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On the Symplectic Volume of the Moduli Space of Polygons*

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Abstract. In this notes, we report some results about the symplectic volume of the moduli space of polygons and raise some questions for further investigations.

Let X be the unit 3-sphere $\S^3$ or the Euclidian space $\mathcal{E}^3$. An polygon in X is specified by its set of vertices $v = (v_1, \dots, v_n)$. This vertices are joined in cyclic order by edges $e_1, \dots, e_n$, where e_i is the directed geodesic segment from v_i to v_{i+1}. Two polygons $P = (v_1, \dots, v_n)$ and $Q = (w_1, \dots, w_n)$ are identified if there exists an orientation preserving isometry sending each v_i to w_i.

The side-length r_i of a polygon is defined to be the length of the geodesic segment e_i. We say that a polygon is *regular* if all of its side-lengths are equal.

Let $r = (r_1, \dots, r_n)$ be a tuple of real numbers such that $0 < r_i < \pi \ \forall i$, following [7], we will denote by $\mathcal{P}_r^X$ the *configuration space of all polygons in X with side-lengths r*. The *moduli space of polygons in X with side-lengths r* is defined to be $\mathcal{P}_r^X/\mathrm{Iso}(X)$, where $\mathrm{Iso}(X)$ is the group of orientation-preserving isometry of X.

The first question that we are interested in is: *for which tuple r the polygon with side-lengths r is the most flexible when it is moved in the space provided that its side-lengths are fixed*? First of all we need to specified what is the exact meaning of "flexible". When a polygon is moved, it always lies in its configuration space. Therefore, it is natural to think that a polygon with side-lengths r is more flexible than another one with side-lengths r' if the corresponding configuration space corresponding to r is "bigger" than that of r' in some sense. Fortunately, there is a standard symplectic structure on the moduli spaces $\mathcal{P}_r^X/\mathrm{Iso}(X)$, see

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[4, 7], and *we will measure the flexibility of a polygon with side-lengths r by the symplectic volume of its moduli space*. The symplectic structure on

$\mathcal{P}_r^X/\mathrm{Iso}(X)$ can be visualize roughly as follows: as the length-sides are fixed, each vert ice of the polygon can move in a 2-sphere with fixed radius and the symplectic structure on the product of these spheres induces the symplectic structure on $\mathcal{P}_r^X/\mathrm{Iso}(X)$. Moreover $\mathcal{P}_r^X/\mathrm{Iso}(X)$ is a Hamiltonian space with the torus action by rotating the part of the polygon consisting of edges $e_i, e_{i+1}, \ldots, e_j, j > i$, around the axis connecting e_i and e_{j+1}.

The main tool for studying the symplectic volume is the following (see [8]):

Theorem 0.1. (Witten's formula) *Denote by $V^{\S^3}(r)$ the symplectic volume of $\mathcal{P}_r^{\S^3}/SO(4)$- the moduli space of polygons of side-lengths $r = (r_1, \ldots, r_n)$. We have:*

$$V^{\S^3}(r) = \frac{2^{n-1}}{\pi}\mathrm{SU}(2)m_{k=1}^{\infty}\frac{\sin kr_1 \ldots \sin kr_n}{k^{n-2}}.$$

The main results of [9] are the followings:

Theorem 0.2. *Among all the spherical or Euclidean polygons with fixed perimeter, the regular polygon is one of the most flexible. Moreover in the case n is even, the regular n-gon is the unique one with this property.*

Theorem 0.3. *Among all the spherical polygons the regular one with side-length $\pi/2$ is the unique one which is the most flexible.*

Let $S_n := S^2\backslash\{p_1, \ldots, p_n\}$ be an n-punctured sphere. For a tuple of numbers $r = (r_1, \ldots, r_n)$ such that $0 < r_i < \pi\ \forall i$, we will denote by $\chi_{\mathrm{SU}(2)}(r)$ the representation variety consists of all representations from $\pi_1(S_n)$ to SU(2) such that the image of the loop around the puncture p_i is conjugate to $\begin{pmatrix} e^{ir_i} & 0 \\ 0 & e^{-ir_i} \end{pmatrix}$, modulo the conjugate action of SU(2):

$$\chi_{\mathrm{SU}(2)}(r) := \{(g_1, \ldots, g_n) \in \mathrm{SU}(2)^n |\, g_1 \cdots g_n = \mathrm{Id},\ \mathrm{Tr}(g_j) = 2\cos(r_j)\ \forall j\}/\mathrm{SU}(2)$$

It is well-known that, see [7], we may identify $\chi_{\mathrm{SU}(2)}(r)$ with the moduli space of spherical polygons with side-lengths r.

The representation variety of S_n into the general unitary group SU(2)n can also be equipped with a natural symplectic structure (see [2, 3]) and its symplectic volume is also given by Witten (see [8, 6]). Therefore it is natural to ask if Theorem can be extended to the representation variety into SU(2)n such that the loops around the punctures are mapped to fixed conjugacy classes $C_1, \ldots, C_n$:

Question 1. For what conjugacy classes $C_1, \ldots, C_n$ does $\chi_{\mathrm{SU}(2)n}(C_1, \ldots, C_n)$ has maximal symplectic volume?

Another interesting question that can be studied via the symplectic volume is:

Question 2. For what conjugacy classes $C_1, \dots, C_n$ is $\chi_{\mathrm{SU}(2)n}(C_1, \dots, C_n)$ non empty?

This question has been answered in [1] and the conditions for such $C_1, \dots, C_n$ are formulated in terms of quantum Schubert calculus. It would be interesting to have a direct answer by studying the support of the volume function of $\chi_{\mathrm{SU}(2)n}(C_1, \dots, C_n)$. By taking a pant decomposition of the punctured sphere S_n, we can reduce this problem to the case of 3-punctured sphere. In the case of SU(2), from Witten's formula we get:

$$V^{\S^3}(r_1, r_2, r_3) = \left\{\frac{r_1 + r_2 + r_3}{2\pi}\right\} + \left\{\frac{r_1 - r_2 - r_3}{2\pi}\right\} + \left\{\frac{-r_1 + r_2 - r_3}{2\pi}\right\} + \left\{\frac{-r_1 - r_2 + r_3}{2\pi}\right\} - 2.$$

Here $\{\}$ denotes the fractional part. It is easy to deduce from this formula that $V^{\S^3}(r_1, r_2, r_3)$ is non-zero if and only if:

$$r_1 + r_2 + r_3 \leqslant 2\pi, |r_1 - r_2| \leqslant r_3 \leqslant r_1 + r_2.$$

This is exactly the famous triangle-inequality condition for $\chi_{\mathrm{SU}(2)}(r_1, r_2, r_3)$ to be non-empty. In conclusion we propose the following variant form of Question 2:

Question 3. Find the conditions on C_1, C_2, C_3 for the volume of $\chi_{\mathrm{SU}(2)n}(C_1, C_2, C_3)$ to be non-zero?

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