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A Family of Hyperbolic Hypersurfaces of Low Degree in $\mathbb{P}^3$ and $\mathbb{P}^4$

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Abstract. In this article, we show a family of hyperbolic surfaces of degree 8 in $\mathbb{P}^3$ and a family of hyperbolic hypersurfaces of degree 28 in $\mathbb{P}^4$.

1. Introduction

In 1970, Kobayashi conjectured that generic hypersurfaces in $\mathbb{P}^n$ of sufficiently high degree are hyperbolic [4]. Some progress has been made towards this conjecture. For instance, Masuda and Noguchi [5] have found an algorithm to produce nonsingular hyperbolic hypersurfaces of every degree $d \geq d(n)$ in $\mathbb{P}^n$. However, $d(n)$ is still very high.

For $n = 3$: Demailly and El Goul [1], and independently Mc Quillan [6] (with a slightly bigger degree estimate) proved that a very generic surface of degree at least 21 in $\mathbb{P}^3$ is hyperbolic. Many examples have been given of low degree hyperbolic surfaces in $\mathbb{P}^3$. The example of hyperbolic surfaces of degree 10 in $\mathbb{P}^3$ was found by Shirosaki [8]. Later, the examples of hyperbolic surfaces in $\mathbb{P}^3$ of lowest degree found to date are of degree 8 and were discovered independently by Duval [2] and Fujimoto [3] and Shiffman-Zaidenberg [11, 3].

For $n = 4$: as far as we know, there are a few hyperbolic hypersurfaces of low degree constructed in $\mathbb{P}^4$. The example of hyperbolic hypersurfaces in $\mathbb{P}^4$ of lowest degree found to date is of degree 16 and was discovered by Fujimoto [3]. Outside Fujimoto's example, the hyperbolic hypersurfaces found are of degree at least 36.

Our main aim in this article is to show another hyperbolic hypersurfaces of degree 8 in $\mathbb{P}^3$ and hyperbolic hypersurfaces of degree 28 in $\mathbb{P}^4$. Furthermore, we would like to emphasize that our construction is much simpler than the constructions used before.

Namely, we prove the following

Theorem 1. *Let $d \geq 4$ be an integer and α, β be positive integers such that $\alpha \leqslant 2d-1$, $\beta \leqslant 2d-1$, $\alpha - \beta \geq d$. Let a_0, a_1, a_2, a_3 be nonzero constants. Define the hypersurface X in $\mathbb{P}^3$ by*

$$w_0^{\alpha} w_1^{2d-\alpha} + w_0^{\beta} w_1^{2d-\beta} - (a_0 w_0^d + a_1 w_1^{d-1} w_2 + a_2 w_2^d + a_3 w_3^d)^2 = 0$$

Then the hypersurface X is hyperbolic.

Theorem 2. *Let $d \geq 7$ be an integer. Let a, b, c, m be nonzero constants. Define the hypersurface X in $\mathbb{P}^4$ by*

$$w_0^{4d} + w_1^{4d} - \left(w_0^d w_2^d + w_2^{2d} + (a w_0^5 w_3^{d-5} + b w_1^3 w_3^{d-3} + c w_2 w_3^{d-1} + m w_4^d)^2\right)^2 = 0$$

Then the hypersurface X is hyperbolic.

2. Proof of Theorem 1

First of all we need the following classical fact in the Nevanlinna theory

Lemma 2.1. *Let f be a nonconstant meromorphic function on $\mathbb{C}$ and $a_j (1 \leqslant j \leqslant q)$ distinct points in $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. If all the zeros of $f - a_j$ have the multiplicities at least m_j for each j, where m_j are arbitrarily fixed positive intergers $(1 \leqslant j \leqslant q)$ and $f - \infty$ means $\dfrac{1}{f}$, then $\sum\limits_{j=1}^{q} \left(1 - \dfrac{1}{m_j}\right) \leqslant 2$.*

We now prove Theorem 1.

By the Brody theorem, it suffices to show that X contains no complex lines.

