Vietnam Journal of Mathematics **34**:1 (2006) 17–30

Vietnam Journal of MATHEMATICS

Closed Weak Supplemented Modules*

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Received June 13, 2004
Revised June 01, 2005

Abstract. A module M is called closed weak supplemented if for any closed submodule N of M, there is a submodule K of M such that $M = K + N$ and $K \cap N \ll M$. Any direct summand of a closed weak supplemented module is also closed weak supplemented. Any finite direct sum of local distributive closed weak supplemented modules is also closed weak supplemented. Any nonsingular homomorphic image of a closed weak supplemented module is closed weak supplemented. R is a closed weak supplemented ring if and only if $M_n(R)$ is also a closed weak supplemented ring for any positive integer n.

1. Introduction

Throughout this paper, unless otherwise stated, all rings are associative rings with identity and all modules are unitary right R-modules.

A submodule N of M is called an essential submodule, denoted by $N \leqslant_e M$, if for any nonzero submodule L of M, $L \cap N \neq 0$. A closed submodule N of M, denoted by $N \leqslant_c M$, is a submodule which has no proper essential extension in M. If $L \leqslant_c N$ and $N \leqslant_c M$, then $L \leqslant_c M$ (see [2]).

A submodule N of M is small in M, denoted by $N \ll M$, if $N + K = M$ implies $K = M$. Let N and K be submodules of M. N is called a supplement of K in M if it is minimal with respect to $M = N + K$, or equivalently, $M = N + K$

*This work was supported by the Natural Science Foundation of Zhejiang Province of China Project(No.102028).

and $N \cap K \ll N$ (see [6]). A module M is called supplemented if for any submodule N of M there is a submodule K of M such that $M = K + N$ and $N \cap K \ll N$ (see [3]). A module M is called weak supplemented if for each submodule N of M, there is a submodule L of M such that $M = N + L$ and $N \cap L \ll M$. A module M is called $\oplus$-supplemented if every submodule N of M has a supplement K in M which is also a direct summand of M (see [8]).

A module M is called extending, or a CS module, if every submodule is essential in a direct summand of M, or equivalently, every closed submodule is a direct summand (see [9]).

Let M be a module and $m \in M$. Then $r(m) = \{r \in R | mr = 0\}$ is called right annihilator of m. First we collect some well-known facts.

Lemma 1.1. [1] *Let M be a module and let $K \leqslant L$ and L_i $(1 \leqslant i \leqslant n)$ be submodules of M, for some positive integer n. Then the following hold.*

(1) *$L \ll M$ if and only if $K \ll M$ and $L/K \ll M/K$;*

(2) *$L_1 + L_2 + ... + L_n \ll M$ if and only if $L_i \ll M$ $(1 \leqslant i \leqslant n)$;*

(3) *If M' is a module and $f : M \to M'$ is a homomorphism, then $f(L) \ll M'$ where $L \ll M$;*

(4) *If L is a direct summand of M, then $K \ll L$ if and only if $K \ll M$;*

(5) *$K_1 \oplus K_2 \ll L_1 \oplus L_2$ if and only if $K_i \ll L_i$ $(i = 1, 2)$.*

Lemma 1.2. *Let N and L be submodules of M such that $N + L$ has a weak supplement H in M and $N \cap (H + L)$ has a weak supplement G in N. Then $H + G$ is a weak supplement of L in M.*

Proof. Similar to the proof of 41.2 of [6]. ■

In this paper, we define closed weak supplemented modules which generalize weak supplemented modules.

In Sec. 2, we give the definition of a closed weak supplemented module and show that any direct summand of a closed weak supplemented module and, with some additional conditions, finite direct sum of closed weak supplemented modules are also closed weak supplemented modules.

In Sec. 3, some conditions of which the homomorphic image of a closed weak supplemented module is a closed weak supplemented module are given.

In Sec. 4, we show that $S = End(F)$ is closed weak supplemented if and only if F is closed weak supplemented, where F is a free right R-module. We also show that R is a closed weak supplemented ring if and only if $M_n(R)$ is also a closed weak supplemented ring for any positive integer n. Let R be a commutative ring and M a finite generated faithful multiplication module. Then R is closed weak supplemented if and only if M is closed weak supplemented.

In Sec. 5, we investigate the relations between (closed) weak supplemented modules and supplemented modules, extending modules, etc,.

