Vietnam Journal of Mathematics **34**:1 (2006) 1–15

Vietnam Journal of MATHEMATICS

The Central Exponent and Asymptotic Stability of Linear Differential Algebraic Equations of Index 1

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Received October 29, 2003
Revised June 20, 2005

Abstract. In this paper, we introduce a concept of the central exponent of linear differential algebraic equations (DAEs) similar to the one of linear ordinary differential equations (ODEs), and use it for investigation of asymptotic stability of the DAEs.

1. Introduction

Differential algebraic equations (DAEs) have been developed as a highly topical subject of applied mathematics. The research on this topic has been carried out by many mathematicians in the world (see [1, 5, 7] and the references therein) for a linear DAE

$$A(t)x' + B(t)x = 0,$$

where $A(t)$ is singular for all $t \in \mathbb{R}^+$. Under certain conditions we are able to transform it into a system consisting of a system of ordinary differential equations (ODEs) and a system of algebraic equations so that we can use methods and results of the theory of ODEs. Many results on stability properties of DAEs were obtained: asymptotical and exponential stability of DAEs which are of index 1 and 2 [6], Floquet theory of periodic DAEs, criteria for the trivial solution of DAEs with small nonlinearities to be asymptotically stable. Similar results for autonomous quasilinear systems are given in [7].

In this paper we are intersted in stability and asymptotical properties of the DAE

$$A(t)x' + B(t)x + f(t, x) = 0,$$

which can be considered as a linear DAE $Ax' + Bx = 0$ perturbed by the term f.

For this aim we introduce a concept of central exponent of linear DAEs similar to that of ODEs (see [2]).

The paper is organized as follows. In Sec. 2 we introduce the notion of the central exponent and some properties of central exponents of linear DAEs of index 1. In Sec. 3 we investigate exponential asymptotic stability of linear DAEs with respect to small linear as well as nonlinear perturbation.

2. The Central Exponent of Linear DAE of Index 1 and Its Properties

In this paper we will consider a linear DAE

$$A(t)x' + B(t)x = 0, \tag{2.1}$$

where $A, B : \mathbb{R}^+ = (0, +\infty) \to L(\mathbb{R}^m, \mathbb{R}^m)$ are bounded continuous $(m \times m)$ matrix functions, $\operatorname{rank} A(t) = r < m$, $N(t) := \ker A(t)$ is of the constant dimension $m - r$ for all $t \in \mathbb{R}^+$ and $N(t)$ is smooth, i.e there exists a continuously differentiable matrix function $Q \in C^1(\mathbb{R}^+, L(\mathbb{R}^m, \mathbb{R}^m))$ such that $Q(t)$ is a projection onto $N(t)$. We shall use the notation $P = I - Q$. We will always assume that (2.1) is of index 1, i.e there exists a C^1-smooth projector $Q \in C^1(\mathbb{R}^+, L(\mathbb{R}^m, \mathbb{R}^m))$ onto $\ker A(t)$ such that the matrix

$$A_1(t) := A(t) + (B(t) - A(t)P'(t))Q(t)$$

(or, equivalently, the matrix $G(t) := A(t) + B(t)Q(t)$) has bounded inverse on each interval $[t_0, T] \subset \mathbb{R}^+$ (see [5, 6]).

For definition of a solution $x(t)$ of the DAE (2.1) one does not require $x(t)$ to be C^1-smooth but only a part of its coordinates be smooth. Namely, we introduce the space

$$C^1_A(0, \infty) = \{x(t) : \mathbb{R}^+ \to \mathbb{R}^m, \ \ x(t) \text{ is continuous and } P(t)x(t) \in C^1\}.$$

A function $x \in C^1_A(0, \infty)$ is said to be a *solution* of (2.1) on $\mathbb{R}^+$ if the identity

$$A(t)\big[(P(t)x(t))' - P'(t)x(t)\big] + B(t)x(t) = 0$$

holds for all $t \in \mathbb{R}^+$. Note that $C^1_A(0, \infty)$ does not depend neither on the choice of P, nor on the definition of a solution of (2.1) above, as solution of DAEs of index 1.

Definition 2.1. *A measurable bounded function $R(\cdot)$ on $\mathbb{R}^+$ is called C-function of system (2.1) if for any $\varepsilon > 0$ there exists a positive number $D_{R,\varepsilon} > 0$ such that the following estimate*

$$\|x(t)\| \leqslant D_{R,\varepsilon}\|x(t_0)\| e^{\int_{t_0}^{t}(R(\tau)+\varepsilon)d\tau} \tag{2.2}$$

holds for all $t \geq t_0 \geq 0$ and any solution $x(\cdot)$ of (2.1).

The set $\mathcal{R}_{A,B}$ of all C-functions of (2.1) is called C-class of (2.1).

For any function $f : \mathbb{R}^+ \to \mathbb{R}$ we denote its upper mean value by $\overline{f}$, i.e.

$$\overline{f} := \limsup_{T\to\infty} \frac{1}{T} \int_0^T f(t)dt.$$

Definition 2.2. *The number*

$$\Omega := \inf_{R\in\mathcal{R}_{A,B}} \overline{R}$$

is called the central exponent of system (2.1).

Let V $(\dim V(t) = d = \text{constant})$ be an invariant subspace of the solution space of system of (2.1), i.e. V is a linear space spanned by solutions of (2.1), $V(t)$ is the section of V at time t. Notice that like a linear ODE, the solutions of the DAE (2.1) form a finite-dimensional linear subspace of the space of continuous $\mathbb{R}^m$-valued functions on $\mathbb{R}^+$, understood also as a subspace of the linear (function) space of solutions.