Assume that $f := (f_0 : f_1 : f_2 : f_3) : \mathbb{C} \to \mathbb{P}^3$ is a complex line such that range $(f) \subset X$, where $f_i : \mathbb{C} \to \mathbb{C}$ are holomophic. Then

$$f_0^{\alpha} f_1^{2d-\alpha} + f_0^{\beta} f_1^{2d-\beta} - (a_0 f_0^d + a_1 f_1^{d-1} f_2 + a_2 f_2^d + a_3 f_3^d)^2 = 0. \tag{1}$$

Case I. $f_0 \equiv 0$.

Then from (1) we get

$$a_1 f_1^{d-1} f_2 + a_2 f_2^d + a_3 f_3^d = 0. \tag{2}$$

(i) If $f_2 \equiv 0$, then $f_3 \equiv 0$, and hence, f is constant.

(ii) If $f_2 \not\equiv 0$, then from (2) we have

$$a_1 g^{d-1} + a_2 = -a_3 \left(\frac{f_3}{f_2}\right)^d, \tag{3}$$

where $g = \dfrac{f_1}{f_2}$.

Remark that the equation $a_1 z^{d-1} + a_2 = 0$ has $d-1$ distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_{d-1}$. For each $j = 1, 2, ..., d-1$, multiplicities of zeros of $g - \alpha_j$ are multiples of d by

(3). The inequality $(d-1)\Big(1-\dfrac{1}{d}\Big) = \dfrac{(d-1)^2}{d} > 2$ and Lemma 2.1 imply that g is constant. Hence $(f_3 : f_2)$ is constant. Thus f is constant.

Case II. $f_1 \equiv 0$

From (1) we get

$$a_0 f_0^d + a_2 f_2^d + a_3 f_3^d = 0 \tag{4}$$

(i) If $f_2 \equiv 0$, then $a_0 f_0^d + a_3 f_3^d = 0$. Trivially $(f_0 : f_3)$ is constant, and hence, f is constant.

(ii) If $f_2 \not\equiv 0$, then from (4) we have $a_0 g^d + a_2 = -a_3\Big(\dfrac{f_3}{f_2}\Big)^d$, where $g = \dfrac{f_0}{f_2}$. Repeating as in Case I(i), it implies that g is constant. Hence $(f_3 : f_2)$ is constant. Thus f is constant.

Case III. $f_0 \not\equiv 0$ and $f_1 \not\equiv 0$

Put $g = \dfrac{f_1}{f_0}$. From (1) we have

$$g^{2d-\alpha} + g^{2d-\beta} = \Big(\frac{a_0 f_0^d + a_1 f_1^{d-1} f_2 + a_2 f_2^d + a_3 f_3^d}{f_0^d}\Big)^2 \tag{5}$$

Remark that the equation $z^{2d-\alpha} + z^{2d-\beta} = 0 \Leftrightarrow z^{2d-\alpha}\big(z^{\alpha-\beta} - 1\big) = 0$ has $(\alpha-\beta+1)$ distinct roots. By the hypothesis, we have $\alpha - \beta + 1 \geq d+1$. Denote by $\alpha_1, \alpha_2, \dots, \alpha_{d+1}$ the $(d+1)$ distinct roots of the above equation. For each $j = 1, 2, \dots, d+1$, multiplicities of zeros of $g - \alpha_j$ are multiples of 2 by (5). The inequality $(d+1)\Big(1-\dfrac{1}{2}\Big) = \dfrac{d+1}{2} > 2$ and Lemma 2.1 imply that g is nonzero constant.

From (1) we have

$$\big(g^{2d-\alpha} + g^{2d-\beta}\big) f_0^{2d} - \big(a_0 f_0^d + a_1 f_1^{d-1} f_2 + a_2 f_2^d + a_3 f_3^d\big)^2 = 0.$$

This follows that $\alpha f_0^d + a_0 f_0^d + a_1 f_1^{d-1} f_2 + a_2 f_2^d + a_3 f_3^d = 0$, where $\alpha^2 = g^{2d-\alpha} + g^{2d-\beta}$, or equivalently,

$$(\alpha + a_0) f_0^d + a_1' f_0^{d-1} f_2 + a_2 f_2^d + a_3 f_3^d = 0, \tag{6}$$

where $a_1' = a_1 . g^{d-1} \neq 0$.