2. Closed Weak Supplemented Modules

In [3], a module M is called weak supplemented if for every submodule N of M there is a submodule K of M such that $M = K + N$ and $N \cap K \ll M$. Also, co-finitely weak supplemented modules have been defined and studied. Now we give the definition of a closed weak supplemented module as follows:

Definition 2.1. *A module M is called closed weak supplemented if for any closed submodule N of M, there is a submodule K of M such that $M = K + N$ and $K \cap N \ll M$. A submodule K of M is called weak supplement if it is a weak supplement of some submodule of M.*

Clearly, any weak supplemented module is closed weak supplemented and any extending module is closed weak supplemented. Since local modules (i.e., the sum of all proper submodules is also a proper submodule) are hollow (i.e., every proper submodule is small) and hollow modules are weak supplemented, hence closed weak supplemented. So we have the following implications:

$$\textit{local} \Rightarrow \textit{hollow} \Rightarrow \textit{weak supplemented} \Rightarrow \textit{closed weak supplemented.}$$

But a closed weak supplemented need not be weak supplemented, in general.

Example 2.2. Let $\mathbb{Z}$ be the ring of all integers. Then $\mathbb{Z}$ is uniform as a $\mathbb{Z}$-module and the summands of $\mathbb{Z}$ are 0 and $\mathbb{Z}$ itself. Since all closed submodules are 0 and $\mathbb{Z}$, it is easy to see that $\mathbb{Z}$ is closed weak supplemented. But $\mathbb{Z}$ is not $\oplus$-supplemented. For $n \geq 2$, $n\mathbb{Z}$ has no supplement in $\mathbb{Z}$. Because for any prime p, $(p, n) = 1$, we have $p\mathbb{Z} + n\mathbb{Z} = \mathbb{Z}$.

Similarly, a closed weak supplemented module need not be an extending module, as following example shows:

Example 2.3. Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{pmatrix}$, where $\mathbb{Z}$ is the ring of all integers. (see [8, Example 6.2]). Then R is not extending as a right R-module. But all right ideals of R are of the form:

$$I = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where A, B, C are ideals of $\mathbb{Z}$ and $A \leqslant B$.

Since $\mathbb{Z}$ is uniform as a $\mathbb{Z}$-module, then, besides 0 and R, all closed right ideals of R are I with $A = 0$ and $B = \mathbb{Z}$, $C = \mathbb{Z}$ or $A = \mathbb{Z}$, $B = \mathbb{Z}$, $C = 0$ or $A = 0$, $B = \mathbb{Z}$, $C = 0$ or $A = 0$, $B = 0$, $C = \mathbb{Z}$. It is easy to see that R is closed weak supplemented.

The direct summand of a weak supplemented module is weak supplemented. For a closed weak supplemented module, we also have the following proposition:

Proposition 2.4. *Let M be a closed weak supplemented module. Then any direct summand of M is closed weak supplemented.*

Proof. Let N be any direct summand of M and L any closed submodule of N. Since N is closed in M, we see that L is closed in M. Then there is a submodule K of M such that $M = K + L$ and $K \cap L \ll M$. Thus $N = N \cap K + L$. Since N is a direct summand of M, then $N \cap K \cap L = K \cap L \ll N,$ by Lemma 1.1(4). Thus N is closed weak supplemented. ∎

Now we consider when the direct sum of closed weak supplemented modules is also closed weak supplemented.

Proposition 2.5. *Let $M = M_1 \oplus M_2$ with each M_i $(i = 1, 2)$ closed weak supplemented. Suppose that* (1) *$M_i \cap (M_j + L) \leqslant_c M_i$ and (2) $M_j \cap (L + K) \leqslant_c M_j$, where K is a weak supplement of $M_i \cap (M_j + L)$ in M_i, $i \neq j$, for any closed submodule L of M. Then M is closed weak supplemented.*

Proof. Let $L \leqslant_c M$, then $M = M_1 + (M_2 + L)$ has a trivial supplement 0 in M. Since $M_1 \cap (M_2 + L) \leqslant_c M_1$ and M_1 is closed weak supplemented, then there is a submodule K of M_1 such that $M_1 = K + M_1 \cap (M_2 + L)$ and $K \cap (M_1 \cap (M_2 + L)) = K \cap (M_2 + L) \ll M_1$. By Lemma 1.2, K is a weak supplement of $M_2 + L$ in M, i.e., $M = K + (M_2 + L)$. Since $M_2 \cap (K + L) \leqslant_c M_2$ and M_2 is closed weak supplemented, then $M_2 \cap (K + L)$ has a weak supplement J in M_2. Again by Lemma 1.2, $K + J$ is a weak supplement of L in M. Hence M is closed weak supplemented. ∎

We define a module M to be local distributive if for any closed submodules L, N, K of M, we have $L \cap (K + N) = L \cap K + L \cap N$. Obviously any distributive module is local distributive, but local distributive module need not be distributive. For example, $\mathbb{Z} \oplus \mathbb{Z}$ is local distributive and is not distributive as $\mathbb{Z}$-module. Since $\mathbb{Z}(2, 3) \cap ((\mathbb{Z} \oplus 0) + (0 \oplus \mathbb{Z})) = Z(2, 3)$, but $\mathbb{Z}(2, 3) \cap (\mathbb{Z} \oplus 0) = \mathbb{Z}(2, 3) \cap (0 \oplus \mathbb{Z}) = (0, 0)$. All closed submodules of $\mathbb{Z} \oplus \mathbb{Z}$ are $0 \oplus \mathbb{Z}$, $\mathbb{Z} \oplus 0$, $0 \oplus 0$ and itself. So $\mathbb{Z} \oplus \mathbb{Z}$ is local distributive.

