Vietnam Journal of Mathematics **34**:1 (2006) 51–61

Vietnam Journal
of
MATHEMATICS

Weighted Estimates of Multilinear Singular Integral Operators with Variable Calderón-Zygmund Kernel for the Extreme Cases*

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Received February 28, 2005

Abstract. The weighted endpoint estimates for the multilinear singular integral operators with variable Calderón-Zygmund kernel on some Hardy and Herz type Hardy spaces are obtained.

1. Introduction

Let $b \in BMO(R^n)$ and T be the Calderón-Zygmund operator. The commutator $[b, T]$ generated by b and T is defined by $[b, T]f(x) = b(x)Tf(x) - T(bf)(x)$. By a classical result of Coifman, Rochberg and Weiss(see [9]), we know that the commutator $[b, T]$ is bounded on $L^p(R^n)$ for $1 < p < \infty$. In [13], the boundedness properties of the commutators for the extreme values of p are proved, and in [3], the weak (H^1, L^1)-boundedness of the multilinear operator related to some singular integral operator are obtained. In [2], Calderón and Zygmund introduce some singular integral operators with variable kernel and discuss their boundedness. In [10], the authors obtain the boundedness for the commutators generated by the singular integral operators with variable kernel and BMO functions. In [16], the authors prove the boundedness for the multilinear oscillatory singular integral operators generated by the operators and BMO functions. In recent years, the theory of Herz space and Herz type Hardy space, as a local version of Lebesgue space and Hardy space, have been developed (see[11,

*Supported by the NNSF (Grant: 10271071).

14, 15]). The main purpose of this paper is to establish the weighted endpoint continuity properties of the multilinear singular integral operators with variable Calderón-Zygmund kernel on Hardy and Herz type Hardy spaces.

2. Notations and Theorems

Throughout this paper, we denote the Muckenhoupt weights by A_p for $1 \leqslant p < \infty$ (see [12]). Q will denote a cube of R^n with sides parallel to the axes. For a cube Q and a locally integrable function f, let $f(Q) = \int_Q f(x)dx$, $f_Q = |Q|^{-1}\int_Q f(x)dx$ and $f^{\#}(x) = \sup_{x\in Q}|Q|^{-1}\int_Q |f(y) - f_Q|dy$. Moreover, f is said to belong to $BMO(R^n)$ if $f^{\#} \in L^\infty(R^n)$ and define that $||f||_{BMO} = ||f^{\#}||_{L^\infty}$. Also, we give the concepts of the atom and weighted H^1 space. A function a is called a $H^1(w)$ atom if there exists a cube Q such that a is supported on Q, $||a||_{L^\infty(w)} \leqslant w(Q)^{-1}$ and $\int_{R^n} a(x)dx = 0$. It is well known that the weighted Hardy space $H^1(w)$ has the atomic decomposition characterization (see [1, 12]).

For $k \in Z$, define $B_k = \{x \in R^n : |x| \leqslant 2^k\}$ and $C_k = B_k \setminus B_{k-1}$. Denote by χ_k the characteristic function of C_k and $\tilde{\chi}_k$ the characteristic function of C_k for $k \geq 1$ and $\tilde{\chi}_0$ the characteristic function of B_0.

Definition 1. *Let $1 < p < \infty$ and w_1, w_2 be two non-negative weight functions on R^n.*

(1) *The homogeneous weighted Herz space is defined by*

$$\dot{K}_p(w_1, w_2; R^n) = \{f \in L^p_{loc}(R^n \setminus \{0\}) : \|f\|_{\dot{K}_p(w_1,w_2)} < \infty\},$$

where

$$\|f\|_{\dot{K}_p(w_1,w_2)} = \sum_{k=-\infty}^{\infty}[w_1(B_k)]^{1-1/p}\|f\chi_k\|_{L^p(w_2)};$$

(2) *The nonhomogeneous weighted Herz space is defined by*

$$K_p(w_1, w_2; R^n) = \{f \in L^p_{loc}(R^n) : \|f\|_{K_p(w_1,w_2)} < \infty\},$$

where

$$\|f\|_{K_p} = \sum_{k=0}^{\infty}[w_1(B_k)]^{1-1/p}\|f\tilde{\chi}_k\|_{L^p(w_2)};$$

