition on the configuration of singularities of B, can one determine the (non-) commutativity of $\pi_1(\mathbb{P}^2 \setminus B, p_o)$?

Question 3.2 is still subtle. To see its subtleness, let us recall the following example:

Example 3.1. There exists a pair of sextics (B_1, B_2) such that
(i) the singularities of B_i are six cusps for $i = 1, 2$,
(ii) $\pi_1(\mathbb{P}^2 \setminus B_1)$ is non-abelian, while $\pi_1(\mathbb{P}^2 \setminus B_2)$ is abelian.

See and [11, 21, 22], for details.

As Example 3.1 shows, in general, just the configuration of singularities is not enough to determine whether $\pi_1(\mathbb{P}^2 \setminus B, p_o)$ is abelian or non-abelian. Nevertheless, there are several cases when we are able to determine it.

Let us begin with the cases when $\pi_1(\mathbb{P}^2 \setminus B, p_o)$ is abelian. The first result is so called *Zariski Conjecture* proved by Deligne and Fulton which is as follows:

Theorem 3.1. *If B has only nodes, then $\pi_1(\mathbb{P}^2 \setminus B, p_o)$ is abelian.*

See [5, 6] for details. For irreducible plane curves having only nodes and cusps, Theorem 3.1 is generalized by Nori [10] in the following way:

Theorem 3.2. *Suppose that B is an irreducible curve of degree d and has only nodes and cusps. Let a and b be the numbers of nodes and cusps, respectively. If $2a + 6b < d^2$, then $\pi_1(\mathbb{P}^2 \setminus B, p_o)$ is abelian.*

Note that Example 3.1 shows that the inequality in Theorem 3.2 is sharp. Here is another result by Shimada [13] as follows:

Theorem 3.3. *Under the same notations and assumption as in Theorem* 0.6, *if $2a \geq d^2 - 5d + 8$, then $\pi_1(\mathbb{P}^2 \setminus B, p_o)$ is abelian.*

We now go on to the case when $\pi_1(\mathbb{P}^2 \setminus B, p_o)$ is non-abelian. We make use of the existence D_{2n}-covers for this purpose. To state our results, we introduce some terminology and notations as follows:
(a) For $x \in \mathrm{Sing}\,(B)$, we denote its Milnor number by μ_x. We define the total Milnor number of B by

$$\mu_B = \sum_{x \in \mathrm{Sing}\,(B)} \mu_x.$$

(b) Let p be an odd prime. For B, we define a non-negative integer l_p as follows:

If $p = 3$, $l_3 =$ the number of singularities of type A_{3k-1} $(k \geq 1)$ and E_6.

If $p \geq 5$, $l_p =$ the number of singularities of type A_{pk-1} $(k \geq 1)$.

See [4], p. 64 for the type of singularities.

Under these notation, we have:

Theorem 3.4. *Let B be a reduced plane curve of even degree with at most simple singularities* (*see* [4] *for their definition*). *Suppose that there exists an odd prime p such that*

$$l_p + \mu_B > d^2 - 3d + 3.$$

Then there exists a D_{2p}-cover of $\mathbb{P}^2$ branched at $2B$.

By Theorem 3.4 and Fact in Sec. 1, we have

Corollary 3.1.

(i) *Let B be a plane curve satisfying the condition of Theorem* 3.4 *for some odd prime p. Then $\pi_1(\mathbb{P}^2 \setminus B, p_o)$ is non-abelian.*

(ii) *Let B be a plane curve of even degree with only nodes and cusps. Let a and b be the number of nodes and cusps, respectively. If $a + 3b > d^2 - 3d + 3$, then $\pi_1(\mathbb{P}^2 \setminus B, p_o)$ is non-abelian.*

Note that Corollary 3.1 (ii) gives a nice contrast to Theorem 3.4. In fact, the inequality Corollary 3.1 is equivalent to $2a + 6b > 2d^2 - 6d + 6$; and the left hand side is the same as that of the inequality in Theorem 3.4. We give some examples for the case when $\deg B = 6$ and $p = 3$ as follows:

Example 3.2. In this case, $d^2 - 3d + 3 = 21$. There exists a sextic curve B for every case in the following table. In each case, the inequality in Theorem 3.4 is satisfied. Hence $\pi_1(\mathbb{P}^2 \setminus B, p_o)$ is non-abelian by Corollary 3.1.

	types of singularities of B	μ_B	l_3
1	$6A_2 + 4A_1$	16	6
2	$3E_6 + A_1$	19	3
3	$2E_6 + 2A_2 + 2A_1$	18	4
4	$E_6 + 4A_2 + 3A_1$	17	5
5	$E_6 + A_5 + 4A_2$	19	6
6	$E_6 + A_{11} + A_2$	19	3
7	$E_6 + A_8 + A_3 + A_2$	19	3
8	$E_6 + A_8 + 2A_2 + A_1$	19	4
9	$E_6 + A_5 + A_4 + 2A_2$	19	3
10	$D_5 + A_8 + 3A_2$	19	4
11	$E_6 + A_5 + A_3 + 2A_2 + A_1$	19	4
12	$E_6 + 2A_5 + A_3$	19	3
13	$D_5 + 2A_5 + 2A_2$	19	4
14	$D_4 + 3A_5$	19	3
15	$D_4 + A_{11} + 2A_2$	19	3
16	$3A_5 + 4A_1$	19	3

As for existence of the curves as above, see [19] and [20] Now our next question is:

Question 3.3. Is the inequality in Theorem 3.4 best possible?

It is known that the inequality is best possible in the cases when $\deg B = 6$ and $p = 3, 5$. See [19] for the case $p = 3$ and [2] for the case $p = 5$.

Remark 3.1. The author does not know any single example of B with $\deg B \geq 8$ satisfying the inequality in Theorem 3.4. The condition may be too strong for curves of higher degree. In fact, in [12], Sakai proved that the number of cusps $\leqslant \frac{5}{16}d^2 - \frac{3}{8}d$, where d is the degree of B. This means that, if B has only cusps, Sakai's inequality implies that there is no B with $3b > d^2 - 3d + 3$ if $d \geq 29$. Hence our inequality seems to be too strong for curves of higher degree.

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