Theorem 2.6. *Let $M = M_1 \oplus M_2$. Suppose that M is local distributive, then M is closed weak supplemented if and only if each M_i is weak supplemented for all $1 \leqslant i \leqslant 2$.*

Proof. The necessity is clear by Proposition 2.4.

Conversely, let L be any closed submodule of M. Then for each i, $L \cap M_i$ is closed in M_i. In fact, suppose that $L \cap M_1 \leqslant_e K \leqslant M_1$. Since $M_2 \cap L \leqslant_e M_2 \cap L$ and M is local distributive, we have that $L = (M_1 \cap L) \oplus (M_2 \cap L) \leqslant_e K \oplus (M_2 \cap L)$. Hence $L = (M_1 \cap L) \oplus (M_2 \cap L) = K \oplus (M_2 \cap L)$, because L is closed in M. So $K = L \cap M_1$ and $L \cap M_1$ is closed in M_1.

So, there is a submodule K_i of M_i such that

$$M_i = K_i + L \cap M_i, \quad \text{and} \quad (L \cap M_i) \cap K_i \ll M_i, \quad i = 1, 2.$$

Hence

$$M = M_1 \oplus M_2 = K_1 \oplus K_2 + ((L \cap M_1) \oplus (L \cap M_2)) = K_1 \oplus K_2 + L$$
$$L \cap (K_1 \oplus K_2) = (L \cap K_1) \oplus (L \cap K_2) \ll (M_1 \oplus M_2) = M.$$

Thus M is closed weak supplemented. ■

A submodule N of M is called a fully invariant submodule if for every $f \in S$, we have $f(N) \subseteq N$, where $S = End_R(M)$. If $M = K \oplus L$ and N is a fully invariant submodule of M, then we have $N = (N \cap K) \oplus (N \cap L)$ and $M/N = K/(N \cap K) \oplus L/(N \cap L)$.

Proposition 2.7. *Let $M = M_1 \oplus M_2$. Suppose that every closed submodule of M is fully invariant, then M is closed weak supplemented if and only if each M_i $(i = 1, 2)$ is closed weak supplemented.*

Proof. Straightforward. ■

Lemma 2.8. [3] *If $f : M \to N$ is a small epimorphism (i.e., $Kerf \ll M$), then a submodule L of M is a weak supplement in M if and only if $f(L)$ is a weak supplement in N.*

Proposition 2.9. *Let $f : M \to N$ be a small epimorphism with N closed weak supplemented. If for any nonzero closed submodule L of M, $Kerf \subseteq L$, then M is closed weak supplemented.*

Proof. Since for any closed submodule L of M, $Kerf \subseteq L$, then $f(L) \cong L/Kerf$ is closed in $M/Kerf \cong N$. By Lemma 2.8, L has a weak supplement in M and M is weak supplemented. ■

Let $f : R \to T$ be a homomorphism of rings and M a right T-module. Then M can be defined to be a right R- module by $mr = mf(r)$ for all $m \in M$ and $r \in R$. Moreover, if f is an epimorphism and M is a right R-module such that $Kerf \subseteq r(M)$, then M can also be defined to be a right T-module by $mt = mr$, where $f(r) = t$. We denote by M_T, M_R that M is a right T-module, right R-module, respectively.

Lemma 2.10. *Let $f : R \to T$ be an epimorphism of rings and M a right R-module. If $Kerf \subseteq r(M)$, then $N_T \leqslant_c M_T$ if and only if $N_R \leqslant_c M_R$.*

Proof. Suppose that $N_T \leqslant_c M_T$ and that $N_R \leqslant_e L_R \leqslant M_R$. Then for any $0 \neq l \in L$, there is $r \in R$, such that $0 \neq lr \in N_R$. Since f is an epimorphism and $Kerf \subseteq r(M)$, so L_R can be defined to be a right T-module by $lt = lr$, while $f(r) = t$. So $0 \neq lr = lf(r) \in N_T$ and $N_T \leqslant_e L_T$. So $N_T = L_T$ and $N_R \leqslant_c M_R$, as required.