Definition 2.3. *A function R is called C-function of (2.1) with respect to V if for any $\varepsilon > 0$, there exists $D_{R,\varepsilon} > 0$ such that for any solution $x(t) \in V$, we have*

$$\|x(t)\| \leqslant D_{R,\varepsilon}\|x(t_0)\|e^{\int_{t_0}^{t}(R(\tau)+\varepsilon)d\tau}, \quad \text{for all} \quad t \geq t_0 \geq 0.$$

Denote by $\mathcal{R}_V$ the collection of all C-functions of V. The number

$$\Omega_V := \inf_{R_V\in\mathcal{R}_V} \overline{R}_V$$

is called central exponent of (2.1) *with respect to V.*

Remark 2.1. If $V_1 \subset V_2$ then $\mathcal{R}_{V_2} \subset \mathcal{R}_{V_1}$, hence $\Omega_{V_1} \leqslant \Omega_{V_2}$. In particular, $\Omega_V \leqslant \Omega_{A,B}$.

Let $X(t) = [x_1(t), ..., x_m(t)]$ be a maximal fundamental solution matrix (FSM) of (2.1), i.e $x_1(t), ..., x_m(t)$ are solutions of (2.1) and they span the solution space $\text{im}P_s(t)$ of (2.1) (see [5]). Here $P_s(t) = I - QA_1^{-1}B$ is the canonical projection of (2.1). Denote by $X(t, t_0)$ the maximal FSM of (2.1) normalized at $t_0, t_0 \in \mathbb{R}^+$ (see [1]), i.e. $X(\cdot, t_0)$ is a maximal FSM satisfying the initial condition

$$A(t_0)(X(t_0, t_0) - I) = 0.$$

Such a FSM exists and is the solution of the initial value problem posed with initial value $X(t_0, t_0) = P_s(t_0)$. Note that the normalized maximal FSMs play the role of the Cauchy matrix for the DAEs.

Lemma 2.1. *Suppose that* (2.1) *is a DAE of index* 1 *and the coe cient matrices $A(t), B(t)$ are continuous and bounded on $\mathbb{R}^+$. Suppose further that the matrices*

A_1^{-1} and P' are bounded on $\mathbb{R}^+$. Then the central exponent Ω of (2.1) satifies the following equality

$$\begin{aligned}\Omega &= \lim_{T\to\infty} \limsup_{n\to\infty} \frac{1}{nT} \sum_{i=1}^{n} \ln \|X(iT,(i-1)T)\| \\ &= \inf_{T>0} \limsup_{n\to\infty} \frac{1}{nT} \sum_{i=1}^{n} \ln \|X(iT,(i-1)T)\|. \end{aligned} \tag{2.3}$$

Proof. The proof is a simple analogue of the ODE case given in [2] (the idea is to use boundedness of A, B, A_1^{-1}, P' and a property of the matrix norm). ∎

Note that formula (2.3) can serve as a definition of the central exponent (as for the central exponent Ω_V we can use the restriction of $X(t,\tau)$ to V instead of X in the above formula). Now we will derive some properties of the central exponent of linear DAE of index 1 and of its corresponding ODE.

Theorem 2.2. *Suppose that* (2.1) *is a linear* DAE *of index 1 and the matrices $A(t)$, $B(t)$, A_1^{-1}, $P'(t)$ are bounded on $\mathbb{R}^+$. Then the central exponent Ω_x of* (2.1) *is smaller than or equal to the central exponent Ω_u of the corresponding* ODE *of* (2.1) *under $P \in C^1$, i.e of the* ODE

$$u' = (P' - PA_1^{-1}B_0)u. \tag{2.4}$$

Proof. Denote by $X(t,s)$ the maximal fundamental solution matrix of (2.1) normalized at s and by $U(t,s)$ the Cauchy matrix of (2.4). Then $X(t,s)$ and $U(t,s)$ are related by the following equality (see [1], p.18)

$$X(t,s) = P_s(t)U(t,s)P(s), \quad \forall t \ge s \ge 0,$$

hence

$$\|X(t,s)\| \leqslant \|P_s(t)\| \, \|U(t,s)\| \, \|P(s)\|.$$

Since the matrices $A(t)$, $B(t)$, $A_1^{-1}(t)$ are bounded on $\mathbb{R}^+$, the projectors $P = A_1^{-1}A$, $Q = I - P$, $Q_s = QA_1^{-1}B$ and $P_s = I - Q_s$ are bounded on $\mathbb{R}^+$, too. Let $\|P_s\| \leqslant b_1$, $\|P\| \leqslant b_2$, we have

$$\|X(t,s)\| \leqslant b_1b_2\|U(t,s)\|.$$

Therefore

$$\ln\|X(t,s)\| \leqslant \ln(b_1b_2) + \ln\|U(t,s)\|.$$

This implies that

$$\sum_{j=1}^{n} \ln\|X(jT,(j-1)T)\| \leqslant n\ln(b_1b_2) + \sum_{j=1}^{n} \ln\|U(jT,(j-1)T)\|.$$

Hence, by (2.3)

$$\begin{aligned}\Omega_x &= \lim_{T\to\infty}\limsup_{n\to\infty}\frac{1}{nT}\sum_{j=1}^{n}\ln\|X(jT,(j-1)T)\|\\ &\leqslant \lim_{T\to\infty}\limsup_{n\to\infty}\frac{1}{nT}\Big(\sum_{j=1}^{n}\ln\|U(jT,(j-1)T)\| + n\ln(b_1b_2)\Big)\\ &\leqslant \lim_{T\to\infty}\limsup_{n\to\infty}\frac{1}{nT}\sum_{j=1}^{n}\ln\|U(jT,(j-1)T)\| = \Omega_u.\end{aligned}$$

Hence $\Omega_x \leqslant \Omega_u$. The theorem is proved. ■

In Definition 2.3 we introduced the notion of central exponent of a DAE with respect to an invariant subspace of the solution space. This can certainly be done for ODEs.