(3) *The homogeneous weighted Herz type Hardy space is defined by*

$$H\dot{K}_p(w_1, w_2; R^n) = \{f \in S'(R^n) : G(f) \in \dot{K}_p(w_1, w_2; R^n)\},$$

where

$$||f||_{H\dot{K}_p(w_1,w_2)} = ||G(f)||_{\dot{K}_p(w_1,w_2)};$$

(4) *The nonhomogeneous weighted Herz type Hardy space is defined by*

$$HK_p(w_1, w_2; R^n) = \{f \in S'(R^n) : G(f) \in K_p(w_1, w_2; R^n)\},$$

where

$$\|f\|_{HK_p(w_1,w_2)} = \|G(f)\|_{K_p(w_1,w_2)}$$

and $G(f)$ is the grand maximal function of f.

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 2. *Let $1 < p < \infty$ and $w_1, w_2 \in A_1$. A function $a(x)$ on R^n is called a central $(n(1-1/p), p; w_1, w_2)$-atom (or a central $(n(1-1/p), p; w_1, w_2)$-atom of restrict type), if*

(1) Supp $a \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$);
(2) $||a||_{L^p(w_2)} \leqslant [w_1(B(0, r))]^{1/p-1}$,
(3) $\int_{R^n} a(x)dx = 0$.

Lemma 1. (see [11, 15]) *Let $w_1, w_2 \in A_1$ and $1 < p < \infty$. A temperate distribution f belongs to $H\dot{K}_p(w_1, w_2; R^n)$(or $HK_p(w_1, w_2; R^n)$) if and only if there exist central $(n(1-1/p), p; w_1, w_2)$-atoms (or central $(n(1-1/p), p; w_1, w_2)$-atoms of restrict type) a_j supported on $B_j = B(0, 2^j)$ and constants λ_j, $\sum_j |\lambda_j| < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$(or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(R^n)$ sense, and*

$$\|f\|_{H\dot{K}_p(w_1,w_2)}(\text{ or } \|f\|_{HK_p(w_1,w_2)}) \approx \sum_j |\lambda_j|.$$

In this paper, we will study a class of multilinear operators related to the singular integral operators with variable kernel, whose definitions are following.

Definition 3. *Let $k(x) = \Omega(x)/|x|^n : R^n \setminus \{0\} \longrightarrow R$. k is said to be a Calderón-Zygmund kernel if*

(a) $\Omega \in C^\infty(R^n \setminus \{0\})$;
(b) *Ω is homogeneous of degree zero;*
(c) $\int_\Sigma \Omega(x)x^\alpha d\sigma(x) = 0$ *for all multi-indices $\alpha \in (N \bigcup \{0\})^n$ with $|\alpha| = N$, where $\Sigma = \{x \in R^n : |x| = 1\}$ is the unit sphere of R^n.*

Definition 4. *Let $k(x, y) = \Omega(x, y)/|y|^n : R^n \times (R^n \setminus \{0\}) \longrightarrow R$. k is said to be a variable Calderón-Zygmund kernel if*

(d) *$k(x, \cdot)$ is a Calderón-Zygmund kernel for a.e. $x \in R^n$;*
(e) $\max_{|\gamma| \leqslant 2n} \left|\left|\frac{\partial^{|\gamma|}}{\partial^\gamma y}\Omega(x, y)\right|\right|_{L^\infty(R^n\times\Sigma)} = M < \infty.$

Let m be a positive integer and A be a function on R^n. Set

$$R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha| \leqslant m} \frac{1}{\alpha!} D^\alpha A(y)(x - y)^\alpha$$

and

$$Q_{m+1}(A;x,y)=R_m(A;x,y)-\sum_{|\alpha|=m}\frac{1}{\alpha!}D^\alpha A(x)(x-y)^\alpha.$$

The multilinear singular integral operators with variable Calderón-Zygmund kernel are defined by

$$\tilde{T}_A(f)(x)=\int_{R^n}\frac{\Omega(x,x-y)}{|x-y|^{n+m}}Q_{m+1}(A;x,y)f(y)dy$$

and

$$T_A(f)(x)=\int_{R^n}\frac{\Omega(x,x-y)}{|x-y|^{n+m}}R_{m+1}(A;x,y)f(y)dy,$$

where $\Omega(x,y)/|y|^n$ is a variable Calderón-Zygmund kernel. We also define

$$T(f)(x)=\int_{R^n}\frac{\Omega(x,x-y)}{|x-y|^n}f(y)dy,$$

which is the singular integral operator with variable Calderón-Zygmund kernel (*see* [2]).

