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On Convergence of Two-Parameter Multivalued Pramarts and Mils

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Abstract. In this paper we give some convergence results for two-parameter multivalued 1-pramarts and 1-mils.

1. Introduction

Real-valued martingales were first introduced and considered by Doob [6], and later systematically extended to the Banach space-valued case by many authors. For the main convergence results for vector-valued martingales and their generalizations, the interested reader is referred to Neveu [14], Millet and Sucheston [12], Talagrand [16], Edgar and Sucheston [5], Luu [9] and etc... On the other hand, martingales, submartingales and laws of large numbers of random sets have been also extensively considered in recent years by Mosco [13], Castaing and Valadier [2], Luu [10, 11], Hess [6], Wang and Xue [17], Choukairi - Dini [3], Wenlong and Zhenpend [18], Lavie [8] and etc... The main aim of the note is to apply some of these results to prove several convergence theorems for multivalued 1-pramarts and 1-mils.

2. Notations and Definitions

Throuthout the paper, we shall denote by $(\Omega, \mathcal{F}, \mathbb{P})$ a complete probabitity space, X a separable (real) Banach space and $wkc(X)$ the collection of all nonempty, weakly compact and convex subsets of X. Further, let denote by $\mathbb{N}$ the set of all nonnegative integers and $J = \mathbb{N} \times \mathbb{N}$. Then it is known that endowed with the usual partial order "$\leqslant$", given by $s = (s_1, s_2) \leqslant t = (t_1, t_2)$ if and only if $s_1 \leqslant t_1$ and $s_2 \leqslant t_2$, J becomes a directed set. Let $(\mathcal{F}_t)_{t \in J}$ be a complete stochastic basis of $(\Omega, \mathcal{F}, \mathbb{P})$, i.e, a nondecreasing family of complete sub-fields of $\mathcal{F}$ with $\mathcal{F} = \bigvee_{t \in J} \mathcal{F}_t$. For each $t = (t_1, t_2) \in J$, we put

$$\mathcal{F}_t^1 = \bigvee_{u \in \mathbb{N}} \mathcal{F}_{(t_1, u)}.$$

A map $\tau : \Omega \to J$ is called *an 1-stopping time*, if $[\tau = t] \in \mathcal{F}_t^1,\ t \in J$. The set of all simple 1-stopping times is denoted by T^1. Then it is also known that equipped with the a.s. order "$\leqslant$", given by

$$\sigma \leqslant \tau \ \text{ if and only if } \ \sigma(\omega) \leqslant \tau(\omega), \ \text{a.s.},$$

T^1 becomes also a directed set, and $\overline{\mathbb{N}} = \{\overline{n} = (n, n),\ n \in \mathbb{N}\}$ and J would be regarded as two special cofinal subsets of T^1. Furthermore, by Proposition 4.2.5 [4], the stochastic basis $(\mathcal{F}_t^1)_{t \in T}$ satisfies the Vitalli condition (V), i.e., for any $A \in \mathcal{F} = \sigma\left(\bigcup_t \mathcal{F}_t^1\right)$, $A_t \in \mathcal{F}_t^1$ with $A \subset \operatorname{esslim\,sup}_{t \in J} A_t$ and $\varepsilon > 0$, there is a finite system $\{t_i,\ i \leqslant m\}$ of J and disjoint sets (B_i) with each $B_i \in \mathcal{F}_{t_i}^1,\ B_i \subset A_{t_i},\ i \leqslant m$ and such that $\mathbb{P}\left(A \backslash \bigcup_{i \leqslant m} B_i\right) < \varepsilon$.

Now let $A, C, A_t \in wkc(X),\ t \in J$. We say that $(A_t)_{t \in J}$ is *weakly convergent to* A, write

$$A_t \xrightarrow{w} A,\ t \in J,$$

if for each $x^* \in X^*$, we have

$$s(x^*, A_t) \to s(x^*, A),\ t \in J,$$

where X^* is the topological dual of X and

$$s(x^*, C) = \sup\{< x^*, x >,\ x \in C\}.$$

Further, $(A_t)_{t \in J}$ is said to be *Wijsman convergent* to A, write

$$A_t \xrightarrow{Wijs} A,\ t \in J,$$

if for each $x \in X$, we have

$$d(x, A_t) \to d(x, A),\ t \in J,$$

where

$$d(x, C) = \inf\{\|x - y\|, y \in C\}.$$

In the particular case, when

$$A_t \xrightarrow{w} A \text{ and } A_t \xrightarrow{Wijs} A,\ t \in J,$$

we shall say that $(A_t)_{t \in J}$ *converges to* A *in linear topology* and write $\tau_L\text{-}\lim_{t \in J} A_t = A$ or

