Vietnam Journal of Mathematics **34**:1 (2006) 41–49

Vietnam Journal
of
MATHEMATICS

Strong Insertion of a Contra - Continuous Function*

Majid Mirmiran

Department of Mathematics, University of Isfahan
Isfahan 81746-73441, Iran

Received February 22, 2004
Revised October 20, 2005

Abstract. Necessary and sufficient conditions in terms of lower cut sets are given for the strong insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that $\Lambda-$sets are open.

1. Introduction

A generalized class of closed sets was considered by Maki in 1986 [9]. He investigated the sets that can be represented as union of closed sets and called them $V-$sets. Complements of $V-$sets, i.e., sets that are intersection of open sets are called $\Lambda-$sets [9].

Results of Katětov [5, 6] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give necessary and sufficient conditions for the insertion of a contra-continuous function between two comparable real-valued functions on such topological spaces that $\Lambda-$sets are open [3].

A real-valued function f defined on a topological space X is called *contra-continuous* if the preimage of every open subset of $\mathbb{R}$ is closed in X.

If g and f are real-valued functions defined on a space X, we write $g \leq f$ in case $g(x) \leq f(x)$ for all x in X.

*This work was supported by University of Isfahan, R.P. 821033 and Centre of Excellence for Mathematics (University of Isfahan).

The following definitions are modifications of conditions considered in [7].

A property P defined relative to a real-valued function on a topological space is a *$cc-$property* provided that any constant function has property P and provided that the sum of a function with property P and any contra-continuous function also has property P. If P_1 and P_2 are $cc-$properties, the following terminology is used: (i) A space X has the *weak $cc-$insertion property for* (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a contra-continuous function h such that $g \leq h \leq f$. (ii) A space X has the *strong $cc-$insertion property for* (P_1, P_2) if and only if for any functions g and f on X such that $g \leq f, g$ has property P_1 and f has property P_2, then there exists a contra-continuous function h such that $g \leq h \leq f$ and such that if $g(x) < f(x)$ for any x in X, then $g(x) < h(x) < f(x)$.

In this paper, for a topological space that $\Lambda-$sets are open, is given a sufficient condition for the weak $cc-$insertion property. Also for a space with the weak $cc-$insertion property, we give necessary and sufficient conditions for the space to have the strong $cc-$insertion property. Several insertion theorems are obtained as corollaries of these results.

2. The Main Results

Before giving a sufficient condition for insertability of a contra-continuous function, the necessary definitions and terminology are stated.

Definition 2.1. *Let A be a subset of a topological space (X, τ). We define the subsets A^Λ and A^V as follows:*
$A^\Lambda = \cap\{O : O \supseteq A, O \in \tau\}$ and $A^V = \cup\{F : F \subseteq A, F^c \in \tau\}$.
In [4, 8], A^Λ is called the *kernel* of A.

The following first two definitions are modifications of conditions considered in [5, 6].

Definition 2.2. *If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x \ \bar{\rho} \ y$ if and only if $y \ \rho \ v$ implies $x \ \rho \ v$ and $u \ \rho \ x$ implies $u \ \rho \ y$ for any u and v in S.*

Definition 2.3. *A binary relation ρ in the power set $P(X)$ of a topological space X is called a strong binary relation in $P(X)$ in case ρ satisfies each of the following conditions:*

1) *If $A_i \ \rho \ B_j$ for any $i \in \{1, \ldots, m\}$ and for any $j \in \{1, \ldots, n\}$, then there exists a set C in $P(X)$ such that $A_i \ \rho \ C$ and $C \ \rho \ B_j$ for any $i \in \{1, \ldots, m\}$ and any $j \in \{1, \ldots, n\}$.*
2) *If $A \subseteq B$, then $A \ \bar{\rho} \ B$.*
3) *If $A \ \rho \ B$, then $A^\Lambda \subseteq B$ and $A \subseteq B^V$.*

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [2] as follows:

Definition 2.4. *If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ \} \subseteq A(f, \ ) \subseteq \{x \in X : f(x) \leq \ \}$ for a real number , then $A(f, \ )$ is called a lower indefinite cut set in the domain of f at the level .*

We now give the following main results:

Theorem 2.1. *Let g and f be real-valued functions on a topological space X, in which $\Lambda-$sets are open, with $g \leq f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\ \rho\ A(g,t_2)$, then there exists a contra-continuous function h defined on X such that $g \leq h \leq f$.*

Proof. Let g and f be real-valued functions defined on X such that $g \leq f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\ \rho\ A(g,t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f,t)$ and $G(t) = A(g,t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1)\ \bar{\rho}\ F(t_2), G(t_1)\ \bar{\rho}\ G(t_2)$, and $F(t_1)\ \rho\ G(t_2)$. By Lemmas 1 and 2 of [6] it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\ \rho\ H(t_2), H(t_1)\ \rho\ H(t_2)$ and $H(t_1)\ \rho\ G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leq h \leq f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x is in $G(t') = A(g,t')$ implies that $g(x) \leq t'$, it follows that $g(x) \leq t$. Hence $g \leq h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f,t')$ implies that $f(x) > t'$, it follows that $f(x) \geq t$. Hence $h \leq f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1,t_2) = H(t_2)^V \setminus H(t_1)^\Lambda$. Hence $h^{-1}(t_1,t_2)$ is closed in X, i.e., h is a contra-continuous function on X. ■

The above proof used the technique of proof of Theorem 1 of [5].

If a space has the strong cc-insertion property for (P_1,P_2), then it has the weak cc-insertion property for (P_1,P_2).The following results use lower cut sets and gives a necessary and sufficient condition for a space satisfying the weak cc-insertion property to satisfy the strong cc-insertion property.

Theorem 2.2. *Let P_1 and P_2 be $cc-$properties and X be a space satisfying the weak $cc-$insertion property for (P_1,P_2). Also assume that g and f are functions on X such that $g \leq f, g$ has property P_1 and f has property P_2. The space X has the strong $cc-$insertion property for (P_1,P_2) if and only if there exist lower cut sets $A(f-g,2^{-n})$ and there exists a sequence $\{F_n\}$ of subsets of X such that (i) for each n, F_n and $A(f-g,2^{-n})$ are completely separated by contra-continuous functions, and (ii)$\{x \in X : (f-g)(x) > 0\} = \bigcup_{n=1}^{\infty} F_n$.*

Proof. Theorem 3.1 of [11]. ■

Theorem 2.3. *Let P_1 and P_2 be $cc-$properties and assume that a space X satisfies the weak $cc-$insertion property for (P_1, P_2). The space X satisfies the strong $cc-$insertion property for (P_1, P_2) if and only if X satisfies the strong $cc-$insertion property for (P_1, cc) and for (cc, P_2).*

Proof. Theorem 3.2 of [11]. ■

3. Applications

Definition 3.1. *A real-valued function f defined on a space X is called upper semi-contra-continuous (resp. lower semi-contra-continuous) if $f^{-1}(-\infty, t)$ (resp. $f^{-1}(t, +\infty)$) is closed for any real number t.*

The abbreviations $usc, lsc, uscc, lscc,$ and cc are used for upper semicontinuous, lower semicontinuous, upper semi-contra-continuous, lower semi-contra-continuous, and contra-continuous, respectively.

Before stating the consequences of Theorems 2.1, 2.2, and 2.3 we suppose that X is a topological space that $\Lambda-$sets are open.

Corollary 3.1. *X is an extremally disconnected space if and only if X has the weak $cc-$insertion property for $(uscc, lscc)$.*

Proof. Let X be an extremally disconnected space and let g and f be real-valued functions defined on the X, such that f is $lscc, g$ is $uscc$, and $g \leq f$. If a binary relation ρ is defined by $A\ \rho\ B$ in case $A^{\Lambda} \subseteq B^{V}$, then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is open and since $\{x \in X : g(x) < t_2\}$ is closed, it follows that $A(f, t_1)^{\Lambda} \subseteq A(g, t_2)^{V}$. Hence $t_1 < t_2$ implies that $A(f, t_1)\ \rho\ A(g, t_2)$. The proof of the first part follows from Theorem 2.1.