(i) If $\alpha + a_0 = 0$, then $a_1' f_0^{d-1} f_2 + a_2 f_2^d + a_3 f_3^d = 0$.

- If $f_2 \equiv 0$, then $f_3 \equiv 0$, and hence, f is constant.
- If $f_2 \not\equiv 0$, then from (6) we have $a_1' . h^{d-1} + a_2 = -a_3\Big(\dfrac{f_3}{f_2}\Big)^d$, where $h = \dfrac{f_0}{f_2}$.

Repeating as in Case I(i), it implies that h is constant. Hence $(f_3 : f_2)$ is constant. Thus f is constant.

(ii) If $\alpha + a_0 \neq 0$:

- If $f_2 \equiv 0$, then from (6) it implies that $(f_0 : f_3)$ is constant. Hence f is constant.
- If $f_2 \not\equiv 0$, then from (6) we have

$$(\alpha + a_0) h^d + a_1' h^{d-1} + a_2 = -a_3\Big(\frac{f_3}{f_2}\Big)^d,$$

where $h = \dfrac{f_0}{f_2}$.

We show that the equation $(\alpha + a_0)z^d + a_1' z^{d-1} + a_2 = 0$ has at least $(d-1)$ distinct roots.

Indeed, let z_0 be a multiple zero of $P(z) = (\alpha + a_0)z^d + a_1' z^{d-1} + a_2$. Then $P'(z_0) = d(\alpha + a_0)z_0^{d-1} + (d-1)a_1' z_0^{d-2} = 0$. Trivially $z_0 \neq 0$ because $P(0) = a_2 \neq 0$. Hence we have $z_0 = -\dfrac{(d-1)a_1'}{d(\alpha + a_0)}$ and

$$P^{''}(z_0) = d(d-1)(\alpha + a_0)z_0^{d-2} + (d-1)(d-2)a_1'.z_0^{d-3} = -(d-1)a_1' z_0^{d-3} \neq 0.$$

Therefore, $P(z)$ has at most one multiple zero, and its multiplicity is $\leqslant 2$, as claimed.

Repeating as in Case I(i) , it implies that h is contant. Hence $(f_3 : f_2)$ is constant. Thus f is constant.

The proof of Theorem 1 is completed. ■

3. Proof of Theorem 2

By the Brody theorem, it sufficies to show that X contains no complex lines.

Assume that $f := (f_0 : f_1 : f_2 : f_3 : f_4) : \mathbb{C} \to \mathbb{P}^4$ is a holomorphic map with $f(\mathbb{C}) \subset X$, where $f_i : \mathbb{C} \to \mathbb{C}$ are holomorphic.

By the hypothesis, we have

$$f_0^{4d} + f_1^{4d} - \big(f_0^d f_2^d + f_2^{2d} + (af_0^5 f_3^{d-5} + bf_1^3 f_3^{d-3} + cf_2 f_3^{d-1} + mf_4^d)^2\big)^2 = 0 \quad (1)$$

Case I. $f_0 \equiv 0$

Then, from (1), we get

$$\alpha f_1^{2d} + f_2^{2d} + \big(bf_1^3 f_3^{d-3} + cf_2 f_3^{d-1} + mf_4^d\big)^2 = 0, \quad (2)$$

where $\alpha = \pm 1$.

(i) If $f_1 \equiv 0$, then from (2) we have

$$f_2^{2d} + (cf_2 f_3^{d-1} + mf_4^d)^2 = 0, \quad \text{i.e} \quad \beta f_2^d + cf_2 f_3^{d-1} + mf_4^d = 0, \quad (3)$$

where $\beta = \pm 1$.

- If $f_3 \equiv 0$ then trivially $(f_2 : f_4)$ is constant, and hence, f is constant.
- If $f_3 \not\equiv 0$ then from (3) we have

$$\beta g^d + cg = -m\Big(\frac{f_4}{f_3}\Big)^d, \quad (4)$$

where $g = \dfrac{f_2}{f_3}$.

It is easy to see that the equation $\beta z^d + cz = 0$ has d distinct roots $\xi_1, \xi_2, ..., \xi_d$. For each $j = 1, 2, ..., d$, multiplicities of zeros of $g - \xi_j$ are multiples of d by (4).