Conversely, suppose that $N_R \leqslant_c M_R$ and $N_T \leqslant_e L_T \leqslant M_T$. Then for any $0 \neq l \in L$, there is $t \in T$, such that $0 \neq lt = lf(r) = lr \in N_R$, where $f(r) = t$. So $N_R \leqslant_e L_R$ and $N_R = L_R$ and $N_T \leqslant_c M_T$, as required. ■

Theorem 2.11. *Let $f : R \to T$ be an epimorphism of rings and M a right R-module with $Kerf \subseteq r(M)$. Then M_R is closed weak supplemented if and only if M_T is closed weak supplemented.*

Proof. Suppose that M_R is closed weak supplemented and $N_T \leqslant_c M_T$. Then $N_R \leqslant_c M_R$. Since M_R is closed weak supplemented, there is a submodule K_R of M_R such that $M_R = N_R + K_R$ and $N_R \cap K_R \ll M_R$. It is easy to see that $K_T \cap N_T \ll M_T$. So M_T is closed weak supplemented.

The converse is similar. ∎

3. The Homomorphic Images

In this section, we will consider the conditions for which the homomorphic images of closed weak supplemented modules are also closed weak supplemented modules.

Lemma 3.1. *Let $f : M \to N$ be an epimorphism of modules and $L \leqslant_c N$. Then $L \cong U/Kerf$ for some $U \leqslant M$. If $r(m) = r(f(m))$ for all $m \in M \backslash Kerf$ or N is torsion-free. Then U is closed in M.*

Proof. Suppose that $Kerf \leqslant U \leqslant_e K \leqslant M$. Then for any $k \in K \backslash Kerf$, $f(k) \neq 0$. There is $r \in R$ such that $0 \neq kr \in U$.

If $r(k) = r(f(k))$, then $f(kr) = f(k)r \neq 0$, so $0 \neq kr + Kerf \in U/Kerf$;

If N is torsion-free, then, since $f(k) \neq 0$, we have $f(k)r = f(kr) \neq 0$.

In either cases, we have that $L \cong U/Kerf \leqslant_e K/Kerf$. Since L is closed in N, it implies that $U = K$ and hence U is closed in M. ∎

Lemma 3.2. *Let $L \leqslant U \leqslant_c M$ with M closed weak supplemented. Then $M/L = U/L + (V + L)/L$ for some submodule V of M and $U/L \cap (V + L)/L \ll M/L$.*

Proof. Firstly, we show that $U/L \leq_c M/L$. Suppose that $U/L \leqslant_e K/L \leqslant M/L$ where $L \leqslant U \leqslant K \leqslant M$. For any $k \in K \backslash U$, then $k \notin L$ and $k + L \neq 0$. Since $U/L \leqslant_e K/L$, there is $r \in R$, such that $0 \neq kr + L \in U/L$. Then there is $u \in U \backslash L$, such that $kr + L = u + L$, that is, $kr - u \in L \leqslant U$. So $0 \neq kr \in U$. Hence $U \leqslant_e K$ and $U = K$. So U/L is closed in M/L.

Since M is closed weak supplemented, there is a submodule V of M, such that $M = V + U$ and $U \cap V \ll M$. So $M/L = U/L + (V + L)/L$.

Now, we show that $U/L \cap (V + L)/L \ll M/L$. It is easy to see that

$$\begin{aligned} U/L \cap (V + L)/L &= ((U \cap (V + L))/L = ((U \cap V) + L)/L \\ &\cong (U \cap V)/(L \cap U \cap V) = (U \cap V)/(L \cap V). \end{aligned}$$

Let $\pi : M \to M/(L \cap V)$ be the canonical epimorphism. Since $U \cap V \ll M$, then $\pi(U \cap V) = (U \cap V)/(L \cap V) \ll M/L$. ∎

Theorem 3.3. *Let $f : M \to N$ be an epimorphism of modules with M closed* *weak supplemented. If $r(m) = r(f(m))$ for all $m \in M \backslash Kerf$ or N is torsion-free, then N is also closed weak supplemented.*

Proof. By Lemma 3.1, for any closed submodule L of N, there is a closed submodule U of M, such that $Kerf \leqslant U \leqslant_c M$, $L \cong U/Kerf$. Then $N \cong M/Kerf = U/Kerf + (V + Kerf)/Kerf$ where $M = V + U$ for some submodule V of M. By Lemma 3.2 N is closed weak supplemented. ∎

Remark 3.4. The converse of Theorem 3.3 is not true, in general. For example, $\mathbb{Z}$ is closed weak supplemented as a $\mathbb{Z}$-module, for any prime p, $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ is a simple $\mathbb{Z}$-module and is closed weak supplemented. But $\mathbb{Z}_p$ is torsion.