On the other hand, let G_1 and G_2 be disjoint open sets. Set $f = \chi_{G_1^c}$ and $g = \chi_{G_2}$, then f is $lscc, g$ is $uscc$, and $g \leq f$. Thus there exists a contra-continuous function h such that $g \leq h \leq f$. Set $F_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $F_2 = \{x \in X : h(x) > \frac{1}{2}\}$, then F_1 and F_2 are disjoint closed sets such that $G_1 \subseteq F_1$ and $G_2 \subseteq F_2$ i.e.,X is an extremally disconnected space. ■

Before stating the consequences of Theorem 2.2, we state and prove some necessary lemmas.

Lemma 3.1. *The following conditions on a space X are equivalent:*
(i) *X is an extremally disconnected space.*

(ii) *If G is an open subset of X which is contained in a closed subset F, then there exists a closed subset H such that $G \subseteq H \subseteq H^{\Lambda} \subseteq F$.*

Proof.
(i) $\Rightarrow$ (ii) Suppose that $G \subseteq F$, where G and F are open subset and closed subset of X, respectively. Hence, F^c is an open set and $G \cap F^c = \varnothing$.

By (i) there exist two disjoint closed sets F_1, F_2 such that, $G \subseteq F_1$ and $F^c \subseteq F_2$. But

$$F^c \subseteq F_2 \Rightarrow F_2^c \subseteq F,$$

and

$$F_1 \cap F_2 = \varnothing \Rightarrow F_1 \subseteq F_2^c,$$

hence

$$G \subseteq F_1 \subseteq F_2^c \subseteq F,$$

and since F_2^c is an open set containing F_1 we conclude that $F_1^{\Lambda} \subseteq F_2^c$, i.e.,

$$G \subseteq F_1 \subseteq F_1^{\Lambda} \subseteq F.$$

By setting $H = F_1$, condition (ii) holds.
(ii) $\Rightarrow$ (i) Suppose that G_1, G_2 are two disjoint open sets of X.

This implies that $G_1 \subseteq G_2^c$ and G_2^c is a closed set. Hence by (ii) there exists a closed set H such that, $G_1 \subseteq H \subseteq H^{\Lambda} \subseteq G_2^c$.

But

$$H \subseteq H^{\Lambda} \Rightarrow H \cap (H^{\Lambda})^c = \varnothing,$$

and

$$H^{\Lambda} \subseteq G_2^c \Rightarrow G_2 \subseteq (H^{\Lambda})^c.$$

Furthermore, $(H^{\Lambda})^c$ is a closed subset of X. Hence $G_1 \subseteq H, G_2 \subseteq (H^{\Lambda})^c$ and $H \cap (H^{\Lambda})^c = \varnothing$. This means that condition (i) holds. ■

Lemma 3.2. *Suppose that X is an extremally disconnected space. If G_1 and G_2 are two disjoint open subsets of X, then there exists a contra-continuous function $h : X \to [0,1]$ such that $h(G_1) = \{0\}$ and $h(G_2) = \{1\}$.*

Proof. Suppose G_1 and G_2 are two disjoint open subsets of X. Since $G_1 \cap G_2 = \varnothing$, hence $G_1 \subseteq G_2^c$. In particular, since G_2^c is a closed subset of X containing G_1, by Lemma 3.1, there exists a closed set $H_{1/2}$ such that,

$$G_1 \subseteq H_{1/2} \subseteq H_{1/2}^{\Lambda} \subseteq G_2^c.$$

Note that $H_{1/2}$ is a closed set and contains G_1, and G_2^c is a closed set and contains $H_{1/2}^{\Lambda}$. Hence, by Lemma 3.1, there exists closed sets $H_{1/4}$ and $H_{3/4}$ such that,

$$G_1 \subseteq H_{1/4} \subseteq H_{1/4}^{\Lambda} \subseteq H_{1/2} \subseteq H_{1/2}^{\Lambda} \subseteq H_{3/4} \subseteq H_{3/4}^{\Lambda} \subseteq G_2^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0,1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain closed sets H_t

with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function h on X by setting $h(x) = \inf\{t : x \in H_t\}$ for $x \notin G_2$ and $h(x) = 1$ for $x \in G_2$.