Note that when $m=0$, T_A is just the commutator of T and A (see [10]). While when $m>0$, T_A is the non-trivial generalizations of the commutator. From [16], we know that T_A is bounded on $L^p(w)$ for $1<p\leqslant\infty$ and $w\in A_1$. In this paper, we will study the weighted endpoint continuity properties of the multilinear operators $\tilde{T}_A$ on Hardy and Herz type Hardy spaces.

We shall prove the following theorems in Sec. 3.

Theorem 1. *Let $w\in A_1$ and $D^\alpha A\in BMO(R^n)$ for all α with $|\alpha|=m$. Then $\tilde{T}_A$ is bounded from $H^1(w)$ to $L^1(w)$.*

Theorem 2. *Let $1<p<\infty$, $w_1,w_2\in A_1$ and $D^\alpha A\in BMO(R^n)$ for all α with $|\alpha|=m$. Then $\tilde{T}_A$ is bounded from $H\dot{K}_p(w_1,w_2;R^n)$ (resp. $HK_p(w_1,w_2;R^n)$) to $\dot{K}_p(w_1,w_2;R^n)$(resp. $HK_p(w_1,w_2;R^n)$).*

3. Proofs of Theorems

To prove the theorems, we need the following lemma.

Lemma 2. (see [7]) *Let A be a function on R^n and $D^\alpha A\in L^q(R^n)$ for $|\alpha|=m$ and some $q>n$. Then*

$$|R_m(A;x,y)|\leqslant C|x-y|^m\sum_{|\alpha|=m}\Big(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)}|D^\alpha A(z)|^q dz\Big)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x-y|$.

Proof of Theorem 1. It suffices to show that there exists a constant $C > 0$ such that for every $H^1(w)$-atom a (that is that a satisfies: supp$a \subset Q = Q(x_0, r)$, $||a||_{L^\infty(w)} \leqslant w(Q)^{-1}$ and $\int a(y)dy = 0$ (see [1])), the following holds:

$$||\tilde{T}_A(a)||_{L^1(w)} \leqslant C.$$

Without loss of generality, we may assume $l = 2$. Write

$$\int_{R^n} \tilde{T}_A(a)(x)w(x)dx = \Big[\int_{2Q} + \int_{(2Q)^c}\Big]\tilde{T}_A(a)(x)w(x)dx := I_1 + I_2.$$

For I_1, by the following equality

$$Q_{m+1}(A; x, y) = R_{m+1}(A; x, y) + \sum_{|\alpha|=m} \frac{1}{\alpha!}(x-y)^\alpha(D^\alpha A(x) - D^\alpha A(y)),$$

we get

$$|\tilde{T}_A(a)(x)| \leqslant |T_A(a)(x)| + C\sum_{|\alpha|=m} |[D^\alpha A, T]a(x)|,$$

thus, $\tilde{T}_A$ is $L^p(w)$-bounded for $1 < p \leqslant \infty$ (see[10, 16]), we see that

$$I_1 \leqslant C||\tilde{T}_A(a)||_{L^\infty(w)}w(2Q) \leqslant C||a||_{L^\infty(w)}w(Q) \leqslant C.$$

To obtain the estimate of I_2, we need to estimate $\tilde{T}_A(a)(x)$ for $x \in (2Q)^c$. Denote $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{2Q}x^\alpha$, then $D^\alpha\tilde{A} = D^\alpha A - (D^\alpha A)_{2Q}$ for $|\alpha| = m$, $Q_m(A; x, y) = Q_m(\tilde{A}; x, y)$ and $Q_{m+1}(A; x, y) = R_m(A; x, y) - \sum_{|\alpha|=m} \frac{1}{\alpha!}D^\alpha A(x)(x-y)^\alpha$. By [4, 10], we know that

$$\begin{aligned}\tilde{T}_A(f)(x) &= \sum_{k=1}^{\infty}\sum_{h=1}^{g_k} a_{hk}(x)\int_{R^n} \frac{Y_{hk}(x-y)}{|x-y|^{n+m}}Q_{m+1}(A; x, y)f(y)dy \\ &:= \sum_{k=1}^{\infty}\sum_{h=1}^{g_k} a_{hk}(x)S_{hk}^A(f)(x),\end{aligned}$$