Recall that a right R-module is called singular if $Z(M) = M$ where $Z(M) = \{m \in M | mI = 0,$ for some essential right ideal I of $R\}$ and non-singular if $Z(M) = 0$. A ring R is called right non-singular if R_R is non-singular and singular if R_R is singular. For a closed weak-supplemented ring R, we have:

Theorem 3.5. *Let R be closed weak supplemented as a right R-module. Then every non-singular cyclic module M is closed weak-supplemented.*

Proof. Let $M = mR$, $m \in M$, then $M \cong R/X$, where $X = r(m)$. For $L \leqslant_c M$, there is $X \leqslant T \leqslant R_R$, such that $L = T/X \leqslant_c R/X$. We show that $T \leqslant_c R_R$.

Suppose that $X \leqslant T \leqslant_e K \leqslant R_R$. Then for any $k \in K \backslash T$, there is an essential right ideal I of R such that $kI \subseteq T$. Since $M \cong R/X$ is non-singular, we have that $kI \subseteq T \backslash X$. Hence there is $i \in I$ such that $0 \neq ki + X \in T/X$, therefore, $T/X \leqslant_e K/X$. Since T/X is closed, so $T/X = K/X$ and $T = K$. Since R is closed weak supplemented, there is a submodule V of R, such that $R = V + T$ and $T \cap V \ll R$. By Lemma 3.2, $M \cong R/X$ is closed weak supplemented. ∎

Corollary 3.6. *Let $f : M \to N$ be an epimorphism with M closed weak supplemented and N non-singular. Then N is closed weak supplemented.*

4. Closed Weak Supplemented Ring

A ring R is called closed weak supplemented if R_R is closed weak supplemented. For example, the ring $\mathbb{Z}$ of all integers is closed weak supplemented. In this section, we will discuss the relations between closed weak supplemented rings and modules.

Let F be a free right R-module and $S = End(F)$. Then Theorem 3.5 in [5] says that there is a one-to-one correspondence between the closed right ideals of S and the closed submodules of F.

Theorem 4.1. *Let F be a free R-module and $S = End_R(F)$. Then F is closed weak supplemented as a left S-module if and only if S is closed weak supplemented as a left R-module.*

Proof. Suppose that S is closed weak supplemented. Let $M \leqslant_c F$ and $K = \{s \in S | sF \subseteq M\}$. Then $KF = M$ and $K \leqslant_c S$ by [5, Theorem 3.5]. Since S is closed weak supplemented, there is a submodule T of S such that $S = T + K$ and $T \cap K \ll S$. Since F is a left S-module defined by $sf = s(f)$ for any $s \in S$ and $f \in F$, it is easy to see that F is a faithful left S-module and that $F = TF + KF = TF + M$. Since $TF \cap KF = (T \cap K)F$ and F is free, we have that $TF \cap KF \ll F$. Therefore F is closed weak supplemented.

Conversely, suppose that F is closed weak supplemented and let $K \leqslant_c S$. Then $KF \leqslant_c F$. Hence, there is a submodule M such that $F = M + KF$ and $M \cap KF \ll F$.

Set $I = \{s \in S | sF \subseteq M\}$. Then $IF = M$. Since $SF = F = IF + KF$ and F is faithful, we have that $S = I + K$. $IF \cap KF = (I \cap K)F \ll IF$ implies $I \cap K \ll S$. Hence S is closed weak supplemented. ■

Next we will show that a ring R is closed weak supplemented if and only if $M_n(R)$, the ring of all $n \times n$ matrices over R, is closed weak supplemented, for any positive integer n.

Lemma 4.2. *Let R be any ring and X a right ideal of R. Then $X \leqslant_e R$ if and only if $M_n(X) \leqslant_e M_n(R)$ for any positive integer n. In particular, if $X \leqslant_c R$, then $M_n(X) \leqslant_c M_n(R)$.*

Proof. The proof involves a case-by-case verification as is illustrated in the following proof for $n = 2$.

Suppose that $X \leqslant_e R$. Let $0 \neq \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$.