Note that the corresponding ODE (2.4) of the DAE (2.1) under some projector $P(t)$ is defined on the whole phase space $\mathbb{R}^m$. The function space spanned by solutions $u(t)$ of (2.4) satisfying $u(t) \in \operatorname{im} P(t)$ for $t \geq 0$, is an invariant subspace of the solution space of that ODE. With an abuse of language we denote that funtion space by $\operatorname{im} P$. We show that the central exponent of this ODE with respect to the function space $\operatorname{im} P$ is closely related to the central exponent of the DAE (2.1) (in the sense that it characterizes better the central exponent of the DAE (2.1)).

Let us consider the corresponding ODE of (2.1) under a projector P

$$u' = (P'P_s - PG^{-1}B)u. \tag{2.5}$$

Similarly to Definition 2.3 we call the number

$$\Omega_{U_{\operatorname{im} P}} := \inf_{R\in\mathcal{R}_{\operatorname{im} P}} \overline{R},$$

where $\mathcal{R}_{\operatorname{im} P}$ is the class of C-functions of the invariant subspace $\operatorname{im} P$ of the solution space of (2.5), *central exponent* of ODE (2.5) with respect to $\operatorname{im} P$. Clearly, $\Omega_{U_{\operatorname{im} P}} \leqslant \Omega_U$.

Denote by $U(t,t_0)$ the Cauchy matrix of (2.5).

Put

$$U_{\operatorname{im} P}(t,t_0) := P(t)U(t,t_0)P(t_0) = U(t,t_0)P(t_0). \tag{2.6}$$

One can see that

$$\Omega_{U_{\operatorname{im} P}} = \lim_{T\to\infty}\limsup_{n\to\infty}\frac{1}{nT}\sum_{i=0}^{n-1}\ln\big\|U_{\operatorname{im} P}\big((i+1)T, iT\big)\big\|.$$

Denote by Ω_x, Ω_u and $\Omega_{U_{\operatorname{im} P}}$ the central exponents of (2.1), (2.5) and of (2.5) with respect to $\operatorname{im} P$.

Theorem 2.3. *Suppose that* (2.1) *is a linear DAE of index 1 with the coe cient matrices $A(t)$, $B(t)$ being continuous and bounded on $\mathbb{R}^+$. Then the following assertions are true*

i) *If the projector P is bounded on $\mathbb{R}^+$ then*

$$\Omega_{U_{\mathrm{im}\,P}} \leqslant \Omega_X,$$

ii) *If projectors P and P_s are bounded on $\mathbb{R}^+$ then*

$$\Omega_{U_{\mathrm{im}\,P}} = \Omega_X.$$

Proof.
i) We have the following relation between $X(t,t_0)$ and $U(t,t_0)$ (see [1])

$$X(t,t_0) = P_s(t)U(t,t_0)P(t_0).$$

Therefore

$$P(t)X(t,t_0) = P(t)P_s(t)U(t,t_0)P(t_0) = P(t)U(t,t_0)P(t_0) = U_{\mathrm{im}\,P}(t,t_0).$$

By assumption P is bounded on $\mathbb{R}^+$, hence $\|P\| \leqslant b$ for some constant $b > 0$. Therefore

$$\left\|U_{\mathrm{im}\,P}\right\| \leqslant \|P\|\,\|X\| \leqslant b\|X\|.$$

Consequently

$$\ln\left\|U_{\mathrm{im}\,P}(jT,(j-1)T)\right\| \leqslant \ln b + \ln\|X(jT,(j-1)T)\|.$$

This implies that

$$\begin{aligned}\Omega_{U_{\mathrm{im}\,P}} &= \lim_{T\to\infty}\limsup_{n\to\infty}\frac{1}{nT}\sum_{j=1}^{n}\ln\left\|U_{\mathrm{im}\,P}(jT,(j-1)T)\right\| \\ &\leqslant \lim_{T\to\infty}\limsup_{n\to\infty}\frac{1}{nT}\Big(n\ln b + \sum_{j=1}^{n}\ln\|X(jT,(j-1)T)\|\Big) \\ &\leqslant \lim_{T\to\infty}\limsup_{n\to\infty}\frac{1}{nT}\sum_{j=1}^{n}\ln\|X(jT,(j-1)T)\| = \Omega_X.\end{aligned}$$

Thus $\Omega_{U_{\mathrm{im}P}} \leqslant \Omega_X$.
ii) The matrices $X(t,t_0)$ and $U(t,t_0)$ are related by the following equality (see [1])

$$X(t,t_0) = P_s(t)U(t,t_0)P(t_0) = P_s(t)U_{\mathrm{im}\,P}(t,t_0).$$

By assumption P_s is bounded on $\mathbb{R}^+$, hence $\|P_s\| \leqslant C$ for constant $C > 0$. Therefore,

$$\|X(t,t_0)\| \leqslant \|P_s(t)\|\,\|U_{\mathrm{im}\,P}(t,t_0)\| \leqslant C\|U_{\mathrm{im}\,P}\|.$$

This implies

$$\ln\|X(t,t_0)\| \leqslant \ln C + \ln\|U_{\mathrm{im}\,P}(t,t_0)\|,$$

hence

$$\sum_{j=1}^{n}\ln\|X(jT,(j-1)T)\| \leqslant n\ln C + \sum_{j=1}^{n}\ln\|U_{\mathrm{im}\,P}(jT,(j-1)T\|.$$

Therefore

$$\Omega_x \leqslant \Omega_{U_{\mathrm{im}\,P}}.$$

By the first part of the theorem, since P is bounded on $\mathbb{R}^+$, $\Omega_{U_{\mathrm{im}\,P}} \leqslant \Omega_x$, hence $\Omega_x = \Omega_{U_{\mathrm{im}P}}$. ∎