where $g_k \leqslant Ck^{n-2}$, $||a_{hk}||_{L^\infty} \leqslant Ck^{-2n}$, $|Y_{hk}(x-y)| \leqslant Ck^{n/2-1}$ and

$$\Big|\frac{Y_{hk}(x-y)}{|x-y|^n} - \frac{Y_{hk}(x-x_0)}{|x-x_0|^n}\Big| \leqslant Ck^{n/2}|x_0-y|/|x-x_0|^{n+1}$$

for $|x-x_0| > 2|x_0-y| > 0$. we write, by the vanishing moment of a and for $x \in (2Q)^c$,

$$S_{hk}^{A}(a)(x)=\int_{R^n}\Big[\frac{Y_{hk}(x-y)}{|x-y|^{m+n}}-\frac{Y_{hk}(x-x_0)}{|x-x_0|^{m+n}}\Big]R_m(\tilde{A};x,y)a(y)dy$$
$$+\int_{R^n}\frac{Y_{hk}(x-x_0)}{|x-x_0|^{m+n}}[R_m(\tilde{A};x,y)-R_m(\tilde{A};x,x_0)]a(y)dy$$
$$-C\sum_{|\alpha|=m}\int_{R^n}\Big[\frac{Y_{hk}(x-y)(x-y)^\alpha}{|x-y|^{m+n}}-\frac{Y_{hk}(x-x_0)(x-x_0)^\alpha}{|x-x_0|^{m+n}}\Big]D^\alpha\tilde{A}(x)a(y)dy$$
$$=I_2^{(1)}(x)+I_2^{(2)}(x)+I_2^{(3)}(x);$$

For $I_2^{(1)}(x)$, by Lemma 1 and the following inequality (see [17])

$$|b_{Q_1}-b_{Q_2}|\leqslant C\log(|Q_2|/|Q_1|)||b||_{BMO} \text{ for } Q_1\subset Q_2,$$

we know that, for $x\in Q$ and $y\in 2^{j+1}Q\setminus 2^jQ(j\geq 1)$,

$$|R_m(\tilde{A};x,y)|\leqslant C|x-y|^m\sum_{|\alpha|=m}(||D^\alpha A||_{BMO}+|(D^\alpha A)_{2Q(x,y)}-(D^\alpha A)_{2Q}|)$$
$$\leqslant Cj|x-y|^m\sum_{|\alpha|=m}||D^\alpha A||_{BMO},$$

note that $|x-y|\sim|x-x_0|$ for $y\in Q$ and $x\in R^n\setminus 2Q$, then

$$|I_2^{(1)}(x)|\leqslant Ck^{n/2}\int_{R^n}\frac{|y-x_0|}{|x-x_0|^{m+n+1}}|R_m(\tilde{A};x,y)||a(y)|dy$$
$$\leqslant Ck^{n/2}\sum_{|\alpha|=m}||D^\alpha A||_{BMO}j\int_Q\frac{|y-x_0|}{|x-x_0|^{n+1}}|a(y)|dy$$
$$\leqslant Ck^{n/2}\sum_{|\alpha|=m}||D^\alpha A||_{BMO}j\frac{|Q|^{1/n+1}}{|x-x_0|^{n+1}}w(Q)^{-1}.$$

For $I_2^{(2)}(x)$, by the formula (see [7]):

$$R_m(\tilde{A};x,y)-R_m(\tilde{A};x,x_0)=\sum_{|\beta|<m}\frac{1}{\beta!}R_{m-|\beta|}(D^\beta\tilde{A};x,x_0)(x-y)^\beta$$

and Lemma 1, we have

$$|R_m(\tilde{A};x,y)-R_m(\tilde{A};x,x_0)|\leqslant C\sum_{|\beta|<m}\sum_{|\alpha|=m}|x-x_0|^{m-|\beta|}|x-y|^{|\beta|}\|D^\alpha A\|_{BMO},$$

then

$$|I_2^{(2)}(x)| \leqslant Ck^{n/2} \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} j \int_Q \frac{|y-x_0|}{|x-x_0|^{n+1}} |a(y)| dy$$
$$\leqslant Ck^{n/2} \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} j \frac{|Q|^{1/n+1}}{|x-x_0|^{n+1}} w(Q)^{-1}.$$