Note that for every $x \in X, 0 \leq h(x) \leq 1$, i.e., h maps X into [0,1]. Also, we note that for any $t \in D, G_1 \subseteq H_t$; hence $h(G_1) = \{0\}$. Furthermore, by definition, $h(G_2) = \{1\}$. It remains only to prove that h is a contra-continuous function on X. For every $\alpha \in \mathbb{R}$, we have if $\alpha \leq 0$ then $\{x \in X : h(x) < \alpha\} = \varnothing$ and if $0 < \alpha$ then $\{x \in X : h(x) < \alpha\} = \cup\{H_t : t < \alpha\}$, hence, they are closed subsets of X. Similarly, if $\alpha < 0$ then $\{x \in X : h(x) > \alpha\} = X$ and if $0 \leq \alpha$ then $\{x \in X : h(x) > \alpha\} = \cup\{(H_t^{\Lambda})^c : t > \alpha\}$ hence, every of them is a closed set. Consequently h is a contra-continuous function. ■

Lemma 3.3. *Suppose that X is an extremally disconnected space. If G_1 and G_2 are two disjoint open subsets of X and G_1 is a countable intersection of closed sets, then there exists a contra-continuous function $h : X \to [0, 1]$ such that $h^{-1}(0) = G_1$ and $h(G_2) = \{1\}$.*

Proof. Suppose that $G_1 = \bigcap_{n=1}^{\infty} F_n$, where F_n is a closed subset of X. We can suppose that $F_n \cap G_2 = \varnothing$, otherwise we can substitute F_n by $F_n \setminus G_2$. By Lemma 3.2, for every $n \in \mathbb{N}$, there exists a contra-continuous function $h_n : X \to [0, 1]$ such that $h_n(G_1) = \{0\}$ and $h_n(X \setminus F_n) = \{1\}$. We set $h(x) = \sum_{n=1}^{\infty} 2^{-n} h_n(x)$.

Since the above series is uniformly convergent, it follows that h is a contra-continuous function from X into $[0, 1]$. Since for every $n \in \mathbb{N}, G_2 \subseteq X \setminus F_n$, therefore $h_n(G_2) = \{1\}$ and consequently $h(G_2) = \{1\}$. Since $h_n(G_1) = \{0\}$, hence $h(G_1) = \{0\}$. It suffices to show that if $x \notin G_1$, then $h(x) \neq 0$.

Now if $x \notin G_1$, since $G_1 = \bigcap_{n=1}^{\infty} F_n$, therefore there exists $n_0 \in \mathbb{N}$ such that $x \notin F_{n_0}$, hence $h_{n_0}(x) = 1$, i.e., $h(x) > 0$. Therefore $h^{-1}(0) = G_1$. ■

Lemma 3.4. *Suppose that X is an extremally disconnected space. The following conditions are equivalent:*

(i) *For every two disjoint open sets G_1 and G_2, there exists a contra-continuous function $h : X \to [0, 1]$ such that $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$.*

(ii) *Every open set is a countable intersection of closed sets.*

(iii) *Every closed set is a countable union of open sets.*

Proof.
(i) ⇒ (ii). Suppose that G is an open set. Since $\varnothing$ is an open set, by (i) there exists a contra-continuous function $h : X \to [0, 1]$ such that $h^{-1}(0) = G$. Set $F_n = \{x \in X : h(x) < \frac{1}{n}\}$. Then for every $n \in \mathbb{N}$, F_n is a closed set and $\bigcap_{n=1}^{\infty} F_n = \{x \in X : h(x) = 0\} = G$.
(ii) ⇒ (i). Suppose that G_1 and G_2 are two disjoint open sets. By Lemma 3.3, there exists a contra-continuous function $f : X \to [0, 1]$ such that $f^{-1}(0) = G_1$ and $f(G_2) = \{1\}$. Set $F = \{x \in X : f(x) < \frac{1}{2}\}$, $G = \{x \in X : f(x) = \frac{1}{2}\}$, and $H = \{x \in X : f(x) > \frac{1}{2}\}$. Then $F \cup G$ and $H \cup G$ are two open sets and $(F \cup G) \cap G_2 = \varnothing$. By Lemma 3.3, there exists a contra-continuous function $g : X \to [\frac{1}{2}, 1]$ such that $g^{-1}(1) = G_2$ and $g(F \cup G) = \{\frac{1}{2}\}$. Define h by setting $h(x) = f(x)$ for $x \in F \cup G$, and $h(x) = g(x)$ for $x \in H \cup G$ .Then h is well-

defined and is a contra-continuous function, since $(F \cup G) \cap (H \cup G) = G$ and for every $x \in G$ we have $f(x) = g(x) = \frac{1}{2}$. Furthermore, $(F \cup G) \cup (H \cup G) = X$, hence h defined on X and maps X into $[0, 1]$. Also, we have $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$.
(ii) ⇔ (iii) By De Morgan laws and noting that the complement of every open set is a closed set and the complement of every closed set is an open set, the equivalence holds. ■