Case 1. If $a \neq 0$, there is $r \in R$ such that $0 \neq as \in X$. Then we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} as & 0 \\ cs & 0 \end{pmatrix}$$

If $cs = 0$, then $0 \neq \begin{pmatrix} as & 0 \\ 0 & 0 \end{pmatrix} \in M_2(X)$. If $cs \neq 0$, then there is $t \in R$ such that $0 \neq cst \in X$. So

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} st & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} ast & 0 \\ cst & 0 \end{pmatrix} \in M_2(X)$$

Case 2. If $b \neq 0$, there is $s \in R$ such that $0 \neq bs \in X$. Hence

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix} = \begin{pmatrix} 0 & bs \\ 0 & ds \end{pmatrix}$$

If $ds = 0$, then $\begin{pmatrix} 0 & bs \\ 0 & ds \end{pmatrix} \in M_2(X)$;
If $ds \neq 0$, there is $t \in R$ such that $0 \neq dst \in X$. So

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & st \end{pmatrix} = \begin{pmatrix} 0 & bst \\ 0 & dst \end{pmatrix} \in M_2(X)$$

Case 3. If $c \neq 0$, this is similar to case 1.

Case 4. If $d \neq 0$, this is similar to case 2.

Thus $M_2(X) \leqslant_c M_2(R)$.

Conversely, assume that $M_2(X) \leqslant_e M_2(R)$.

For any $0 \neq s \in R$, there is $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(R)$ such that

$$0 \neq \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} sa & sb \\ sc & sd \end{pmatrix} \in M_2(X).$$

Hence at least one of $sa,\ sb,\ sc,\ sd \in X$ is not zero. So $X \leqslant_e R$. ∎

Lemma 4.3. *Let X be a right ideal of $M_n(R)$. Then there are right ideals $I_1,\ I_2,\ ...,\ I_n$ of R such that*

$$X = \begin{pmatrix} I_1 & I_1 & \dots & I_1 \\ I_2 & I_2 & \dots & I_2 \\ \vdots & \vdots & \ddots & \vdots \\ I_n & I_n & \dots & I_n \end{pmatrix} = \left\{ \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} | a_{ij} \in I_i, \right.$$
$$\left. 1 \leqslant j \leqslant n,\ 1 \leqslant i \leqslant n \right\}.$$

Proof. Set $X_{ij} = \{a_{ij} \in R | (a_{ij}) \in X\},\ 1 \leq i,\ j \leqslant n$. It is easy to see that each X_{ij} is a right ideal of R and that $X_{i1} = X_{i2} = \dots = X_{in}$ for any $1 \leqslant i \leqslant n$. So we set $X_{i1} = I_i$ for all $1 \leqslant i \leq n$, as required. ∎

Lemma 4.4. *Let R be any ring and I a right ideal of R. If $I \ll R$ as a right R-module, then $M_n(I) \ll M_n(R)$ for any positive integer n.*

Proof. The proof is routine. ∎

Lemma 4.5. *Let R be any ring and $M_n(R)$ the matrix ring over R. Let X be an essential right ideal of $M_n(R)$. Then there are essential right ideals $I_1,\ I_2,\ \dots\ I_n$ of R such that*

$$X = \begin{pmatrix} I_1 & I_1 & \dots & I_1 \\ I_2 & I_2 & \dots & I_2 \\ \vdots & \vdots & \ddots & \vdots \\ I_n & I_n & \dots & I_n \end{pmatrix}.$$

Moreover, if X is closed (small) in $M_n(R)$, then I_i are closed (small) in R for all $1 \leqslant i \leqslant n$.

Proof. The proof is routine and omitted. ∎

Theorem 4.6. *Let R be any ring. Then R is closed weak supplemented ring if and only if $M_n(R)$ is also closed weak supplemented ring for any positive integer n.*

Proof. Suppose that R is closed weak supplemented. Let X be any closed right ideal of $M_n(R)$, then by Lemma 4.5, there are closed right ideals $I_1, I_2, \ldots, I_n$ of R such that

$$X = \begin{pmatrix} I_1 & I_1 & \ldots & I_1 \\ I_2 & I_2 & \ldots & I_2 \\ \vdots & \vdots & \ddots & \vdots \\ I_n & I_n & \ldots & I_n \end{pmatrix}.$$

Since R is closed weak supplemented, there is submodule J_i of R such that $R = I_i + J_i$ and $I_i \cap J_i \ll R$ for all $1 \leqslant i \leqslant n$. Set

$$Y = \begin{pmatrix} J_1 & J_1 & \ldots & J_1 \\ J_2 & J_2 & \ldots & J_2 \\ \vdots & \vdots & \ddots & \vdots \\ J_n & J_n & \ldots & J_n \end{pmatrix}.$$

It is easy to see that $M_n(R) = Y + X$ and $X \cap Y \ll M_n(R)$. Hence $M_n(R)$ is closed weak supplemented.