The inequality $d\Big(1-\dfrac{1}{d}\Big) > 2$ and Lemma 2.1 imply that g is constant. Hence $(f_4 : f_3)$ is constant. Thus f is constant.

(ii) If $f_1 \not\equiv 0$

• If $f_2 \equiv 0$ then, from (2), we have

$$\alpha f_1^{2d} + \big(bf_1^3 f_3^{d-3} + mf_4^d\big)^2 = 0,$$

and hence,

$$\gamma f_1^d + bf_1^3 f_3^{d-3} + mf_4^d = 0, \qquad (5)$$

where $\gamma \neq 0$ with $\gamma^2 = -\alpha$.

If $f_3 \equiv 0$ then $(f_1 : f_4)$ is constant, and hence, f is constant.

If $f_3 \not\equiv 0$ then, from (5), we have

$$\gamma h^d + bh^3 = -m\Big(\frac{f_4}{f_3}\Big)^d, \qquad (6)$$

where $h = \dfrac{f_1}{f_3}$. Since the equation $\gamma z^d + bz^3 = 0$ has $(d-2)$ distinct roots $\zeta_1, \zeta_2, ..., \zeta_{d-2}$, it follows that, for each $1 \leqslant j \leqslant d-2$, multiplicities of zeros of $h - \zeta_j$ are multiples of d by (6). The inequality $(d-2)\Big(1-\dfrac{1}{d}\Big) > 2$ and Lemma 2.1 imply that h is constant. Hence $(f_4 : f_3)$ is constant. Thus f is constant.

• If $f_2 \not\equiv 0$ then, from (2), we have

$$\alpha\widetilde{g}^{2d} + 1 = -\Big(\frac{bf_1^3 f_3^{d-3} + cf_2 f_3^{d-1} + mf_4^d}{f_2^d}\Big)^2, \qquad (7)$$

where $\widetilde{g} = \dfrac{f_1}{f_2}$.

Since the equation $\alpha z^{2d} + 1 = 0$ has $2d$ distinct roots $\nu_1, ..., \nu_{2d}$, it follows that, for each $1 \leqslant j \leqslant 2d$, multiplicities of zeros of $\widetilde{g} - \nu_j$ are multiples of 2 by (7). The inequality $2d\Big(1-\dfrac{1}{2}\Big) > 2$ and Lemma 2.1 imply that $\widetilde{g}$ is constant, $\widetilde{g} \neq 0$.

From (2), we have

$$(\alpha\widetilde{g}^{2d} + 1)f_2^{2d} + (b\widetilde{g}^3 f_2^3 f_3^{d-3} + cf_2 f_3^{d-1} + mf_4^d)^2 = 0$$
$$\Leftrightarrow (\alpha\widetilde{g}^{2d} + 1)f_2^{2d} + (b' f_2^3 f_3^{d-3} + cf_2 f_3^{d-1} + mf_4^d)^2 = 0,$$

where $b' = b\widetilde{g}^3 \neq 0$.

Hence

$$\eta f_2^d + b' f_2^3 f_3^{d-3} + cf_2 f_3^{d-1} + mf_4^d = 0, \qquad (8)$$

where $\eta^2 = -(\alpha\widetilde{g}^{2d} + 1)$.

If $f_3 \equiv 0$ then, from (8), $(f_4 : f_2)$ is constant and hence, f is constant.

If $f_3 \not\equiv 0$ then, from (8), we have

$$\eta.\widetilde{h}^d + b'.\widetilde{h}^3 + c\widetilde{h} = -m\Big(\frac{f_4}{f_3}\Big)^d, \qquad (9)$$

where $\widetilde{h} = \dfrac{f_2}{f_3}$.

We show that the equation

$$\eta z^d + b' z^3 + cz = 0 \tag{10}$$

has at least 3 distinct roots. Indeed, (10) $\Leftrightarrow$ $z(\eta z^{d-1} + b'z^2 + c) = 0$. The equation $\eta z^{d-1} + b'z^2 + c = 0$ has at leat 2 distinct non-zero roots. Thus (10) has at least 3 distinct μ_1, μ_2, μ_3.

