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On the Symmetric and Rees Algebras of Some Binomial Ideals

Ha Minh Lam[1] **and Morales Marcel**[1,2]

[1]*Université de Grenoble I, Institut Fourier,*
URA 188, B.P.74, 38402 Saint-Martin D'Hères Cedex, France
[2]*IUFM de Lyon, 5 rue Anselme, 69317 Lyon Cedex, France*

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Abstract. We give an explicit form of the presentation ideal of the Rees algebra and a primary decomposition of the presentation ideal of the Symmetric algebra for some binomial ideals generated by four elements, without any assumption on the finiteness and the characteristic of the ground field.

Introduction

In this paper we consider a binomial ideal I in the polynomial ring $\mathcal{K}[x_1, x_2, \dots, x_n]$, minimally generated by four binomials, such that each binomial is a difference of monomials without common factors. Codimension 2 lattice ideals generated by four elements are a particular case. We study the Rees algebra and the Symmetric algebra associated to I.

The *Rees algebra* $\mathcal{R}(I)$ of I is defined to be the graded ring $\mathcal{R}[It] = \bigoplus_{k\geq 0} I^k t^k$. By introducing four independent variables, called $\underline{T} = \{T_1, T_2, T_3, T_4\}$, and considering the ideal $\mathcal{J} = \ker \pi$, where

$$\begin{array}{ccccc} \mathcal{R}[\underline{T}] & \xrightarrow{\pi} & \mathcal{R}[It] & \longrightarrow & 0 \\ T_i & \longrightarrow & f_i t & & \end{array},$$

we have a presentation $\mathcal{R}[It] \simeq \mathcal{K}[\underline{x}, \underline{T}]/\mathcal{J}$ of the Rees algebra. The *Symmetric algebra* $\mathrm{Sym}(I)$ of I is $\mathrm{Sym}(I) = \mathcal{K}[\underline{x}, \underline{T}]/\mathcal{L}$, where $\mathcal{L}$ is the ideal generated by the first syzygies of I.

In this paper, an explicit form of the presentation ideal $\mathcal{J}$ will be given in Theorem 2.1. We obtain also a primary decomposition of the presentation ideal of the Symmetric algebra Sym(I) in Theorem 3.1. All these results are independent of the characteristic and of the cardinal of $\mathcal{K}$.

1. Preliminaries

Let f_u and f_v be two arbitrary binomials in the polynomial ring $\mathcal{K}[x_1, x_2, \dots, x_n]$, such that the greatest common divisor (g.c.d. for short) of two terms of each binomial is 1. Denote by x^p the g.c.d. of the first term of f_u and the first term of f_v, by x^t the g.c.d. of the first term of f_u and the second term of f_v, by x^r the g.c.d. of the second term of f_u and the second term of f_v, and by x^s the g.c.d. of the second term of f_u and the first term of f_v. We have

$$\begin{aligned} f_u &= \alpha_1 x^p x^t x^{\mu_+} - \beta_1 x^r x^s x^{\mu_-} \\ f_v &= \alpha_2 x^p x^s x^{\nu_+} - \beta_2 x^r x^t x^{\nu_-} \end{aligned} \tag{1}$$

where α_1, α_2, β_1, β_2 are non-zero elements in the field $\mathcal{K}$.

Remark 1.
- The monomials $x^p,\ x^t,\ x^r,\ x^s$ are pairwise coprime.
- The monomials $x^{\mu_+},\ x^{\nu_+},\ x^{\mu_-},\ x^{\nu_-}$ are pairwise coprime.
- $(x^t, x^{\mu_-}) = (x^t, x^{\nu_+}) = 1$, and $(x^s, x^{\mu_+}) = (x^s, x^{\nu_-}) = 1$, and $(x^p, x^{\mu_-}) = (x^p, x^{\nu_-}) = 1$, and $(x^r, x^{\mu_+}) = (x^r, x^{\nu_+}) = 1$.