$$A_t \xrightarrow{\tau_L} A, \ t \in J.$$

Now, we define

$$s\text{-}\liminf_{t\in J} A_t = \{x \in X : \lim_{t\in J} d(x, A_t) = 0\}$$

and

$$w\text{-}\limsup_{t\in J} = \left\{ x \in X : x_k \xrightarrow{w} x, \ x_k \in A_{t_k} \right\}$$

where $(t_k)_{k\in\mathbb{N}}$ is a cofinal subsequence of J. Finally, $(A_t)_{t\in J}$ is said to be *convergent to A in the Mosco sense*, write $M\text{-}\lim_{t\in J} A_t = A$, if

$$w\text{-}\limsup_{t\in J} A_t = A = s\text{-}\liminf_{t\in J} A_t.$$

It is easily checked that if $\tau_L\text{-}\lim_{t\in J} A_t = A$ then $M\text{-}\lim_{t\in J} A_t = A$ (see also Lemma 5.4. [6]).

For other related notations and definitions, the reader is referred to Castaing and Valadier [2].

3. Main Results

From now on, let $\mathcal{L}^1_{wkc(X)}$ denote the complete metric space of all integrably bounded multifunctions $F : \Omega \to wkc(X)$ (see [7]). It is clear that if $F, G \in \mathcal{L}^1_{wkc(X)}$ then both real-valued functions

$$|F|(\omega) = \sup\{\|x\|, \ x \in F(\omega)\}$$

and

$$h(F, G)(\omega) = h(F(\omega), G(\omega)), \ \omega \in \Omega$$

are also integrable, where

$$h(A, C) = \max\left(\sup_{x\in A} d(x, C), \ \sup_{y\in C} d(y, A) \right).$$

Unless otherwise stated, we shall consider in the note only processes $(F_t)_{t\in J}$ in $\mathcal{L}^1_{wkc(X)}$ such that each F_t is $\mathcal{F}_t$-measurable. Note that the first convergence result we shall prove is connected with the following notion.

Definition 3.1. *We say that $(F_t)_{t\in J}$ is an 1-pramart, if for every $\varepsilon > 0$, there is $\sigma_0 \in T^1$ such that*

$$\mathbb{P}\left(h(F_\sigma, \mathcal{E}(F_\tau|\mathcal{F}^1_\sigma)) > \varepsilon \right) < \varepsilon, \quad \forall \sigma, \tau \in T^1, \ \sigma_0 \leqslant \sigma \leqslant \tau.$$

Remark 1. It is worth noting that in the case, when $(\mathcal{F}_t)$ satisfies the usual conditional independence condition F_4, every real-valued L^1-bounded martingale (F_t) is an 1-amart (cf. [5], Remark 9.4.12). This with Theorem 4.2.10 [5] guarantees that (F_t) converges a.s., hence it should be an 1-pramart.

The following theorem seems to be the first convergence result for multivalued 1-pramarts.

Theorem 3.1. *Let* $((F_t)_{t\in T}$ *be an 1-pramart such that*

a) $\overline{co} \bigcup_t F_t(\omega) \in wkc(X), \ \forall \omega \in \Omega,$

b) $\sup_{t\in J} E|F_t| < \infty.$

Then there exists an integrably bounded multifunction F *such that*

$$\tau_L\text{-}\lim_{t\in J} F_t = F \text{ a.s..}$$

Proof. We denote by D (D^*) a countable subset which is dense for the norm (Mackey) topology in the closed unit ball B (B^*) of X $(X^*$, respectively) and by D_1^* the set of all rational linear combinations of members of D^*.