By (9), multiplicities of zeros of $\widetilde{h} - \mu_j (1 \leqslant j \leqslant 3)$ are multiples of d. The inequality $3\Big(1 - \frac{1}{d}\Big) > 2$ and Lemma 2.1 imply that $\widetilde{h}$ is constant. Hence $(f_4 : f_3)$ is constant. Thus f is constant.

Case II. $f_0 \not\equiv 0$ and $f_1 \equiv 0$

Then, from (1), we get

$$\alpha f_0^{2d} + f_0^d f_2^d + f_2^{2d} + \big(a f_0^5 f_3^{d-5} + c f_2 f_3^{d-1} + m f_4^d\big)^2 = 0, \tag{11}$$

where $\alpha = \pm 1$.

(i) If $f_2 \equiv 0$ then, from (11), we have $\alpha f_0^{2d} + (a f_0^5 f_3^{d-5} + m f_4^d)^2 = 0$. Hence

$$\beta f_0^d + a f_0^5 f_3^{d-5} + m f_4^d = 0, \tag{12}$$

where $\beta^3 = -\alpha, \beta \neq 0$.

- If $f_3 \equiv 0$ then $(f_4 : f_0)$ is constant, and hence, f is constant.
- If $f_3 \not\equiv 0$ then, from (12), we have

$$\beta g^d + a g^5 = -m\Big(\frac{f_4}{f_3}\Big)^d, \tag{13}$$

where $g = \dfrac{f_0}{f_3}$.

Since $d \geq 7$, the equation $\beta z^d + a z^5 = 0 \Leftrightarrow z^5(\beta z^{d-5} + a) = 0$ has at least 3 distinct roots ξ_1, ξ_2, ξ_3. By (13), multiplicities of zeros of $g - \xi_j$, $j = 1, 2, 3$, are multiples of d. The inequality $3\Big(1 - \dfrac{1}{d}\Big) > 2$ and Lemma 2.1 imply that g is constant. Hence $(f_4 : f_3)$ is constant. Hence f is constant.

(ii) If $f_2 \not\equiv 0$ then, from (11), we get

$$\alpha\Big(\frac{f_0}{f_2}\Big)^{2d} + \Big(\frac{f_0}{f_2}\Big)^d + 1 = -\Big(\frac{a_0 f_0^5 f_3^{d-5} + c f_2 f_3^{d-1} + m f_4^d}{f_2^d}\Big)^2 \tag{14}$$

We show that the equation $\alpha z^{2d} + z^d + 1 = 0$ has $2d$ distinct roots $\zeta_1, \zeta_2, ..., \zeta_{2d}$. Indeed, let z_0 be a multiple zero of $P(z) = \alpha z^{2d} + z^d + 1$. Then

$$P'(z_0) = 2d\alpha z_0^{2d-1} + d z_0^{d-1} = d z_0^{d-1}(2\alpha z_0^d + 1) = 0$$

Trivially $z_0 \neq 0$, because $P(0) = 1 \neq 0$. Hence we have $z_0^d = -\dfrac{1}{2\alpha}$. Thus

$$P(z_0) = \alpha (z_0^d)^2 + z_0^d + 1 = \alpha\Big(-\frac{1}{2\alpha}\Big)^2 - \frac{1}{2\alpha} + 1 = \frac{1}{4\alpha} - \frac{1}{2\alpha} + 1 = 1 - \frac{1}{4\alpha} \neq 0,$$

because $\alpha = \pm 1$. This is a contradiction.

By (14), multiplicities of zeros of $\frac{f_0}{f_2} - \zeta_j$, $1 \leqslant j \leqslant 2d$, are multiples of 2. The inequality $2d.\Big(1 - \frac{1}{2}\Big) > 2$ and Lemma 2.1 imply that $\frac{f_0}{f_2}$ is constant. Put $f_2 = kf_0, k \neq 0$.

From (11) we get

$$(\alpha + k^d + k^{2d})f_0^{2d} + (af_0^5 f_3^{d-5} + ckf_0 f_3^{d-1} + mf_4^d)^2 = 0,$$

and hence,

$$\eta.f_0^d + af_0^5 f_3^{d-5} + c' f_0.f_3^{d-1} + m.f_4^d = 0, \tag{15}$$

where $\eta^2 = -(\alpha + k^d + k^{2d})$ and $c' = ck \neq 0$.