Corollary 3.2. *For every two disjoint open sets G_1 and G_2, there exists a contra-continuous function $h : X \to [0, 1]$ such that $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$ if and only if X has the strong $cc-$insertion property for $(uscc, lscc)$.*

Proof. Since for every two disjoint open sets G_1 and G_2, there exists a contra-continuous function $h : X \to [0, 1]$ such that $h^{-1}(0) = G_1$ and $h^{-1}(1) = G_2$, define $F_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $F_2 = \{x \in X : h(x) > \frac{1}{2}\}$. Then F_1 and F_2 are two disjoint closed sets that contain G_1 and G_2, respectively. This means that,X is an extremally disconnected space. Hence by Corollary 3.1, X has the weak $cc-$insertion property for $(uscc, lscc)$. Now, assume that g and f are functions on X such that $g \leq f, g$ is $uscc$ and f is $lscc$. Since $f - g$ is $lscc$, therefore the lower cut set $A(f - g, 2^{-n}) = \{x \in X : (f - g)(x) \leq 2^{-n}\}$ is an open set. By Lemma 3.4, we can choose a sequence $\{G_n\}$ of open sets such that $\{x \in X : (f - g)(x) > 0\} = \bigcup_{n=1}^{\infty} G_n$ and for every $n \in \mathbb{N}, G_n$ and $A(f - g, 2^{-n})$ are disjoint. By Lemma 3.2, G_n and $A(f - g, 2^{-n})$ can be completely separated by contra-continuous functions. Hence by Theorem 2.2, X has the strong $cc-$insertion property for $(uscc, lscc)$.

On the other hand, suppose that G_1 and G_2 are two disjoint open sets. Since $G_1 \cap G_2 = \varnothing$, hence $G_2 \subseteq G_1^c$. Set $g = \chi_{G_2}$ and $f = \chi_{G_1^c}$. Then f is $lscc$ and g is $uscc$ and furthermore $g \leq f$. By hypothesis, there exists a contra-continuous function h on X such that $g \leq h \leq f$ and whenever $g(x) < f(x)$ we have $g(x) < h(x) < f(x)$. By definitions of f and g, we have $h^{-1}(1) = G_2 \cap G_1^c = G_2$ and $h^{-1}(0) = G_1 \cap G_2^c = G_1$. ■

Corollary 3.3. *X is a normal space if and only if X has the weak $cc-$insertion property for $(lscc, uscc)$.*

Proof. Let X be a normal space and let g and f be real-valued functions defined on the X, such that f is $lscc, g$ is $uscc$, and $f \leq g$. If a binary relation ρ is defined by $A \,\rho\, B$ in case $A^{\Lambda} \subseteq F \subseteq F^{\Lambda} \subseteq B^{V}$ for some closed set F in X, then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(g, t_1) = \{x \in X : g(x) < t_1\} \subseteq \{x \in X : f(x) \leq t_2\} = A(f, t_2);$$

since $\{x \in X : g(x) < t_1\}$ is a closed set and since $\{x \in X : f(x) \leq t_2\}$ is an open set, by hypothesis it follows that $A(g, t_1) \,\rho\, A(f, t_2)$. The proof of the first part follows from Theorem 2.1.

On the other hand, let F_1 and F_2 be disjoint closed sets. Set $f = \chi_{F_2}$ and $g = \chi_{F_1^c}$, then f is $lscc, g$ is $uscc$, and $f \leq g$.