Firstly, because for all $x^* \in D^*,\ \sigma, \tau \in T^1,\ \sigma \leqslant \tau$ we have

$$\begin{aligned}|s(x^*, F_\sigma) - \mathcal{E}(s(x^*, F_\tau)|\mathcal{F}_\sigma^1)| &= |s(x^*, F_\sigma) - s(x^*, \mathcal{E}(F_\tau|\mathcal{F}_\sigma^1))| \\ &\leqslant h(F_\sigma, \mathcal{E}(F_\tau|\mathcal{F}_\sigma^1))\end{aligned}$$

and $(F_t)_{t\in J}$ is an 1-pramart, $(s(x^*, F_t))_{t\in J}$ is a real 1-pramart. Further, by Proposition 4.2.5 [5], the stochastic basis $(\mathcal{F}_t^1)_{t\in J}$ satisfies the Vitali condition V and by b), Doob's condition

$$\sup_{t\in J} E|s(x^*, F_t)| \leqslant ||x^*|| \sup_{t\in J} E|F_t| < \infty$$

is satisfied, so by Theorem 4.4 or Theorem 5.1 in [12], the real 1-pramart $(s(x^*, F_t))_{t\in J}$ converges almost surely. Therefore, by Lemma 5.2 [6], there exist a measurable multifunction F with values in $wkc(X)$ and a negligible subset N_1 such that

$$\lim_{t\in J} s(x^*, F_t(\omega)) = s(x^*, F(\omega)), \quad \forall x^* \in D_1^*, \ \ \forall \omega \notin N_1.$$

It follows that

$$F_t(\omega) \xrightarrow{w} F(\omega), \ t \in J, \ \omega \notin N_1. \tag{3.1}$$

Secondly we prove that $(F_t)_{t\in J}$ is Wijsman convergent to F (a.s.). For the purpose, let us fix $x \in X$, and put

$$Z_t^{x^*} = < x^*, x > - s(x^*, F_t), \ x^* \in X^*, \ t \in J.$$

We show that the process $\{(Z_t^{x^*}, \mathcal{F}_t^1)_{t\in J},\ x^* \in D^*\}$ is a uniform sequence of real-valued pramarts, i.e., for every $\varepsilon > 0$, there exists $\sigma_0 \in T^1$ such that for every $\sigma, \tau \in T^1$ with $\sigma_0 \leqslant \sigma \leqslant \tau$, we have

$$\mathbb{P}[\sup_{x^*\in D^*} |Z_\sigma^{x^*} - \mathcal{E}(Z_\tau^{x^*}|\mathcal{F}_\sigma^1)| > \varepsilon] < \varepsilon. \tag{3.2}$$

Indeed, since $(F_t, \mathcal{F}_t^1)_{t\in J}$ is a pramart, hence for each $\varepsilon > 0$, there exists $\sigma_0 \in T^1$ such that for any $\sigma, \tau \in T^1,\ \sigma_0 \leqslant \sigma \leqslant \tau,$ we have

$$\mathbb{P}[h(F_\sigma, \mathcal{E}(F_\tau|\mathcal{F}^1_\sigma)) > \ ] < \ . \tag{3.3}$$

On the other hand,

$$\begin{aligned}\sup_{x^*\in D^*} |Z^{x^*}_\sigma - \mathcal{E}(Z^{x^*}_\tau|\mathcal{F}^1_\sigma)| &= \sup_{x^*\in D^*} |\mathcal{E}[s(x^*, F_\tau)|\mathcal{F}^1_\sigma) - s(x^*, F^1_\sigma)| \\ &= \sup_{x^*\in D^*} |s(x^*, F_\sigma) - s(x^*, \mathcal{E}(F_\tau|\mathcal{F}^1_\sigma)| \\ &\leqslant h(F_\sigma, \mathcal{E}(F_\tau|\mathcal{F}^1_\sigma)). \end{aligned} \tag{3.4}$$

Then (3.3) and (3.4) imply (3.2).

But

$$\sup_{t\in J} \mathcal{E}(\sup_{x^*\in D^*} |Z^{x^*}_t|) \leqslant ||x|| + \sup_{t\in J} \mathcal{E}|F_t| < \infty,$$

it follows that for each $x^* \in D^*$ $(Z^{x^*}_t)_{t\in J}$ converges a.s. to some real integrable function Z^{x^*} (cf. [12, Theorem 5.1]), $R^{x^*}_t = \operatorname{ess}\inf_{\sigma\in T^1(t)} (Z^{x^*}_\sigma|\mathcal{F}^1_t)$ is finite a.s. and $(R^{x^*}_t, \mathcal{F}^1_t)_{t\in J}$ is a generalized sub-martingale (cf. [12, Proposition 3.3]).