Consider two new binomials, denoted by f_{u+v} and f_{u-v}, obtained from f_u and f_v as follows

$$\begin{aligned} f_{u+v} &= \alpha_1\alpha_2 x^{2p} x^{\mu_+} x^{\nu_+} - \beta_1\beta_2 x^{2r} x^{\mu_-} x^{\nu_-} \\ f_{u-v} &= \alpha_1\beta_2 x^{2t} x^{\mu_+} x^{\nu_-} - \alpha_2\beta_1 x^{2s} x^{\mu_-} x^{\nu_+} \end{aligned} \tag{2}$$

We denote by I the ideal $(f_u, f_v, f_{u+v}, f_{u-v})$.

Example Let $\mathcal{L}$ be a lattice in $\mathbb{Z}^n$. The lattice ideal $I_{\mathcal{L}}$ associated to $\mathcal{L}$ is defined as follows

$$I_{\mathcal{L}} := (f_v := \underline{x}^{v_+} - \underline{x}^{v_-} \mid v = v_+ - v_- \in \mathcal{L}) \subset \mathcal{R} := \mathcal{K}[x_1, \dots, x_n].$$

If $I_{\mathcal{L}}$ is of codimension 2 and is generated by four elements then it is known that $I_{\mathcal{L}}$ is generated by four binomials of the type $f_u,\ f_v,\ f_{u+v},\ f_{u-v}$ as in our case. Moreover, these four binomials are determined by the Hilbert basis $\{u, v, u+v, u-v\}$ of $\mathcal{L}$.

Proposition 1.1. *If one of four monomials $x^p,\ x^t,\ x^r,\ x^s$ is a unit, then I is of codimension 2, and is either a complete intersection or an almost complete intersection. In both cases, the Rees algebra and the Symmetric algebra are isomorphic.*

Proof. Assume that one of four monomials $x^p,\ x^t,\ x^r,\ x^s$ is a unit. Because the role of these four monomials is the same, we can assume that $x^p = 1$. In this case, we have $f_{u-v} = \beta_2 x^{\nu_-} x^t f_u - \beta_1 x^{\mu_-} x^s f_v$, and I becomes the ideal generated by all the 2×2 minors of the matrix

$$\begin{pmatrix} \beta_2 x^{\nu_-} x^r & -\alpha_2 x^{\nu_+} \\ \alpha_1 x^{\mu_+} & -\beta_1 x^{\mu_-} x^r \\ -x^s & x^t \end{pmatrix}.$$

Hence, we have the following relations

$$\beta_2 x^{\nu_-} x^r f_u + \alpha_1 x^{\mu_+} f_v - x^s f_{u+v} = 0,$$
$$\alpha_2 x^{\nu_+} f_u + \beta_1 x^{\mu_-} x^r f_v - x^t f_{u+v} = 0.$$

Let us remark that if either $x^s = 1$ or $x^t = 1$ then I is a complete intersection ideal, generated by f_u and f_v. In this case, it is known that $\mathcal{R}(I) = \mathrm{Sym}(I) = \mathcal{K}[\underline{x}, \underline{T}]/(f_u T_v - f_v T_u)$. Consider the case where both x^s and x^t are non units.

Set $L_1 = \beta_2 x^{\nu_-} x^r T_u + \alpha_1 x^{\mu_+} T_v - x^s T_{u+v}$, and $L_2 = \alpha_2 x^{\nu_+} T_u + \beta_1 x^{\mu_-} x^r T_v - x^t T_{u+v}$. We have that all these forms are in the presentation ideal $\mathcal{J}$ of the Rees ring of I. Denote by $\mathcal{A}$ the ideal (L_1, L_2). It is clear that L_1, L_2 is a regular sequence in $\mathcal{K}[\underline{x}, \underline{T}]$, and that $\mathrm{codim}(\mathcal{A}) = \mathrm{codim}(\mathcal{J}) = 2$. In particular, the ideal $\mathcal{A}$ is unmixed.