Corollary 2.4. *Given a linear* DAE *of index 1 in Kronecker normal form, i.e*

$$A(t)x' + B(t)x = 0, \tag{2.7}$$

where

$$A(t) = \begin{pmatrix} W(t) & 0 \\ 0 & 0 \end{pmatrix}, \quad B(t) = \begin{pmatrix} B_1(t) & 0 \\ 0 & I_{m-r} \end{pmatrix},$$

where $W(t)$ is a continuous $(r \times r)$ nonsingular matrix and the matrices $W^{-1}(t)$, $B_1(t)$ are continuous and bounded on $\mathbb{R}^+$. If the central exponent Ω_x of (2.7) *is positive then Ω_x coincides with the central exponent Ω_u of the corresponding* ODE (2.4) *of* (2.7) *with Q being the orthogonal projector onto $\ker A(t)$.*

Proof. Since $P = P_s = I - Q$ are constant, Theorem 2.3 (ii) is applicable. ∎

Corollary 2.5. *Suppose that* (2.1) *is a linear* DAE *of index 1 and the matrices $A(t)$, $B(t), G^{-1}(t)$ are continuous and bounded on $\mathbb{R}^+$, then $\Omega_x = \Omega_{U_{\mathrm{im}\,P}}$.*

Proof. Since A, B and G^{-1} are bounded on $\mathbb{R}^+$, the matrices $P = G^{-1}A$, $P_s = I - QG^{-1}B$ are bounded on $\mathbb{R}^+$, therefore by Theorem 2.3 we have $\Omega_x = \Omega_{U_{\mathrm{im}\,P}}$. ∎

3. The Exponential Asymptotic Stability of Linear DAEs with Respect to Small Perturbations

In this section we shall use the central exponent for investigation of asymptotic stability of DAEs. Let us consider the index 1 linear DAE (2.1).

Assume that G^{-1} is bounded on $\mathbb{R}^+$. We shall be interested in the following small nonlinear perturbations of (2.1)

$$A(t)x' + B(t)x + f(t,x) = 0. \tag{3.1}$$

The perturbation $f(t,x)$ is assumed to be small in the following sense

$$\|f(t,x)\| \leqslant \delta(t)\|x\|, \quad \text{for all} \quad t \in \mathbb{R}^+, \quad x \in \mathbb{R}^m \tag{3.2}$$

for some function $\delta : \mathbb{R}^+ \to \mathbb{R}^+$. We will usually assume that $\delta(t) \leqslant \delta_0$ for all $t \in \mathbb{R}^+$ and some constant $\delta_0 > 0$. We assume additionally that the following inequality

$$\|f'_x(t,x)\| \leqslant \frac{\alpha}{\|G^{-1}\|.\|Q\|}$$

holds for some constant $0 < \alpha < 1$, where $\|G^{-1}\| := \sup_{t\in\mathbb{R}^+} \|G^{-1}(t)\|$ and $\|Q\| := \sup_{t\in\mathbb{R}^+} \|Q(t)\|$ (note that $\|G^{-1}\|.\|Q\| < \infty$ since G^{-1} is bounded on $\mathbb{R}^+$).

Similarly to the theory of ODE we show that each solution of (3.1) is a solution of a linear equation of the form

$$A(t)x' + B(t)x + F(t)x = 0. \tag{3.3}$$

Theorem 3.1. *Any nontrivial solution $x_0(t)$ of the perturbed system* (3.1) *is a solution of some linear system of form* (3.3)*, where $F(t)x$ is of the same order of smallness as $f(t,x)$, i.e.*

$$\|F(t)\| \leqslant \delta(t), \quad \forall t \in \mathbb{R}^+. \tag{3.4}$$

Proof. From (3.2) it follows that $f(t,0) = 0$, $\forall t \in \mathbb{R}^+$. Hence $x \equiv 0$ is the trivial solution of (3.1). By the assumption on $\|f'_x(t,x)\|$, the equation (3.1) is of index 1, hence solution of initial value problem of (3.1) is unique (see [5], Th.15, p. 36). Therefore, a nontrivial solution $x_0(t)$ of (3.1) does not vanish at any $t \in \mathbb{R}^+$.

Put

$$\widetilde{F}(t,x) := \frac{(x, x_0(t))}{\|x_0(t)\|^2} f(t, x_0(t)).$$

Clearly $\widetilde{F}(t,x)$ is linear in the second variable, so that $\widetilde{F}(t,x) = F(t)x$ for some $F(t)$. Furthermore, for any $x \in \mathbb{R}^m$ we have

$$\left\|F(t)x\right\| = \left\|\widetilde{F}(t,x)\right\| \leqslant \frac{\|x\| \, \|x_0(t)\|}{\|x_0(t)\|^2}\|f(t, x_0(t))\| \leqslant \delta(t)\|x\|.$$

This implies that $\|F(t)\| \leqslant \delta(t)$.

Moreover

$$F(t)x_0(t) = \widetilde{F}(t, x_0(t)) = \frac{(x_0(t), x_0(t))}{\|x_0(t)\|^2} f(t, x_0(t)) = f(t, x_0(t)).$$

Therefore $x_0(t)$ is a nontrivial solution of system (3.3). ■

Remark 2.2. (i) We have used a restrictive condition on $\|f'_x(t,x)\|$ to ensure that the DAE (3.1) is of index 1. In some cases this condition can be easily verified. Note that this condition can be replaced by a weaker condition "the initial value problem of (3.1) has a unique solution".
(ii) Different solutions of (3.1) lead to different coefficient matrices of (3.3), hence they are solutions of different linear DAEs of type (3.3).