Similarly,

$$|I_2^{(3)}(x)| \leqslant Ck^{n/2} \sum_{|\alpha|=m} \frac{|Q|^{1/n+1}}{|x-x_0|^{n+1}} w(Q)^{-1} |D^\alpha \tilde{A}(x)|;$$

Thus, for $x \in 2^{j+1}Q \setminus 2^j Q (j \geq 1)$,

$$|\tilde{T}_A(a)(x)| \leqslant C \sum_{k=1}^{\infty} \sum_{h=1}^{g_k} |a_{hk}(x)| k^{n/2} \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} j \frac{|Q|^{1/n+1}}{|x-x_0|^{n+1}} w(Q)^{-1}$$
$$+ C \sum_{k=1}^{\infty} \sum_{h=1}^{g_k} |a_{hk}(x)| k^{n/2} \sum_{|\alpha|=m} \frac{|Q|^{1/n+1}}{|x-x_0|^{n+1}} w(Q)^{-1} |D^\alpha \tilde{A}(x)|$$
$$\leqslant C \sum_{k=1}^{\infty} k^{-2n+n/2+n-2} \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} j \frac{|Q|^{1/n+1}}{|x-x_0|^{n+1}} w(Q)^{-1}$$
$$+ C \sum_{k=1}^{\infty} k^{-2n+n/2+n-2} \sum_{|\alpha|=m} \frac{|Q|^{1/n+1}}{|x-x_0|^{n+1}} w(Q)^{-1} |D^\alpha \tilde{A}(x)|$$
$$\leqslant C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} j \frac{|Q|^{1/n+1}}{|x-x_0|^{n+1}} w(Q)^{-1}$$
$$+ C \sum_{|\alpha|=m} \frac{|Q|^{1/n+1}}{|x-x_0|^{n+1}} w(Q)^{-1} |D^\alpha \tilde{A}(x)|;$$

Notice that if $w \in A_1$, then

$$\frac{w(Q_2)}{|Q_2|} \frac{|Q_1|}{w(Q_1)} \leqslant C$$

for all cubes Q_1, Q_2 with $Q_1 \subset Q_2$, and w satisfies the reverse of Hölder' inequality:

$$\Big(\frac{1}{|Q|} \int_Q w(x)^q dx\Big)^{1/q} \leqslant \frac{C}{|Q|} \int_Q w(x) dx$$

for all cube Q and some $1 < q < \infty$(see[12]). Thus, by Hölder's inequality, we obtain

$$
\begin{aligned}
I_2 &\leqslant \sum_{j=1}^{\infty} \int_{2^{j+1}Q\setminus 2^jQ} |\tilde{T}_A(a)(x)|dx \\
&\leqslant C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \sum_{j=1}^{\infty} j2^{-j} \frac{|Q|}{w(Q)} \frac{w(2^{j+1}Q)}{|2^{j+1}Q|} \\
&+ C \sum_{|\alpha|=m} \sum_{j=1}^{\infty} 2^{-j} \frac{|Q|}{w(Q)} \Big(\frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |D^\alpha \tilde{A}(x)|^{q'} dx \Big)^{1/q'} \\
&\times \Big(\frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} w(x)^q dx \Big)^{1/q} \\
&\leqslant C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO} \sum_{j=1}^{\infty} j2^{-j} \frac{w(2^{j+1}Q)}{|2^{j+1}Q|} \frac{|Q|}{w(Q)} \\
&\leqslant C \sum_{|\alpha|=m} \|D^\alpha A\|_{BMO}.
\end{aligned}
$$