• If $\eta = 0$ then, from (15), we have

$$af_0^5 f_3^{d-5} + c' f_0 f_3^{d-1} + mf_4^d = 0 \tag{16}$$

If $f_3 \equiv 0$ then $f_4 \equiv 0$, and hence, f is constant.

If $f_3 \not\equiv 0$ then, from (16), we get

$$a\Big(\frac{f_0}{f_3}\Big)^5 + c'\Big(\frac{f_0}{f_3}\Big) = -m\Big(\frac{f_4}{f_3}\Big)^d \tag{17}$$

Since the equation $az^5 + c'z = 0 \Leftrightarrow z(az^4 + c') = 0$ has 5 distinct roots $\nu_1, \nu_2, \nu_3, \nu_4, \nu_5$ and by (17), multiplicities of zeros of $\frac{f_0}{f_3} - \nu_j$, $1 \leqslant j \leqslant 5$ are multiples of d. The inequality $5\Big(1 - \frac{1}{d}\Big) > 2$ and Lemma 2.1 imply that $\frac{f_0}{f_3}$ is constant. Hence $\frac{f_4}{f_3}$ is constant. Thus f is constant.

• If $\eta \neq 0$:

If $f_3 \equiv 0$ then, from (15), trivially $(f_0 : f_4)$ is constant. Hence f is constant.

If $f_3 \not\equiv 0$ then, from (15), we have

$$\eta.\Big(\frac{f_0}{f_3}\Big)^d + a.\Big(\frac{f_0}{f_3}\Big)^5 + c'.\Big(\frac{f_0}{f_3}\Big) = -m.\Big(\frac{f_4}{f_3}\Big)^d \tag{18}$$

It is easy to see that the equation $\eta z^d + az^5 + c'z = 0$ has at least 3 distinct roots μ_1, μ_2, μ_3. By (18), multiplicities of zeros of $\frac{f_0}{f_3} - \mu_j$, $j = 1, 2, 3$, are multiples of d. The inequality $3\Big(1 - \frac{1}{d}\Big) > 2$ and Lemma 2.1 imply that $\frac{f_0}{f_3}$ is constant, and hence $\frac{f_4}{f_3}$ is constant. Thus f is constant.

Case III. $f_0 \not\equiv 0$ and $f_1 \not\equiv 0$.

From (1) we have

$$\Big(\frac{f_1}{f_0}\Big)^{4d} + 1 = \Big(\frac{f_0^d f_2^d + f_2^{2d} + (af_0^5 f_3^{d-5} + bf_1^3 f_3^{d-3} + cf_2 f_3^{d-1} + mf_4^d)^2}{f_0^{2d}}\Big)^2$$

We see that the equation $z^{4d}+1=0$ has $4d$ distinct roots $\xi_1,...,\xi_{4d}$ and multiplicities of zeros of $\frac{f_1}{f_0}-\xi_j$, $1\leqslant j\leqslant 4d$, are multiples of 2. The inequality $4d\Big(1-\frac{1}{2}\Big)>2$ and Lemma 2.1 imply that $\frac{f_1}{f_0}$ is constant. Put $\frac{f_1}{f_0}=\alpha\neq 0$.

The equality (1) follows that

$$(1+\alpha^{4d})f_0^{4d}-\Big(f_0^d f_2^d+f_2^{2d}+(af_0^5f_3^{d-5}+b\alpha^3f_0^3f_3^{d-3}+cf_2f_3^{d-1}+mf_4^d)^2\Big)^2=0$$

Hence

$$\beta f_0^{2d}+f_0^d f_2^d+f_2^{2d}+(af_0^5f_3^{d-5}+b'f_0^3f_3^{d-3}+cf_2f_3^{d-1}+mf_4^d)^2=0, \qquad (19)$$

where $\beta^2=1+\alpha^{4d}, b'=b.\alpha^3\neq 0$.

(i) If $f_2\equiv 0$ then, from (19), we have

$$\beta f_0^{2d}+(af_0^5f_3^{d-5}+b'f_0^3f_3^{d-3}+mf_4^d)^2=0.$$

Hence

$$\gamma f_0^d+af_0^5f_3^{d-5}+b'f_0^3f_3^{d-3}+mf_4^d=0, \qquad (20)$$

where $\gamma^2=\beta$.