Conversely, suppose that $M_n(R)$ is closed weak supplemented and I a closed right ideal of R. Then $X = M_n(I)$ is a closed right ideal of $M_n(R)$ by Lemma 4.2. There is a submodule Y of $M_n(R)$ such that $M_n(R) = X + Y$ and $X \cap Y \ll M_n(R)$. Since

$$Y = \begin{pmatrix} J_1 & J_1 & \ldots & J_1 \\ J_2 & J_2 & \ldots & J_2 \\ \vdots & \vdots & \ddots & \vdots \\ J_n & J_n & \ldots & J_n \end{pmatrix}.$$

for some submodule J_i of R for $1 \leqslant i \leqslant n$. Hence $R = I + J_i$ and $I \cap J_i \ll R$ for $1 \leqslant i \leqslant n$. So R is closed weak supplemented. ■

Similarly, we have:

Corollary 4.7. *Let R be any ring. Then R is a weak supplemented ring if and only if $M_n(R)$ is a weak supplemented ring for any positive integer n.*

Let R be a commutative ring. A module M is called a multiplication module if for any submodule N of M, there is an ideal I of R such that $N = MI$ (see [4]). A module M is called faithful if $r(M) = 0$. For a finitely generated faithfully multiplication module M, we have that $MI \subseteq MJ$ if and only if $I \subseteq J$, where I, J are ideals of R. Now we will show that for a commutative ring R and a finitely generated faithfully multiplication module M, R is closed weak-supplemented if and only if M is closed weak-supplemented. In the following of this section, R is a commutative ring.

Lemma 4.8. *Let $N \leqslant_e M$ with M a finitely generated faithfully multiplication module. Then $I = (N : M) = \{r \in R | Mr \subseteq N\} \leqslant_e R$.*

Proof. Suppose that there is an ideal J of R such that $I \cap J = 0$. Since M is a finite generated faithful multiplication module, we have that $MI \cap MJ = 0$. In fact, if $MI \cap MJ \neq 0$, then there is a unique ideal K of R, such that $MI \cap MJ = MK$. Hence $MK \subseteq MI$, $MK \subseteq MJ$, so $K \leqslant I$ and $K \leqslant J$, hence $K = 0$ and $MJ = 0$, because $N = MI \leqslant_e M$. So $J = 0$ and $I \leqslant_e R$. ■

Lemma 4.9. *Let N_1, N_2 be submodules of a finite generated faithfully multiplication module M. Then $N_1 \leqslant_e N_2$ if and only if $I_1 \leqslant_e I_2$, where $N_i = MI_i$, $i = 1, 2$.*

Proof. Obvious. ■

Lemma 4.10. *Let M be a finite generated faithful multiplication module and $N = MI$ a submodule of M. Then N is closed in M if and only if I is closed in R.*

Proof. This is a consequence of Lemma 4.9. ■

Lemma 4.11. *Let M be a finite generated faithful multiplication module and $N = MI$ a submodule of M. Then $N \ll M$ if and only if $I \ll R$.*

Proof. Suppose that $N \ll M$ and $I + J = R$. Then $M = MR = MI + MJ$. Since $N \ll M$, we have that $MJ = M = MR$ and $J = R$. So $I \ll R$.

Conversely, suppose that $I \ll R$ and $N + L = M$. Then there is a unique ideal J of R such that $L = MJ$. So $MR = M = N + L = MI + MJ = M(I + J)$ and $R = I + J$. Hence $J = R$ and $L = M$. ■

The following theorem is a consequence of the lemmas above.

Theorem 4.12. *Let R be a commutative ring and M a finite generated faithful multiplication module. Then R is closed weak supplemented if and only if M is closed weak supplemented.*

5. The Relations

In this section, we will investigate the relations between closed weak supplemented modules and other modules, such as, extending modules, weak supplemented modules, hollow modules, etc..

A module M is called refinable if for any submodules U, V of M with $U + V = M$, there is a direct summand U' of M with $U' \subseteq U$ and $U' + V = M$ (see [7]).

Proposition 5.1. *Let M be a refinable module. Then the following are equivalent:*

(1) *M is $\oplus$-supplemented;*
(2) *M is supplemented;*
(3) *M is weak supplemented.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

(3) $\Rightarrow$ (1) Suppose that M is weak supplemented. Let N be any submodule of M, there is a submodule K of M such that $M = K + N$ and $N \cap K \ll M$. Since M is refinable, there is a direct summand L of M such that $L \leqslant K$ and $M = L + N$. So we have $N \cap L \leqslant N \cap K \ll L$. Thus M is $\oplus$-supplemented. ■

Proposition 5.2. *Let M be an R-module with $Rad(M) = 0$. Then the following are equivalent:*

(1) *M is a closed weak supplemented module.*

(2) *M is extending.*

Proof. (1) $\Rightarrow$ (2). Suppose that M is closed weak supplemented and that N is closed in M. Then there is a submodule K of M such that $M = K + N$ and $K \cap N \ll M$ and therefore $K \cap N = 0$. Hence N is a direct summand of M, i.e., M is extending.