We claim that $\mathcal{A}$ is not contained in the ideal (x^t, x^s). Assume the opposite that $\mathcal{A} \subset (x^t, x^s)$. Denote by x^{t_1} the g.c.d. of x^t and x^{ν_-}, by x^{t_2} the g.c.d. of x^t and x^{μ_+}. Since x^{μ_+}, x^{ν_-} are pairwise coprime, so are x^{t_1} and x^{t_2}. We write $x^{t_3} = \dfrac{x^t}{x^{t_1} x^{t_2}}, x^{\nu'_-} = \dfrac{x^{\nu_-}}{x^{t_1}}, x^{\mu'_+} = \dfrac{x^{\mu_+}}{x^{t_2}}$. First, we will prove that $x^{t_1} = x^{t_2} = 1$. We have $L_1 \in (x^t, x^s)$. Hence,

$$\beta_2 x^{\nu'_-} x^{t_1} x^r T_u = -\alpha_1 x^{\mu'_+} x^{t_2} T_v + a x^s + b x^{t_1} x^{t_2} x^{t_3},$$

with some $a, b \in \mathcal{K}[\underline{x}, \underline{T}]$. Suppose that $x^{t_1} \neq 1$. Setting to 0 all variables appearing in x^{t_1} and all those in x^s, we have $-\alpha_1 x^{\mu'_+} x^{t_2} T_v = 0$. It is a contradiction, so $x^{t_1} = 1$. Similarly, we have $x^{t_2} = 1$. It follows that x^{μ_+}, x^{ν_-}, x^t are pairwise coprime. In addition, by annulling all variables in the monomial x^t and all those in x^s, we obtain $\beta_2 x^{\nu_-} x^r T_u = -\alpha_1 x^{\mu_+} T_v$. Since the two terms of this binomial are pairwise coprime, we have a contradiction. The claim is done.

Consider a minimal prime ideal $\mathfrak{p}$ of $\mathcal{A}$. We have $\mathfrak{p} \not\supseteq (x^t, x^s)$. Assume that $x^s \notin \mathfrak{p}$. After localising at $\mathfrak{p}$, x^s becomes a unit. It is easy to verify that $\mathcal{A}_{\mathfrak{p}} = \mathcal{J}_{\mathfrak{p}}$. Therefore, $\mathcal{A} = \mathcal{J}$. ■

Remark 2. Similar results to the above proposition appeared in [7], and [6], but our proof is elementary, direct, and without any assumption on the finiteness of the ground field $\mathcal{K}$.

From now on, we assume that x^p, x^r, x^t, x^s are non units.

2. Main Theorem

Consider the following sequence

$$0 \longrightarrow \mathcal{R} \xrightarrow{\begin{pmatrix} -x^s \\ x^t \\ x^r \\ -x^p \end{pmatrix}} \mathcal{R}^4 \xrightarrow{\begin{pmatrix} \alpha_2 x^{\nu+} x^p & \beta_2 x^{\nu-} x^r & -\beta_2 x^{\nu-} x^t & -\alpha_2 x^{\nu+} x^s \\ \beta_1 x^{\mu-} x^r & \alpha_1 x^{\mu+} x^p & \beta_1 x^{\mu-} x^s & \alpha_1 x^{\mu+} x^t \\ -x^t & -x^s & 0 & 0 \\ 0 & 0 & x^p & x^r \end{pmatrix}} \mathcal{R}^4 \xrightarrow{(f_u\ f_v\ f_{u+v}\ f_{u-v})} \mathcal{R}.$$

It is easy to verify that this sequence is exact (see for example [1]), and then it is a minimal free resolution of I.