Theorem 3.2. *Suppose that* (2.1) *is a* DAE *of index 1 and matrices $A(t)$, $B(t)$, $A_1^{-1}(t)$, $P'(t)$ are continuous and bounded on $\mathbb{R}^+$, $R(t)$ is a C-function of* (2.1)*. Suppose further that the perturbation term of the linear perturbed* DAE

$$A(t)x' + B(t)x + F(t)x = 0, \tag{3.5}$$

satisfies the condition

$$\|F(t)\| \leqslant \delta(t) \leqslant \delta_0, \tag{3.6}$$

for some $\delta_0 \in \mathbb{R}^+$.

Then for any $\varepsilon > 0$ there exists a constant $D_{R,\varepsilon}$ depending only on R, ε and the DAE (2.1) *such that any solution $x(t)$ of* (3.5) *satisfies the inequality*

$$\|x(t)\| \leqslant D_{R,\varepsilon}\|x(t_0)\| e^{\int_{t_0}^{t}(R(\tau)+\varepsilon+D_{R,\varepsilon}\delta(\tau))d\tau}.$$

Moreover, for any $\varepsilon > 0$ *there exists* $\delta_0 > 0$ *such that if* $F(t)$ *satisfies* (3.6) *then the central exponent* Ω_{δ_0} *of* (3.5) *satisfies the inequality*

$$\Omega_{\delta_0} < \Omega + \varepsilon,$$

where Ω *is the central exponent of* (2.1).

Proof. Since $A_1 = A + B_0Q$, where $B_0 := B - AP'$, we have $A_1^{-1}A = P$, hence P and $Q = I - P$ are bounded on $\mathbb{R}^+$. The DAE (3.5) is equivalent to

$$u' + (PA_1^{-1}B_0 - P')u + PA_1^{-1}F(u+v) = 0, \tag{3.7}$$
$$v + QA_1^{-1}B_0u + QA_1^{-1}F(u+v) = 0, \tag{3.8}$$

where $u = Px$, $v = Qx$. For $\delta_0 < \dfrac{1}{2\sup\limits_{t\in\mathbb{R}^+}\|Q(t)A_1^{-1}(t)\|}$, from (3.8) we can find for v the representation

$$v = -(I + QA_1^{-1}F)^{-1}Q_su - (I + QA_1^{-1}F)^{-1}QA_1^{-1}Fu, \tag{3.9}$$

where $Q_s := QA_1^{-1}B_0$.

Substituting (3.9) into (3.7) we get

$$u' + (PA_1^{-1}B_0 - P')u + PA_1^{-1}F[I - (I + QA_1^{-1}F)^{-1}Q_s$$
$$- (I + QA_1^{-1}F)^{-1}QA_1^{-1}F]u = 0.$$

This is a linear ODE with bounded continuous coefficients, hence it has a unique solution of the initial value problem. Therefore, the system (3.7) - (3.8) has a unique solution of the initial value problem.

Moreover, from (3.8) we have

$$\|v\| \leqslant \|Q_s\|\,\|u\| + \|QA_1^{-1}\|\,\|F\|(\|u\| + \|v\|) \leqslant \|Q_s\|\,\|u\| + \|QA_1^{-1}\|\delta_0(\|u\| + \|v\|),$$

hence for $\delta_0 < \dfrac{1}{2\sup\limits_{t\in\mathbb{R}^+}\|Q(t)A_1^{-1}(t)\|}$ we have

$$\|v\| \leqslant \frac{\|Q_s\| + \|QA_1^{-1}\|\delta_0}{1 - \|QA_1^{-1}\|\delta_0}\|u\| < \frac{\|Q_s\| + 1/2}{1 - \|QA_1^{-1}\|\delta_0}\|u\| \leqslant C_1\|u\|, \tag{3.10}$$

where

$$C_1 := \sup_{t\in\mathbb{R}^+}(2\|Q_s\| + 1) \geq \frac{\|Q_s\| + 1/2}{1 - 1/2} \geq \frac{\|Q_s\| + 1/2}{1 - \|QA_1^{-1}\|\delta_0} \geq \frac{\|Q_s\| + \|QA_1^{-1}\|\delta_0}{1 - \|QA_1^{-1}\|\delta_0}.$$

Using (3.10) we have

$$\left\|PA_1^{-1}F(u+v)\right\| \leqslant \left\|PA_1^{-1}\right\|\,\|F\|(\|u\| + \|v\|) \leqslant \|PA_1^{-1}\|(1 + C_1)\delta\|u\| \leqslant k\delta\|u\|$$

where $k := \sup\limits_{t\in\mathbb{R}^+}\|P(t)A_1^{-1}(t)\|(1 + C_1)$ is a positive constant.

Put

$$\overline{F}(t)u := P(t)A_1^{-1}(t)F(t)(u+v)$$
$$= P(t)A_1^{-1}(t)F(t)\big[I - (I + QA_1^{-1}F)^{-1}QA_1^{-1}F - (I + QA_1^{-1}F)^{-1}Q_s\big](t)u,$$

then the norm of $\overline{F}(t)$ can be estimated as

$$\|\overline{F}(t)\| \leqslant \|PA_1^{-1}\|(1+C_1)\delta \leqslant k\delta \leqslant k\delta_0.$$

Let us consider the linear ODE

$$u' = (P' - PA_1^{-1}B_0)u, \quad u \in \mathbb{R}^m. \tag{3.11}$$

Suppose that R_1 is a C-function of the invariant subspace $\operatorname{im} P$ of the solution space of (3.11), $U(t, t_0)$ is the Cauchy matrix of (3.11) and $u_\delta(t)$ is a solution of the perturbed ODE

$$u' = (P' - PA_1^{-1}B_0)u - \overline{F}u \tag{3.12}$$

with the initial value $u_\delta(t_0) \in \operatorname{im} P(t_0)$.