(2) $\Rightarrow$ (1). Obvious. ■

Corollary 5.3. *Let R be a semiprimitive ring. Then the following are equivalent:*

(1) *R is a closed weak supplemented ring;*

(2) *R is extending.*

A ring R is a V-ring if and only if $Rad(M) = 0$ for all R-modules M. Hence we have:

Corollary 5.4. *Let R be a V-ring. Then the following are equivalent for any R-module M:*

(1) *M is a closed weak supplement;*

(2) *M is extending.*

Next, we will study the relation between closed weak supplemented modules and weak supplemented modules. Let M be a module. If every submodule is closed in M, (for example, M is semi-simple), then M is closed weak supplemented if and only if M is weak supplemented. For other cases, we give:

Lemma 5.5. *Let M be closed weak supplemented and $N \leqslant_c M$. Suppose that $T \ll M$. Then there is a submodule K of M such that $M = K + N = K + N + T$ and $K \cap N \ll M$, $K \cap (N + T) \ll M$.*

Proof. Since M is closed weak supplemented, there is a submodule K of M such that $M = K + N$, and $K \cap N \ll M$.

Let $f : M \to (M/N) \oplus (M/K)$ which is defined by $f(m) = (m + N, m + K)$ and $g : (M/N) \oplus (M/K) \to (M/(N + T)) \oplus (M/K)$ which is defined by $g(m + N, m' + K) = (m + N + T, m' + K)$. Since $M = N + K$, then we have that f is an epimorphism and that $Kerf = N \cap K \ll M$, i.e., $N \cap K \ll M$. Since $Kerg = ((N + T)/N) \oplus 0$ and $(N + T)/N = \pi(T) \ll M/N$, while $\pi : M \to M/N$ is the canonical epimorphism, we have that g is a small epimorphism. So gf is a small epimorphism and $Ker g f = (T+N)\cap K \ll M$. Clearly, $M = (N+T)+K$. ∎

Theorem 5.6. *Let M be a R-module. Suppose that for any submodule N of M, there is a closed submodule L (depending on N) of M such that $N = L + T$ or $L = N + T$ for some $T \ll M$. Then M is weak supplemented if and only if M is closed weak supplemented.*

Proof. Suppose that M is closed weak supplemented and N any submodule of M.

Case 1. Suppose that there is a closed submodule L such that $N = L + T$ for some $T \ll M$. Then this is a consequence of Lemma 5.5.

Case 2. Suppose that there is closed submodule L of M such that $L = N+T$ for some $T \ll M$. Since M is closed weak-supplemented, there is a submodule K of M such that $M = K+L$ and $K\cap L \ll M$. So $M = K+N+T$, hence $M = K+N$, since $T \ll M$. $K \cap N \leqslant K \cap L \ll M$. Thus M is weak supplemented.

The converse is trivial. ∎

Combining this theorem with Proposition 5.1, we have:

Corollary 5.7. *Let M be a refinable module. Suppose that for any submodule N of M, there is a closed submodule L (depending on N) of M such that $N = L+T$ or $L = N + T$ for some $T \ll M$. Then the following are equivalent:*

(1) *M is $\oplus$-supplemented;*
(2) *M is supplemented;*
(3) *M is weak supplemented;*
(4) *M is closed weak supplemented.*

Lemma 5.8. *Let U and K be submodules of M such that K is a weak supplement of a maximal submodule N of M. If $K + U$ has a weak supplement X in M, then U has a weak supplement in M.*

Proof. Since X is a weak supplement of $K + U$ in M, then $X \cap (K + U) \ll M$. If $K \cap (X + U) \subseteq K \cap N \ll M$, then

$$U \cap (K + X) \leqslant X \cap (K + U) + K \cap (X + U) \ll M,$$

hence $K + X$ is a weak supplement of U in M.

Suppose that $K \cap (X + U)$ is not contained in $K \cap N$. Since $K/(K \cap N) \cong (K + N)/N = M/N$, $K \cap N$ is a maximal submodule of K. Therefore, $K \cap N + K \cap (X + U) = K$ and since $K \cap N \ll M$, we have

$$M = X + U + K = X + U + K \cap N + K \cap (X + U) = X + U.$$

Since $U \cap X \ll (K + U) \cap X \ll X$, then X is a weak supplement of U in M. ∎

The following proposition is an immediate consequence of this lemma:

Proposition 5.9. *Suppose that for any submodule U of M, there is a submodule K, which is a weak supplement of some maximal submodule N of M, such that $K+U \leqslant_c M$ has a weak supplement X in M. Then M is closed weak supplemented if and only if M is weak supplemented.*

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