This completes the proof of Theorem 1. ■

Proof of Theorem 2. We only give the proof for case of homogeneous Herz type Hardy space. Without loss of generality, we may assume $l = 2$. Let $f \in H\dot{K}_p(w_1, w_2; R^n)$, by Lemma 1, $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, where $a_j's$ are the central $(n(1-1/p), p; w_1, w_2)$-atom with supp$a_j \subset B_j = B(0, 2^j)$ and $||f||_{H\dot{K}_p(w_1,w_2)} \approx \sum_j |\lambda_j|$. Write

$$
\begin{aligned}
||\tilde{T}_A(f)||_{\dot{K}_p(w_1,w_2)} &= \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1-1/p} ||\chi_k \tilde{T}_A(f)||_{L^p(w_2)} \\
&\leqslant \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1-1/p} \sum_{j=-\infty}^{k-1} |\lambda_j| ||\chi_k \tilde{T}_A(a_j)||_{L^p(w_2)} \\
&+ \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1-1/p} \sum_{j=k}^{\infty} |\lambda_j| ||\chi_k \tilde{T}_A(a_j)||_{L^p(w_2)} \\
&= J_1 + J_2.
\end{aligned}
$$

For J_2, by the $L^p(w)$-boundedness of $\tilde{T}_A$ for $1 < p < \infty$ and $w \in A_1$, we get

$$
\begin{aligned}
J_2 &\leqslant C \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1-1/p} \sum_{j=k}^{\infty} |\lambda_j| ||a_j||_{L^p(w_2)} \\
&\leqslant C \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1-1/p} \sum_{j=k}^{\infty} |\lambda_j| [w_1(B_j)]^{-(1-1/p)} \\
&\leqslant C \sum_{j=-\infty}^{\infty} |\lambda_j| \sum_{k=-\infty}^{j} \Big[\frac{w_1(B_k)}{w_1(B_j)} \Big]^{1-1/p} \leqslant C \sum_{j=-\infty}^{\infty} |\lambda_j| \leqslant C||f||_{H\dot{K}_p(w_1,w_2)}.
\end{aligned}
$$

To estimate J_1, we denote that $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!}(D^\alpha A)_{2Q} x^\alpha$. Then $Q_m(A; x, y) = Q_m(\tilde{A}; x, y)$. Similar to the proof of Theorem 1, we know

$$\begin{aligned}\tilde{T}_A(a_j)(x) &= \sum_{k=1}^{\infty}\sum_{h=1}^{g_k} a_{hk}(x) \int_{R^n} \frac{Y_{hk}(x-y)}{|x-y|^{n+m}} Q_{m+1}(A; x, y) a_j(y) dy \\ &= \sum_{k=1}^{\infty}\sum_{h=1}^{g_k} a_{hk}(x) S_{hk}^A(a_j)(x),\end{aligned}$$

and write, by the vanishing moment of a_j, for $x \in (2Q)^c$,

$$\begin{aligned}S_{hk}^A(a_j)(x) = &\int_{R^n} \Big[\frac{Y_{hk}(x-y)}{|x-y|^{m+n}} - \frac{Y_{hk}(x)}{|x|^{m+n}}\Big] R_m(\tilde{A}; x, y) a_j(y) dy \\ &+ \int_{R^n} \frac{Y_{hk}(x)}{|x|^{m+n}} [R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x, 0)] a(y) dy \\ &- C \sum_{|\alpha|=m} \int_{R^n} \Big[\frac{Y_{hk}(x-y)(x-y)^\alpha}{|x-y|^{m+n}} - \frac{Y_{hk}(x) x^\alpha}{|x|^{m+n}}\Big] D^\alpha \tilde{A}(x) a_j(y) dy;\end{aligned}$$

Similar to the proof of Theorem 1, we obtain

$$\begin{aligned}|\tilde{T}_A(a_j)(x)| &\leqslant C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} \frac{2^j}{2^{k(n+1)}} \int_{B_j} |a_j(y)| dy \\ &\quad + C \sum_{|\alpha|=m} |D^\alpha \tilde{A}(x)| \frac{2^j}{2^{k(n+1)}} \int_{B_j} |a_j(y)| dy \\ &\leqslant C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} \frac{2^j}{2^{k(n+1)}} ||a_j||_{L^p(w_2)} \Big(\int_{B_j} w_2(y)^{-1/(p-1)} dy\Big)^{(p-1)/p} \\ &\quad + C \sum_{|\alpha|=m} |D^\alpha \tilde{A}(x)| \frac{2^j}{2^{k(n+1)}} ||a_j||_{L^p(w_2)} \Big(\int_{B_j} w_2(y)^{-1/(p-1)} dy\Big)^{(p-1)/p} \\ &\leqslant C \sum_{|\alpha|=m} ||D^\alpha A||_{BMO} \frac{2^j}{2^{k(n+1)}} [w_1(B_j)]^{1/p-1} \Big(\int_{B_j} w_2(y)^{-1/(p-1)} dy\Big)^{(p-1)/p} \\ &\quad + C \sum_{|\alpha|=m} \frac{2^j}{2^{k(n+1)}} |D^\alpha \tilde{A}(x)| [w_1(B_j)]^{1/p-1} \Big(\int_{B_j} w_2(y)^{-1/(p-1)} dy\Big)^{(p-1)/p}.\end{aligned}$$