Notice that

$$\overline{F}(t)u = P(t)A_1^{-1}(t)F(t)\big[I - (I + QA_1^{-1}F)^{-1}QA_1^{-1}F - (I + QA_1^{-1}F)^{-1}Q_s\big](t)u$$

belongs to $\operatorname{im} P(t)$ then multiplying the differential equation (3.12) by Q we have

$$Qu' = QP'u, \quad (Qu)' = Q'(Qu),$$

hence, if the initial condition of (3.12) satisfies $Q(t_0)u_\delta(t_0) = 0$ then the solution $u_\delta(t)$ of (3.12) satisfies the condition $Q(t)u_\delta(t) = 0$, i.e $u_\delta(t)$ belongs to $\operatorname{im} P(t)$.

By the solution formula of an nonhomogeneous linear ODE, we have

$$u_\delta(t) = U(t, t_0)u_\delta(t_0) - \int_{t_0}^{t} U(t, s)\overline{F}(s)u_\delta(s)ds.$$

Scaling this equation by $P(t)$ and taking norms, we obtain

$$\|u_\delta(t)\| \leqslant \|P(t)U(t, t_0)\| \, \|u_\delta(t_0)\| + \int_{t_0}^{t} \|P(t)U(t, s)\| \, \|\overline{F}(s)\| \, \|u_\delta(s)\|ds.$$

Let $U(t, t_0) = [u_1(t), ..., u_m(t)]$. Put for $i = 1, ..., m$

$$\widetilde{u}_i(t) := P(t)u_i(t) \in \operatorname{im} P(t),$$
$$\widetilde{v}_i(t) := u_i(t) - \widetilde{u}_i(t) \in \ker P(t).$$

Since R_1 is a C-function of the invariant subspace $\operatorname{im} P$ of the solution space of (3.11), for any $\varepsilon > 0$, there exists a positive constant $D_{R_1,\varepsilon}$ depending on R_1 and ε such that for all $0 \leqslant t_0 \leqslant t$ we have

$$\|P(t)U(t,t_0)\| \leqslant D_{R_1,\varepsilon} e^{\int_{t_0}^{t}(R_1(\tau)+\varepsilon)d\tau},$$

therefore

$$\|u_\delta(t)\| \leqslant D_{R_1,\varepsilon} e^{\int_{t_0}^{t}(R_1(\tau)+\varepsilon)d\tau} \|u_\delta(t_0)\| + D_{R_1,\varepsilon} k \int_{t_0}^{t} e^{\int_{s}^{t}(R_1(\tau)+\varepsilon)d\tau} \delta(s)\|u_\delta(s)\| ds.$$

Put

$$y(t) = \|u_\delta(t)\| e^{-\int_{t_0}^{t}(R_1(\tau)+\varepsilon)d\tau},$$

we have

$$y(t) \leqslant D_{R_1,\varepsilon}\|u_\delta(t_0)\| + kD_{R_1,\varepsilon}\int_{t_0}^{t}\delta(s)y(s)ds.$$

Using the Lemma of Gronwall - Bellman [4. p.108], we have

$$y(t) \leqslant D_{R_1,\varepsilon}\|u_\delta(t_0)\| e^{kD_{R_1,\varepsilon}\int_{t_0}^{t}\delta(s)ds},$$

therefore

$$\|u_\delta(t)\| \leqslant D_{R_1,\varepsilon}\|u_\delta(t_0)\| e^{\int_{t_0}^{t}\left(R_1(\tau)+\varepsilon+kD_{R_1,\varepsilon}\delta(\tau)\right)d\tau}. \tag{3.13}$$

Recall that $X(t,t_0)$ denotes the maximal FSM of (2.1) normalized at t_0. We know that $X(t,t_0)$ and $U(t,t_0)$ are related by the following equality (see [1], p.20-21)

$$X(t,t_0) = P_s(t)U(t,t_0)P(t_0) = P_s(t)P(t)U(t,t_0)P(t_0)$$

for all $t \geq t_0 \geq 0$. Since the matrices P, P_s are bounded on $\mathbb{R}^+$, say by a positive constant $M > 0$, we have

$$\begin{aligned}\|X(t,t_0)\| &\leqslant \|P_s(t)\ \|P(t)U(t,t_0)\|\ \|P(t_0)\| \\ &\leqslant M^2 D_{R_1,\varepsilon} e^{\int_{t_0}^{t}(R_1(\tau)+\varepsilon)d\tau}\end{aligned}$$

for all $t \geq t_0 \geq 0$. Therefore, R_1 is a C-function of (2.1).

Conversely, let R be a C-function of (2.1) and $x(t)$, $u(t)$ are corresponding solutions of (2.1) and (3.11) respectively (thus $u(t) \in \operatorname{im} P(t)$), then we have $x(t) = P_s(t)u(t)$ and $u(t) = P(t)x(t)$. Since $R(t)$ is a C-function of (2.1), for any $\varepsilon > 0$ there exists $D_{R,\varepsilon}$ depending on R and ε such that

$$\|x(t)\| \leqslant D_{R,\varepsilon}\|x(t_0)\| e^{\int_{t_0}^{t}(R(\tau)+\varepsilon)d\tau}.$$

Therefore, since P and P_s are bounded on $\mathbb{R}^+$ by $M > 0$, we have

$$\|u(t)\| = \|P(t)x(t)\| \leqslant \|P(t)\|\|x(t)\| \leqslant M D_{R,\varepsilon}\|x(t_0)\|e^{\int_{t_0}^{t}(R(\tau)+\varepsilon)d\tau}$$
$$\leqslant M^2 D_{R,\varepsilon}\|u(t_0)\|e^{\int_{t_0}^{t}(R(\tau)+\varepsilon)d\tau}.$$

This implies that $R(t)$ is a C-function of $\operatorname{im} P(t)$ of (3.11). Thus, C-classes $\mathcal{R}$ of (2.1) and $\mathcal{R}_{U_{\operatorname{im} P}}$ of the invariant subspace $\operatorname{im} P$ of the solution space of (3.11) coincide.