- If $f_3\equiv 0$ then $(f_4:f_0)$ is constant, and hence, f is constant.
- If $f_3\not\equiv 0$ then, from (20), we have

$$\gamma\Big(\frac{f_0}{f_3}\Big)^d+a\Big(\frac{f_0}{f_3}\Big)^5+b'\Big(\frac{f_0}{f_3}\Big)^3=-m\Big(\frac{f_4}{f_3}\Big)^d. \qquad (21)$$

We see that the equation

$$\gamma z^d+az^5+b'z^3=0\Leftrightarrow z^3(\gamma z^{d-3}+az^2+b')=0$$

has at least 3 distinct roots ζ_1,ζ_2,ζ_3 and, by (21), multiplicities of zeros of $\frac{f_0}{f_3}-\zeta_j$, $j=1,2,3$, are multiples of d. The inequality $3\Big(1-\frac{1}{d}\Big)>2$ and Lemma 2.1 imply that $\frac{f_0}{f_3}$ is constant. Hence $\frac{f_4}{f_3}$ is constant. Thus f is constant also.

(ii) If $f_2\not\equiv 0$ then, from (19), we have

$$\beta\Big(\frac{f_0}{f_2}\Big)^{2d}+\Big(\frac{f_0}{f_2}\Big)^d+1=-\Big(\frac{af_0^5f_3^{d-5}+b'f_0^3f_3^{d-3}+cf_2f_3^{d-1}+mf_4^d}{f_2^d}\Big)^2.$$

Repeating as in Case II (ii), we see that the equation $\beta z^{2d}+z^d+1=0$ has at least d distinct roots $\mu_1,\mu_2,...,\mu_d$. Moreover, by the above equality, it implies that multiplicities of zeros of $\frac{f_0}{f_2}-\mu_j$, $1\leqslant j\leqslant d$, are multiples of 2. The inequality $d\Big(1-\frac{1}{2}\Big)>2$ and Lemma 2.1 follow that $\frac{f_0}{f_2}$ is constant. Put $\frac{f_2}{f_0}=\eta\neq 0$.

From (19), we have

$$(\beta + \eta^d + \eta^{2d}) f_0^{2d} + (a f_0^5 f_3^{d-5} + b' f_0^3 f_3^{d-3} + c\eta f_0 f_3^{d-1} + m f_4^d)^2 = 0.$$

Hence

$$\widetilde{\eta}.f_0^d + a.f_0^5.f_3^{d-5} + b'.f_0^3.f_3^{d-3} + c'.f_0.f_3^{d-1} + m.f_4^d = 0, \tag{22}$$

where $\widetilde{\eta}^2 = -(\beta + \eta^d + \eta^{2d})$ and $c' = c\eta \neq 0$.

- If $f_3 \equiv 0$ then $(f_4 : f_0)$ is constant, and hence, f is constant.
- If $f_3 \not\equiv 0$ then, from (22), we have

$$\widetilde{\eta}\Big(\frac{f_0}{f_3}\Big)^d + a\Big(\frac{f_0}{f_3}\Big)^5 + b'\Big(\frac{f_0}{f_3}\Big)^3 + c'\Big(\frac{f_0}{f_3}\Big) = -m\Big(\frac{f_4}{f_3}\Big)^d. \tag{23}$$

It is easy to see that the equation $\widetilde{\eta}z^d + az^5 + b'z^3 + c'z = 0 \Leftrightarrow z(\widetilde{\eta}z^{d-1} + az^4 + b'z^2 + c') = 0$ has at least 3 distinct roots ν_1, ν_2, ν_3. Moreover, by (23), multiplicities of zeros of $\frac{f_0}{f_3} - \nu_j$, $j = 1, 2, 3$, are multiples of d. The inequality $3\Big(1 - \frac{1}{d}\Big) > 2$ and Lemma 2.1 imply that $\frac{f_0}{f_3}$ is constant. Hence $\frac{f_4}{f_3}$ is constant. Thus f is constant.

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