The first syzygy matrix gives us some relations in the ideal $\mathcal{J}$:

$$\begin{aligned} L_1 &:= \alpha_2 x^{\nu+} x^p T_u + \beta_1 x^{\mu-} x^r T_v - x^t T_{u+v}, \\ L_2 &:= \beta_2 x^{\nu-} x^r T_u + \alpha_1 x^{\mu+} x^p T_v - x^s T_{u+v}, \\ L_3 &:= -\beta_2 x^{\nu-} x^t T_u + \beta_1 x^{\mu-} x^s T_v + x^p T_{u-v}, \\ L_4 &:= -\alpha_2 x^{\nu+} x^s T_u + \alpha_1 x^{\mu+} x^t T_v + x^r T_{u-v}. \end{aligned}$$

In addition, by computing the Plücker relation of the following matrix

$$\begin{pmatrix} x^p & \beta_1 x^s x^{\mu-} & \beta_2 x^t x^{\nu-} & \beta_1\beta_2 x^r x^{\mu-} x^{\nu-} \\ x^r & \alpha_1 x^t x^{\mu+} & \alpha_2 x^s x^{\nu+} & \alpha_1\alpha_2 x^p x^{\mu+} x^{\nu+} \end{pmatrix}$$

we obtain $\alpha_2\beta_2 x^{\nu+} x^{\nu-} f_u^2 - \alpha_1\beta_1 x^{\mu+} x^{\mu-} f_v^2 - f_{u+v} f_{u-v} = 0$. Hence, it follows that $Q := \alpha_2\beta_2 x^{\nu+} x^{\nu-} T_u^2 - \alpha_1\beta_1 x^{\mu+} x^{\mu-} T_v^2 - T_{u+v} T_{u-v}$ is also in $\mathcal{J}$. In fact, we have the following

Theorem 2.1. *The Rees ring $\mathcal{R}(I)$ is equal to $\mathcal{K}[\underline{x}, \underline{T}]/(L_1, L_2, L_3, L_4, Q)$, i.e.*

$$\mathcal{J} = (L_1, L_2, L_3, L_4, Q).$$

Denote by $\mathcal{A}$ the ideal (L_1, L_2, L_3, L_4, Q). Let us first remark that

$$x^s L_1 - x^t L_2 = x^r L_3 - x^p L_4 = f_v T_u - f_u T_v.$$

Hence, the polynomial $f_v T_u - f_u T_v$ is in $\mathcal{A}$. We set $L_5 = f_v T_u - f_u T_v$.

Lemma 2.1. *The set $\{L_1, L_2, L_3, L_4, L_5, Q\}$ is a Gröbner basis of $\mathcal{A}$, with respect to the lexicographic order $<$:*

$$x_1 < x_2 < \cdots < x_n < T_u < T_v < T_{u-v} < T_{u+v}.$$

Proof. With the order as above, the leading terms are $in(Q) = T_{u+v}T_{u-v}$, $in(L_1) = x^t T_{u+v}$, $in(L_2) = x^s T_{u+v}$, $in(L_3) = x^p T_{u-v}$, $in(L_4) = x^r T_{u-v}$ and $in(L_5) = in(f_u)T_v$. It is easy to verify that for all $F, G \in \{L_1, L_2, L_3, L_4, L_5, Q\}$, the term $s(F, G) := \dfrac{in(F)G - in(G)F}{\gcd(in(F), in(G))}$ is in $\mathcal{A} = (L_1, L_2, L_3, L_4, L_5, Q)$. By Buchberger's algorithm, it follows that $\{L_1, L_2, L_3, L_4, L_5, Q\}$ is a Gröbner basis of $\mathcal{A}$. ■

Proposition 2.1. *The ring $\mathcal{K}[\underline{x}, \underline{T}]/\mathcal{A}$ is Gorenstein of codimension 3.*

Proof. Since the Rees ring $\mathcal{R}(I) = \mathcal{K}[\underline{x}, \underline{T}]/\mathcal{J}$ is of dimension $n + 1$, then

$$\operatorname{codim}(\mathcal{K}[\underline{x},\underline{T}]/\mathcal{J}) = (n+4)-(n+1) = 3.$$

In addition, the sequence

$$0 \longrightarrow \mathcal{J}/\mathcal{A} \longrightarrow \mathcal{K}[\underline{x},\underline{T}]/\mathcal{A} \longrightarrow \mathcal{K}[\underline{x},\underline{T}]/\mathcal{J} \longrightarrow 0$$

is exact. This yields that $\operatorname{codim}(\mathcal{K}[\underline{x},\underline{T}]/\mathcal{A}) \leqslant 3$.