Moreover, we can prove that

$$\sup_{\tau\in T^1(\sigma)} \mathbb{P}(\sup_{x^*\in D^*} (Z^{x^*}_\sigma - \mathcal{E}(Z^{x^*}_\tau|\mathcal{F}^1_\sigma)) > \ ) = \mathbb{P}(\sup_{x^*\in D^*} (Z^{x^*}_\sigma - R^{x^*}_\sigma) > \ ), \tag{3.5}$$

where $T^1(\sigma) = \{\tau \in T^1,\ \tau \geqslant \sigma\}$. Indeed, by Proposition 4.1.14 in [5], for each $x^* \in D^*$ we can choose a nondecreasing cofinal sequence $(\tau^{x^*}_n) \subset T^1(\sigma)$ such that $\mathcal{E}(Z^{x^*}_{\tau^{x^*}_n}|\mathcal{F}^1_\sigma) \downarrow R^{x^*}_\sigma$. Then

$$\begin{aligned} &\operatorname{esssup}_{\tau\in T^1(\sigma)} \mathbb{P}\Big(\sup_{x^*\in D^*} (Z^{x^*}_\sigma - \mathcal{E}(Z^{x^*}_\tau|\mathcal{F}^1_\sigma)\Big) > \ ) \\ &\leqslant \mathbb{P}\Big(\operatorname{esssup}_{\tau\in T^1(\sigma)} \sup_{x^*\in D^*} (Z^{x^*}_\sigma - \mathcal{E}(Z^{x^*}_\tau|\mathcal{F}^1_\sigma)) > \ \Big) \\ &= \mathbb{P}\Big(\sup_{x^*\in D^*} \operatorname{esssup}_{\tau\in T^1(\sigma)} (Z^{x^*}_\sigma - \mathcal{E}(Z^{x^*}_\tau|\mathcal{F}^1_\sigma)) > \ \Big) \\ &= \mathbb{P}\Big(\sup_{x^*\in D^*} (Z^{x^*}_\sigma - R^{x^*}_\sigma)) > \ \Big) \\ &= \mathbb{P}\Big(\sup_{x^*\in D^*} \sup_n (Z^{x^*}_\sigma - \mathcal{E}(Z^{x^*}_{\tau^{x^*}_n}|\mathcal{F}^1_\sigma)) > \ \Big) \\ &= \mathbb{P}\Big(\sup_n (\sup_{x^*\in D^*} (Z^{x^*}_\sigma - \mathcal{E}(Z^{x^*}_{\tau^{x^*}_n}|\mathcal{F}^1_\sigma)) > \ \Big) \\ &= \sup_n \mathbb{P}\Big(\sup_{x^*\in D^*} (Z^{x^*}_\sigma - \mathcal{E}(Z^{x^*}_{\tau^{x^*}_n}|\mathcal{F}^1_\sigma)) > \ \Big) \\ &\leqslant \operatorname{esssup}_{\tau\in T^1(\sigma)} \mathbb{P}\Big(\sup_{x^*\in D^*} (Z^{x^*}_\sigma - \mathcal{E}(Z^{x^*}_\tau|\mathcal{F}^1_\sigma)) > \ \Big). \end{aligned}$$

Thus, (3.5) is proved. ∎

But $Z^{x^*}_\sigma \geqslant R^{x^*}_\sigma$, a.s., it follows from Theorem 4.2 [12] that

$$\lim_{t\in J}\Big(\sup_{x^*\in D^*}|Z_t^{x^*}-R_t^{x^*}|\Big)=0 \quad \text{a.s.},$$

and thus for each $x^*\in D^*$, the nets $(Z_t^{x^*})_{t\in J}$ and $(R_t^{x^*})_{t\in J}$ converge almost surely to the same limit Z^{x^*}

Applying the proof of Neveu's Lemma [15, Lemma V.2.9] for the submartingale $(R_t^{x^*})_{t\in J}$ and Wang-Xue's Lemma [17, Lemma 2.2] we obtain

$$\lim_t\Big(\sup_{x^*\in D^*}Z_t^{x^*}(\omega)\Big)=\sup_{x^*\in D^*}\Big(\lim_t Z_t^{x^*}(\omega)\Big)=\sup_{x^*\in D^*}Z^{x^*}(\omega)$$

for every $x\in D$ and $\omega\notin N_x$ $(P(N_x)=0)$. Thus, for each $x^*\in D_1^*,\ x\in D$ and $\omega\notin[N_1\cup N_x]$

$$\lim_{t\in J}Z_t^{x^*}=\lim_{t\in J}\Big(<x^*,x>-s(x^*,F_t(\omega))\Big)=<x^*,x>-s(x^*,F(\omega)).$$