Notice that $w_2 \in A_1 \subseteq A_p$, w_2 satisfies

$$\sup_Q \Big(\frac{1}{|Q|} \int_Q w_2(x) dx\Big) \Big(\frac{1}{|Q|} \int_Q w_2(x)^{-1/(p-1)} dx\Big)^{(p-1)/p} \leqslant C$$

and the reverse of Hölder' inequality for some $1 < q < \infty$ (see [12]), thus

$$
\begin{aligned}
J_1 &\leqslant C \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1-1/p} \sum_{j=-\infty}^{k-1} |\lambda_j| \frac{2^j}{2^{k(n+1)}} [w_1(B_j)]^{-(1-1/p)} \\
&\times \Big(\int_{B_j} w_2(y)^{-1/(p-1)} dy \Big)^{(p-1)/p} \\
&\times \Big[[w_2(B_k)]^{1/p} + \sum_{|\alpha|=m} \Big(\int_{B_k} |D^\alpha \tilde{A}(x)|^p w_2(x) dx \Big)^{1/p} \Big] \\
&\leqslant C \sum_{k=-\infty}^{\infty} [w_1(B_k)]^{1-1/p} \sum_{j=-\infty}^{k-1} |\lambda_j| \frac{2^j}{2^{k(n+1)}} [w_1(B_j)]^{-(1-1/p)} \\
&\times \Big(\int_{B_j} w_2(y)^{-1/(p-1)} dy \Big)^{(p-1)/p} \Big[[w_2(B_k)]^{1/p} + \sum_{|\alpha|=m} \Big(\frac{1}{|B_k|} \\
&\times \int_{B_k} |D^\alpha \tilde{A}(x)|^{pq'} dx \Big)^{1/pq'} \Big(\frac{1}{|B_k|} \int_{B_k} w_2(x)^q dx \Big)^{1/pq} |B_k|^{1/p} \Big] \\
&\leqslant C \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k-1} |\lambda_j| \frac{2^j}{2^{k(n+1)}} \Big[\frac{w_1(B_k)}{w_1(B_j)} \Big]^{1-1/p} \\
&\times \Big(\int_{B_j} w_2(x)^{-1/(p-1)} dx \Big)^{(p-1)/p} [w_2(B_k)]^{1/p} \\
&\leqslant C \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k-1} |\lambda_j| \frac{2^j}{2^{k(n+1)}} \Big[\frac{w_1(B_k)}{w_1(B_j)} \Big]^{1-1/p} \\
&\times \Big[\frac{w_2(B_k)}{w_2(B_j)} \Big]^{1/p} \\
&\times |B_j| \Big(\frac{1}{|B_j|} \int_{B_j} w_2(x) dx \Big)^{1/p} \Big(\frac{1}{|B_j|} \int_{B_j} w_2(y)^{-1/(p-1)} dy \Big)^{(p-1)/p} \\
&\leqslant C \sum_{j=-\infty}^{\infty} |\lambda_j| \sum_{k=j+1}^{\infty} \frac{2^j}{2^{k(n+1)}} \Big[\frac{w_1(B_k)}{w_1(B_j)} \frac{|B_j|}{|B_k|} \Big]^{1-1/p} \\
&\times \Big[\frac{w_2(B_k)}{w_2(B_j)} \frac{|B_j|}{|B_k|} \Big]^{1/p} |B_k| \\
&\leqslant C \sum_{j=-\infty}^{\infty} |\lambda_j| \sum_{k=j+1}^{\infty} 2^{j-k} \\
&\leqslant C \sum_{j=-\infty}^{\infty} |\lambda_j| \leqslant C ||f||_{H\dot{K}_p(w_1,w_2)}.
\end{aligned}
$$

This completes the proof of Theorem 2. ∎

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