However, as we have seen $in(L_5) = in(f_u)T_v$ is the leading term of $\alpha_1 x^p x^t x^{\mu_+} T_v - \beta_1 x^r x^s x^{\mu_-} T_v$. Without loss of generality, we can assume that it is $x^p x^t x^{\mu_+} T_v$. By Remark 1, the set $\{x^s T_{u+v},\ x^r T_{u-v},\ x^p x^t x^{\mu_+} T_v\}$ forms a regular sequence of the initial ideal $in(\mathcal{A})$. It implies that $\operatorname{codim}(\mathcal{K}[\underline{x},\underline{T}]/in(\mathcal{A})) \geq 3$, and so is codimension of $\mathcal{K}[\underline{x},\underline{T}]/\mathcal{A}$.

Therefore, we obtain $\operatorname{codim}(\mathcal{K}[\underline{x},\underline{T}]/\mathcal{A}) = 3$.

Moreover, it is easy to check that $\mathcal{A}$ is generated by the 4×4 Pfaffians of the following 5×5 matrix

$$M = \begin{pmatrix} 0 & -T_{u+v} & \beta_1 x^{\mu_-} T_v & -\beta_2 x^{\nu_-} T_u & x^p \\ T_{u+v} & 0 & \alpha_2 x^{\nu_+} T_u & -\alpha_1 x^{\mu_+} T_v & -x^r \\ -\beta_1 x^{\mu_-} T_v & -\alpha_2 x^{\nu_+} T_u & 0 & -T_{u-v} & x^t \\ \beta_2 x^{\nu_-} T_u & \alpha_1 x^{\mu_+} T_v & T_{u-v} & 0 & -x^s \\ -x^p & x^r & -x^t & x^s & 0 \end{pmatrix}.$$

Due to [1], we have that $\mathcal{K}[\underline{x},\underline{T}]/\mathcal{A}$ is Gorenstein. ∎

As a consequence, we have the following corollary.

Corollary 2.1. *$\mathcal{A}$ is unmixed. More precisely, every primary composition $\mathfrak{q}$ of $\mathcal{A}$ is of height* 3.

Now we will prove Theorem 2.1. The reader should remark that [3, Proposition 2.9] cannot be applied to our situation.

Proof of Theorem 2.1.
The theorem is proved once we show that localisation of $\mathcal{A}$ and of $\mathcal{J}$ at any prime ideal $\mathfrak{p}$ coincide. Let $\mathfrak{p}$ be an arbitrary associated prime ideal of $\mathcal{A}$. It is sufficient to show $\mathcal{A}_\mathfrak{p} = \mathcal{J}_\mathfrak{p}$. Recall that for all associated prime ideal $\mathfrak{p}$ of $\mathcal{A}$, the height of $\mathfrak{p}$ equals 3, while the ideal (x^p, x^t, x^r, x^s) is of height 4 in $\mathcal{K}[\underline{x},\underline{T}]$, since x^p, x^t, x^r, x^s are non units and pairwise coprime. We deduce that $\mathfrak{p} \not\supseteq (x^p, x^t, x^r, x^s)$. The fact $ht(x^p, x^t, x^r, x^s) = 4$ implies also that one of these four elements is a non zero-divisor in $\mathcal{K}[\underline{x},\underline{T}]/\mathcal{A}$. Assume that it is x^s and then $x^s \notin \mathfrak{p}$. After localising at $\mathfrak{p}$, the term x^s becomes a unit. But we have the following relations:

$$x^s L_1 = x^t L_2 + x^r L_3 - x^p L_4,$$
$$x^s Q = T_{u-v} L_2 - x^{\mu_+} T_v L_3 - x^{\nu_-} T_u L_4,$$

then $\mathcal{A}_\mathfrak{p} = (L_2, L_3, L_4)_\mathfrak{p}$, and it is easy to verify that

$$(\mathcal{K}[\underline{x},\underline{T}]/(L_2, L_3, L_4))_{(x^s)} \cong (\mathcal{K}[\underline{x}, T_u, T_v, T_{u-v}]/(L_3, L_4))_{(x^s)}. \qquad (*)$$

On the other hand, we consider the ideal $I' := (f_u, f_v, f_{u-v})$. Since I' is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} -\alpha_2 x^{v_+} x^s & -\beta_2 x^{v_-} x^t \\ \alpha_1 x^{\mu_+} x^t & \beta_1 x^{\mu_-} x^s \\ x^r & x^p \end{pmatrix},$$

then I' is Cohen-Macaulay of codimension 2, and hence it is almost complete intersection. Due to Remark 2, the Rees algebra and the Symmetric algebra are isomorphic:

$$\mathrm{Sym}(I') \cong \mathcal{R}(I') =: \mathcal{K}[\underline{x}, T_u, T_v, T_{u-v}]/\mathcal{J}'.$$

Remark that $\mathrm{Sym}(I') = \mathcal{K}[\underline{x}, T_u, T_v, T_{u-v}]/(L_3, L_4)$. We get then $(L_3, L_4) = \mathcal{J}'$. Hence, the ideal (L_3, L_4) is prime in $\mathcal{K}[\underline{x}, T_u, T_v, T_{u-v}]$. Combine with $(*)$, we deduce that $\mathcal{A}_\mathfrak{p}$ is prime. Since $\mathcal{A}_\mathfrak{p}$ and $\mathcal{J}_\mathfrak{p}$ are prime ideals with the same codimension, and $\mathcal{A}_\mathfrak{p} \subset \mathcal{J}_\mathfrak{p}$, they have to coincide. ∎

Corollary 2.2. *The analytic spread of I is 3, and the Fiber cone $\mathcal{F}(I)$ is as follows*

$$\mathcal{F}(I) = \mathcal{K}[T_u, T_v, T_{u+v}, T_{u-v}]/(\widetilde{Q}),$$

where $\widetilde{Q}$ is the image modulo $\mathfrak{m}$ of Q.

Example 2. In $\mathbb{Z}^4$, we consider the lattice

$$\mathcal{L} = \begin{pmatrix} 2 & 2 & -2 & -2 \\ 4 & 0 & -3 & -1 \end{pmatrix}$$

generated by two vectors $u = (2, 2, -2, -2)$, and $v = (4, 0, -3, -1)$. The ideal $I_\mathcal{L} \subset \mathcal{K}[x, y, z, w]$ associated to this lattice has codimension 2. In [6], we treat all codimension 2 radical lattice ideals in case where the field $\mathcal{K}$ is infinite, and we prove that their Rees rings are Cohen-Macaulay and are generated by forms of degree at most 3, and their analytic spreads are 3. In this example, due to Theorem 2.1, we always have that the analytic spread of $I_\mathcal{L}$ is 3 independently of the characteristic and of the cardinal of $\mathcal{K}$, and even in the case $I_\mathcal{L}$ is not radical. In fact, if $\mathrm{char}(\mathcal{K}) = 2$, then $\sqrt{I_\mathcal{L}} = I_{\mathcal{L}_{sat}}$, where the latter one is the lattice ideal associated to the saturated lattice $\mathcal{L}_{sat}$ of $\mathcal{L}$

$$\mathcal{L}_{sat} = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 4 & 0 & -3 & -1 \end{pmatrix}.$$