Suppose that $x_\delta(t)$ is a solution of the perturbed system (3.5) of (2.1). We have

$$x_\delta(t) = P(t)x_\delta(t) + Q(t)x_\delta(t) =: u_\delta(t) + v_\delta(t),$$

where $u_\delta(t)$, $v_\delta(t)$ are solutions of (3.7) and (3.8), respectively. Because of (3.10), $\|v_\delta\| \leqslant C_1\|u_\delta\|$, hence using (3.13) we have

$$\|x_\delta(t)\| \leqslant \|u_\delta(t)\| + \|v_\delta(t)\| \leqslant \|u_\delta(t)\| + C_1\|u_\delta(t)\| = (1 + C_1)\|u_\delta(t)\|$$
$$\leqslant (1 + C_1)D_{R,\varepsilon}\|u_\delta(t_0)\|e^{\int_{t_0}^{t}(R(\tau)+\varepsilon+D_{R,\varepsilon}k\delta(\tau))d\tau}$$
$$\leqslant D'_{R,\varepsilon}\|x_\delta(t_0)\|e^{\int_{t_0}^{t}(R(\tau)+\varepsilon+D'_{R,\varepsilon}\delta(\tau))d\tau},$$

where $D'_{R,\varepsilon} := \max_{t\in\mathbb{R}^+}\left\{(1 + C_1)M D_{R,\varepsilon}, kD_{R,\varepsilon}\right\}$, $D'_{R,\varepsilon}$ depends only on R and ε. Thus we have for all $0 \leqslant t_0 \leqslant t$

$$\frac{\|x_\delta(t)\|}{\|x_\delta(t_0)\|} \leqslant D'_{R,\varepsilon}e^{\int_{t_0}^{t}(R(\tau)+\varepsilon+D'_{R,\varepsilon}\delta(\tau))d\tau}$$

for any solution $x_\delta(t)$ of (3.5). The first assertion of the theorem is proved.

Hence

$$\|X_\delta(t, t_0)\| = \max_{x_\delta}\frac{\|x_\delta(t)\|}{\|x_\delta(t_0)\|} \leqslant D'_{R,\varepsilon}e^{\int_{t_0}^{t}(R(\tau)+\varepsilon+D'_{R,\varepsilon}\delta(\tau))d\tau}.$$

Moreover, $R_\delta(t) := R(t)+\varepsilon+D'_{R,\varepsilon}\delta(t)$ is a C-function of (3.5) for any C-function R of (2.1) and any fixed $\varepsilon > 0$. It is easily seen that

$$\overline{R}_\delta \leqslant \overline{R} + D'_{R,\varepsilon}\overline{\delta} + \varepsilon.$$

For a given $\varepsilon > 0$ we choose R such that $\overline{R} < \Omega + \varepsilon$ and δ_0 satisfying $D'_{R,\varepsilon}\delta_0 = \varepsilon$ then we have

$$\Omega_\delta \leqslant \overline{R}_\delta \leqslant \Omega + 3\varepsilon.$$

The theorem is proved. ∎

Now we consider again the case of nonlinear perturbation of the DAE (2.1)

$$A(t)x' + B(t)x + f(t,x) = 0 \tag{3.14}$$

where $f(t,x)$ is a small nonlinear perturbation having norm δ_0 ($\delta_0 > 0$):

$$\|f(t,x)\| \leqslant \delta(t)\|x\|,\ \delta(t) \leqslant \delta_0\ ,\ \|f'_x(t,x)\| \leqslant \frac{\alpha}{\|G^{-1}\|.\|Q\|} \tag{3.15}$$

for all $t \in \mathbb{R}^+$ and some $0 < \alpha < 1$.

Theorem 3.3. *Suppose that* (2.1) *is a linear* DAE *of index 1 and the matrices* $A(t)$, $B(t)$, $A_1^{-1}(t)$ *and* $P'(t)$ *are continuous and bounded on* $\mathbb{R}^+$. *Suppose further that the perturbation term* $f(t,x)$ *in* (3.14) *satisfies condition* (3.15). *Then for any* $\varepsilon > 0$ *there exist* $\delta_1 > 0$ *and* $D = D(\varepsilon) > 0$ *such that if* $\delta_0 \leqslant \delta_1$ *for any solution* $x(t)$ *of* (3.14) *the following inequality holds*

$$\|x(t)\| \leqslant D\|x(t_0)\|e^{(\Omega+\varepsilon)(t-t_0)}, \tag{3.16}$$

where Ω *is the central exponent of the linear* DAE (2.1).

Proof. Let $x(t)$ be an arbitrary solution of the perturbed system (3.14). By Theorems 3.1 and 3.2, for a C-function R of (2.1) and any $\varepsilon > 0$ there exists a constant $D_{R,\varepsilon}$ such that

$$\|x(t)\| \leqslant D_{R,\varepsilon}\|x(t_0)\|e^{\int_{t_0}^{t}(R(\tau)+\varepsilon+D_{R,\varepsilon}\delta(\tau))d\tau}.$$

Choose $\delta_0 = \dfrac{\varepsilon}{D_{R,\varepsilon}}$, we have

$$\|x(t)\| \leqslant D_{R,\varepsilon}\|x(t_0)\|e^{\int_{t_0}^{t}(R(\tau)+2\varepsilon)d\tau}.$$