Thus there exists a contra-continuous function h such that $f \leq h \leq g$. Set $G_1 = \{x \in X : h(x) \leq \frac{1}{3}\}$ and $G_2 = \{x \in X : h(x) \geq 2/3\}$ then G_1 and G_2 are disjoint open sets such that $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$. Hence X is a normal space. ∎

Corollary 3.4. *Every closed set is an open set if and only if X has the strong $cc-$insertion property for $(lscc, uscc)$.*

Proof. Suppose that every closed set in X is open, then X is a normal space. Hence by Corollary 3.3, X has the weak $cc-$insertion property for $(lscc, uscc)$. Now, assume that g and f are functions on X such that $g \leq f, g$ is $lscc$ and f is cc. Set $A(f-g, 2^{-n}) = \{x \in X : (f-g)(x) < 2^{-n}\}$. Then, since $f-g$ is $uscc$, we can say that $A(f-g, 2^{-n})$ is a closed set. By hypothesis, $A(f-g, 2^{-n})$ is an open set. Set $F_n = X \setminus A(f-g, 2^{-n})$. Then F_n is a closed set. This means that F_n and $A(f-g, 2^{-n})$ are disjoint closed sets and also are two disjoint open sets. Therefore F_n and $A(f-g, 2^{-n})$ can be completely separated by contra-continuous functions. Now, we have $\bigcup_{n=1}^{\infty} F_n = \{x \in X : (f-g)(x) > 0\}$. By Theorem 2.2, X has the strong $cc-$insertion property for $(lscc, cc)$. By an analogous argument, we can prove that X has the strong $cc-$insertion property for $(cc, uscc)$. Hence, by Theorem 2.3, X has the strong $cc-$insertion property for $(lscc, uscc)$.

On the other hand, suppose that X has the strong $cc-$insertion property for $(lscc, uscc)$. Also, suppose that F is a closed set. Set $f = 1$ and $g = \chi_F$. Then f is $uscc, g$ is $lscc$ and $g \leq f$. By hypothesis, there exists a contra-continuous function h on X such that $g \leq h \leq f$ and whenever $g(x) < f(x)$, we have $g(x) < h(x) < f(x)$. It is clear that $h(F) = \{1\}$ and for $x \in X \setminus F$ we have $0 < h(x) < 1$. Since h is a contra-continuous function, therefore $\{x \in X : h(x) \geq 1\} = F$ is an open set, i.e., F is an open set. ∎

Remark 1. [5, 6]. A space X has the weak $c-$insertion property for (usc, lsc) if and only if X is normal.

Remark 2. [10] . A space X has the strong $c-$insertion property for (usc, lsc) if and only if X is perfectly normal.

Remark 3. [12]. A space X has the weak $c-$insertion property for (lsc, usc) if and only if X is extremally disconnected.

Remark 4. [1]. A space X has the strong $c-$insertion property for (lsc, usc) if and only if each open subset of X is closed.

References

1. J. Blatter and G. L. Seever, Interposition of semicontinuous functions by continuous functions, In: *Analyse Fonctionelle et Applications (Comptes Rendus du colloque d' Analyse, Rio de Janeiro 1972)*, Hermann, Paris, 1975, 27-51.

2. F. Brooks, Indefinite cut sets for real functions, *Amer. Math. Monthly* **78** (1971) 1007–1010.
3. J. Dontchev, Contra-continuous functions and strongly S-closed space, *Internat. J. Math. Sci.* **19** (1996) 303–310.
4. J. Dontchev and H. Maki, On sg-closed sets and semi$-\lambda-$closed sets, *Questions Answers Gen. Topology* **15** (1997) 259–266.
5. M. Katětov, On real-valued functions in topological spaces, *Fund. Math.* **38** (1951) 85–91.
6. M. Katětov, Correction to "On real-valued functions in topological spaces", *Fund. Math.* **40** (1953) 203–205.
7. E. Lane, Insertion of a continuous function, *Pacific J. Math.* **66** (1976) 181–190.
8. S. N. Maheshwari and R. Prasad, On $R_{Os}-$spaces, *Portugal. Math.* **34** (1975) 213–217.
9. H. Maki, *Generalized* $\Lambda-$*sets and the associated closure operator*, The special Issue in commemoration of Prof. Kazuada IKEDA's Retirement, 1986, 139–146.
10. E. Michael, Continuous selections I, *Ann. Math.* **63** (1956) 361–382.
11. M. Mirmiran, Insertion of a function belonging to a certain subclass of $\mathbb{R}^X$, *Bull. Iran. Math. Soc.* **28** (2002) 19–27.
12. M. H. Stone, Boundedness properties in function-lattices, *Canad. J. Math.* **1** (1949) 176–189.