More precisely, the ideal $I_{\mathcal{L}_{sat}}$ is the definition ideal of the curve (s^4, s^3t, st^3, t^4), and $I_\mathcal{L} = (xz^2 - y^2w, y^2z^2 - x^2w^2, z^4 - xw^3, y^4 - x^3w)$. Applying Theorem 2.1, we obtain

$$\mathcal{R}(I) = \mathcal{K}[x, y, z, t, T_1, T_2, T_3, T_4]/\mathcal{J},$$

where the ideal $\mathcal{J}$ is generated by $z^2T_1 + wT_2 + xT_3$, $y^2T_1 + xT_2 + wT_4$, $xw^2T_1 + z^2T_2 + y^2T_3$, $x^2wT_1 + y^2T_2 + z^2T_4$, $xwT_1^2 + T_2^2 + T_3T_4$. From this it follows that

$$\mathcal{F}(I) = \mathcal{K}[T_1, T_2, T_3, T_4]/(T_2^2 + T_3T_4).$$

3. Symmetric Algebra

We have the Symmetric Algebra of I is $\mathrm{Sym}(I) = k[\underline{x}, \underline{T}]/(L_1, L_2, L_3, L_4)$. Denote by $\mathcal{J}(1)$ the presentation ideal (L_1, L_2, L_3, L_4) of $\mathrm{Sym}(I)$. It should be remark that $\mathcal{J}(1) \subset \mathcal{J} \cap (x^p, x^t, x^r, x^s)$. This section is aiming to prove the equality.

Theorem 3.1. *The presentation ideal of the Symmetric Algebra of I admits a primary decomposition as*

$$\mathcal{J}(1) = \mathcal{J} \cap (x^p, x^t, x^r, x^s).$$

In particular, we have $\dim(\mathrm{Sym}(I)) = n + 1 = \dim(\mathcal{R}(I))$.

Proof. It suffices to show that $(x^p, x^t, x^r, x^s) \cap (Q) \subset (L_1, L_2, L_3, L_4)$.

Choose an arbitrary element $a \in k[\underline{x}, \underline{T}]$ such that $aQ \in (x^p, x^t, x^r, x^s)$. We have

$$\begin{aligned} aQ = a_p x^p + a_t x^t + a_s x^s + a_r x^r \text{ for some } a_p, a_t, a_r, a_s \in k[\underline{x}, \underline{T}]. \\ \iff \qquad aQ - a_p x^p - a_t x^t - a_s x^s - a_r x^r = 0. \end{aligned} \tag{$*$}$$

It is easy to show that $\{Q, x^p, x^t, x^r, x^s\}$ is a Grobner basis respected to the order $<$ defined in Sec. 2 of the ideal $\mathfrak{a}$ generated by them. So $(*)$ is a syzygy of $\mathfrak{a}$. By the fact that $\mathrm{in}(Q)$, x^p, x^t, x^r, x^s are pairwise coprime and by Buchberger's algorithm, we deduce that $a \in (x^p, x^t, x^r, x^s)$. It means that $a = b_p x^p + b_t x^t + b_s x^s + b_r x^r$ for some $b_p, b_t, b_r, b_s \in k[\underline{x}, \underline{T}]$. Consider the term $x^p Q$.

$$\begin{aligned} x^p Q &= x^p(\alpha_2\beta_2 x^{\nu+} x^{\nu-} T_u^2 - \alpha_1\beta_1 x^{\mu+} x^{\mu-} T_v^2 - T_{u+v} T_{u-v}) \\ &= -T_{u+v} L_3 - x^{\mu-} T_v L_2 + x^{\nu-} T_u L_1. \end{aligned}$$

Then $x^p Q$ is in (L_1, L_2, L_3, L_4).

For $x^r Q, x^t Q, x^s Q$, we get similar situations. Therefore $aQ \in (L_1, L_2, L_3, L_4)$. It implies that $(x^p, x^t, x^r, x^s) \cap (Q) \subset (L_1, L_2, L_3, L_4)$. ∎

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