On the other hand, since

$$d(x,A)=\sup_{x^*\in D^*}[<x^*,x>-s<x^*,A>],\ A\in wkc(X)$$

(cf [17, p. 815] or [6, p. 190]), we get

$$\lim_{t\in J}d(x,F_t(\omega))=d(x,F(\omega))$$

for all $x\in D$ and $\omega\notin N_1\cup(\cup_{y\in D}N_y)$. Thus, by putting $N_0=N_1\cup(\cup_{x\in D}N_x)$ we get

$$F_t(\omega)\xrightarrow{Wijs}F(\omega),\ t\in J,\ \omega\notin N_0.$$

This with (3.1) implies

$$\tau_L\text{-}\lim F_t(\omega)=F(\omega)\quad\forall\omega\notin N.$$

Finally, since

$$|F(\omega)|=\sup\{||x||;x\in F(\omega)\}=\sup\{s(x^*,F(\omega)):x^*\in D^*\}$$

we have

$$|F(\omega)|\leqslant\liminf_{t\in J}|F_t(\omega)|,\quad\forall\omega\notin N_0.$$

Hence by Fatou's Lemma

$$E|F|\leqslant\liminf_{t\in J}E|F_t|<\infty.$$

In other words, F is integrably bounded, it completes the proof. ■

Related to the constructive results of Talagrand [16] for vector-valued mils, we propose the following.

Definition 3.2. *Let $(F_t)_{t\in J}$ be an adapted sequence of integrably bounded $wkc(X)$-valued multifunctions. We say that $(F_t)_{t\in J}$ is an 1-mil, if $(F_t,\mathcal{F}_t^1)_{t\in J}$*

is a mil, i.e., for every $\varepsilon > 0$*, there exists* $p \in \mathbb{N}$ *such that for any* $n \in \mathbb{N}$*,* $\tau \in T^1, \overline{p} \leqslant \tau \leqslant \overline{n}$*, we have*

$$\mathbb{P}(h(X_\tau, \mathcal{E}(X_{\overline{n}}|\mathcal{F}_\tau^1)) > \varepsilon) < \varepsilon,$$

where $\overline{n} = (n, n) \in \overline{\mathbb{N}}$*.*

Remark 2. It is easy to see that every 1-pramart is an 1-mil. Furthermore, restricted to the one-parameter discrete case, the notion of 1-mils coincides with the original notion of mils introduced by Talagrand [16].

The following lemma will be needed in the proof of the next weak convergence result.

Lemma 3.1. *Let* $(X_t)_{t\in J}$ *be a uniformly integrable, real 1-mil. Then* $(X_t)_{t\in J}$ *converges a.s.*

Proof. Let $(X_t)_{t\in J}$ be as given in the lemma. Then $(X_{\overline{n}}, \mathcal{F}_{\overline{n}}^1)_{n\geqslant 0}$ is also a mil in the sense of Talagrand. Hence, by ([16, Theorem 4]) and the uniform integrability of $(X_t)_{t\in T}$, $(X_{\overline{n}})$ converges to some X a.s. and in L^1. Consequently $(X_{\overline{n}})_{n\in\mathbb{N}}$ is written uniquely in the form $X_{\overline{n}} = Y_{\overline{n}} + Z_{\overline{n}}$ where $(Y_{\overline{n}})_{n\in\mathbb{N}}$ is a regular martingale: $Y_{\overline{n}} = \mathcal{E}(X|\mathcal{F}_{\overline{n}}^1)$ and $(Z_{\overline{n}})_{n\in\mathbb{N}}$ is a mil with $Z_{\overline{n}} \to 0$ a.s. and in L^1, $n \in \mathbb{N}$.

Put $Y_t = \mathcal{E}(X|\mathcal{F}_t^1)$, $Z_t = X_t - Y_t$, $t \in J$. Since $(Y_t, \mathcal{F}_t^1)_{t\in J}$ is a regular martingale, hence by ([12, Theorem 4.3]), (Y_t) converges to X a.s. and in L^1.