Since $\Omega = \inf_{\mathcal{R}} \overline{R}$, for any $\varepsilon > 0$ we may find a C-function R such that

$$\overline{R} < \Omega + \varepsilon.$$

Hence

$$\limsup_{t\to\infty} \frac{1}{t-t_0}\int_{t_0}^{t} R(\tau)d\tau < \Omega + \varepsilon,$$

therefore there exists $M > 0$ such that

$$\int_{t_0}^{t} R(\tau)d\tau < (\Omega+\varepsilon)(t-t_0) + \ln M.$$

This implies

$$\|x(t)\| \leqslant D_{R,\varepsilon}\|x(t_0)\|M.e^{(\Omega+3\varepsilon)(t-t_0)}.$$

Since $\varepsilon > 0$ is arbitrary the theorem is proved. ∎

Lemma 3.4. *If $\|B(t)-C(t)\| \to 0$, then the C-classes, hence central exponents, of the systems*

$$A(t)x' + B(t)x = 0 \tag{3.17}$$

and

$$A(t)x' + C(t)x = 0 \tag{3.18}$$

coincide, provided they are both of index 1.

Proof. We rewrite the DAE (3.18) in the form

$$A(t)x' + B(t)x + (C(t) - B(t))x = 0$$

and consider it as a perturbed system of (3.17).

Since $\delta(t) = \|C(t) - B(t)\| \to 0$, for any $\beta > 0$, there exists $D_\beta > 0$ such that

$$\int_{t_0}^{t} \delta(\tau)d\tau < \beta(t-t_0) + \ln D_\beta, \quad \text{for all} \quad t \geq t_0.$$

Let $R(t)$ be a C-function of (3.17) and $\varepsilon > 0$ be arbitrary. Then there exists a constant $D_{R,\varepsilon} > 0$ such that any solution $x(t)$ of the perturbed system (3.18) satisfies the inequality

$$\begin{aligned}\|x(t)\| &\leqslant D_{R,\varepsilon}\|x(t_0)\|e^{\int_{t_0}^{t}(R(\tau)+\varepsilon+D_{R,\varepsilon}\delta(\tau))d\tau} \\ &\leqslant D_{R,\varepsilon}\|x(t_0)\|e^{\int_{t_0}^{t}(R(\tau)+\varepsilon+D_{R,\varepsilon}\beta)d\tau}.\end{aligned}$$

Since $D_{R,\varepsilon}\beta$ may be chosen arbitrarily small, the above inequality proves that $R(t)$ is a C-function of the perturbed system (3.18).

Now, by changing the role of (3.17) and (3.18) in the above argument we have that if $R(t)$ is a C-function of (3.18) then it is a C-function of (3.17). Thus, the C-classes of (3.17) and (3.18) coincide. Consequently, the central exponents of (3.17) and (3.18) coincide. ∎

Corollary 3.5. *Suppose that* (2.1) *is a linear DAE of index 1 and the matrices* $A(t)$, $B(t)$, $A_1^{-1}(t)$, $P'(t)$ *are continuous and bounded on* $\mathbb{R}^+$. *Suppose that the condition* (3.15) *holds. If the central exponent* Ω *of* (2.1) *is negative then there exists* $\delta_1 > 0$ *such that if* $\delta_0 < \delta_1$ *then there exist positive numbers* $D, \gamma > 0$ *such that any solution* $x(t)$ *of* (2.1) *satisfies the inequality*

$$\|x(t)\| \leqslant D\|x(0)\|e^{-\gamma t}, \quad \text{for all} \quad t \geq 0.$$

Thus, the trivial solution of perturbed system (3.14) *is exponentially stable.*

Proof. The corollary follows immediately from (3.16) and $\Omega < 0$. ∎

Corollary 3.6. *Suppose that* (2.1) *is a linear DAE of index* 1 *and the matrices* $A(t)$, $B(t)$, $G^{-1}(t)$, $P'(t)$ *are continuous and bounded on* $\mathbb{R}^+$. *Suppose further that the central exponent* Ω_u *of the corresponding ODE* (2.5) *is negative and the condition* (3.15) *is satisfied for* $\delta_0 > 0$. *Then there exists* $\delta_1 > 0$, *such that if* $\delta_0 < \delta_1$ *the trivial solution of perturbed equation* (3.14) *is exponentially stable.*

Proof. By Theorem 2.2 we have $\Omega_x \leqslant \Omega_u < 0$. By Corollary 3.5, this implies that the trivial solution of perturbed equation (3.14) is exponentially stable. ■

Acknowledgement. The author would like to thank Prof. Dr. Nguyen Dinh Cong for suggesting the problem and for the help during the work on this paper.

References

1. K. Balla and R. März, Linear differential algebraic equations of index 1 and their adjoint equations, *Results Math.* **37** (2000) 13–35.
2. B. Ph. Bylov, E. R. Vynograd, D. M. Grobman, and V. V. Nemytxki, *Theory of Lyapunov Exponents,* Nauka, Moscow, 1966 (in Russian).
3. N. D. Cong and H. Nam, Lyapunov's inequality for linear differential algebraic equation, *Acta Math. Vietnam.* **28** (2003) 73–88.
4. B. P. Demidovich *Lectures on Mathematical Theory of Stability*, Nauka, Moscow, 1967 (in Russian).
5. E. Griepentrog and R. März, *Di erential Algebraic Equations and Their Numerical Treatment,* Teubner - Text Math. **88**, Leipzig, 1986.
6. M. Hanke and A. Rodriguetz, Asymptotic properties of regularized differential algebraic equation, Humboldts - Universität zu Berlin, Berlin, Germany, Preprint, N95-6, 1995.
7. C. Tischendort, On stability of solutions of autonomous index 1 tractable and quasilinear index 2 tractable DAEs,*Circuits Systems Signal Process* **13** (1994) 139–154.