Now we prove that the mil $(Z_t, \mathcal{F}_t^1)_{t\in J}$ is convergent to 0 a.s. By Theorem 4.2 in [12] (see also [19], Lemma 2), it is sufficient to prove that the net $(Z_\tau)_{\tau\in T^1}$ converges to 0 in probability. Since $(Z_t, \mathcal{F}_t^1)_{t\in J}$ is a mil, for any $\varepsilon > 0$ there is $p \in \mathbb{N}$ such that for every $\tau \in T^1$, $n_1 \in \mathbb{N}$ with $\overline{p} \leqslant \tau \leqslant \overline{n}_1$, we have

$$\mathbb{P}(|Z_\tau - \mathcal{E}(Z_{\overline{n}}|\mathcal{F}_\tau^1)| > \varepsilon) < \varepsilon. \tag{3.6}$$

On the other hand, since $Z_{\overline{n}} \to 0$ in L^1 as $n \uparrow \infty$, it follows that there is $n_2 \geqslant n_1, n_2 \in \mathbb{N}$ such that

$$E|Z_{\overline{n}}| < \varepsilon^2, \; n \geqslant n_2. \tag{3.7}$$

Thus, by (3.6), (3.7) and Chebyshev's inequality, for any $\tau \in T^1$ and $n \in \mathbb{N}$ satisfying $\tau \geqslant \overline{p}$, $n \geqslant n_2$, we have

$$\begin{aligned}\mathbb{P}(|Z_\tau| > 2\varepsilon) &\leqslant \mathbb{P}\left(\left|Z_\tau - \mathcal{E}(Z_{\overline{n}}|\mathcal{F}_\tau^1)\right| > \varepsilon\right) + \mathbb{P}\left(\left|\mathcal{E}(Z_{\overline{n}}|\mathcal{F}_\tau^1)\right| > \varepsilon\right)\\ &\leqslant \varepsilon + \frac{E|Z_{\overline{n}}|}{\varepsilon} \leqslant \varepsilon + \frac{\varepsilon^2}{\varepsilon} = 2\varepsilon.\end{aligned}$$

It means that $(Z_\tau)_{\tau\in T^1}$ converges to 0 in probability.
This completes the proof. ■

For multivalued 1-mils, we get the following weak convergence result.

Theorem 3.2. *Let* $(F_t, t \in N)$ *be a uniformly integrable* $wkc(X)$*-valued 1-mil. Suppose that*

$$\overline{co}\left(\bigcup_{t\in J} F_t(\omega)\right) \in wkc(X), \quad \omega \in \Omega.$$

Then there exists a multifunction F of $\mathcal{L}^1_{wkc(X)}$ such that

$$w\text{-}\lim_{t\in J} F_t = F \ a.s.$$

Proof. Let $(F_t)_{t\in J}$ be as given in the theorem. Since for each $x^* \in X^*$

$$|s(x^*, F_t(\omega))| \leqslant ||x^*|||F_t(\omega)|,$$

the set $\{s(x^*, F_t)\}_{t\in J}$ is uniformly integrable.

But if $\|x^*\| \leqslant 1$, we have

$$\begin{aligned}&\left|s(x^*, F_\sigma) - \mathcal{E}(s(x^*, F_{\overline{n}})|\mathcal{F}^1_\sigma)\right| \\ =& \left|s(x^*, F_\sigma) - s(x^*, \mathcal{E}(F_{\overline{n}}|\mathcal{F}^1_\sigma))\right| \\ \leqslant& h(F_\sigma, \mathcal{E}(F_{\overline{n}}|\mathcal{F}^1_\sigma)) \quad \text{for any} \quad \sigma \leqslant \overline{n},\end{aligned}$$

then for each $x^* \in X^*$, the process $\{s(x^*, F_t\}_{t\in J}$ is also a uniformly integrable real 1-mil. It follows from Lemma 3.1 that for each $x^* \in D^*_1$ there exists a negligible subset N_{x^*} such that $\lim_t s(x^*, F_t(\omega))$ exists for any $\omega \in \Omega \backslash N_{x^*}$. This with the same argument used in Lemma 5.2 [6] entails the existence of a multifunction F with values in $wkc(X)$ which satisfies

$$s(x^*, F(\omega)) = \lim_t s(x^*, F_t(\omega)), \ \forall \omega \in N, \ \forall x^* \in D^*_1$$

where $N = \bigcup_{x^*\in D^*_1} N_{x^*}$. But D^*_1 is countable and dense in X^* for the Mackey topology, it guarantees that

$$F_t \xrightarrow{w} F, \ t \in J.$$

The proof is thus complete. ∎

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