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An Extension of Uniqueness Theorems for Meromorphic Mappings

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Abstract. In this paper, we give some results on the number of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ under a condition on the inverse images of hyperplanes in $\mathbb{C}P^n$. At the same time, we give an answer for an open question posed by H. Fujimoto in 1998.

1. Introduction

In 1926, Nevanlinna showed that for two nonconstant meromorphic functions f and g on the complex plane $\mathbb{C}$, if they have the same inverse images for five distinct values, then $f = g$, and that g is a special type of a linear fractional tranformation of f if they have the same inverse images, counted with multiplicities, for four distinct values.

In 1975, Fujimoto [2] generalized Nevanlinna's result to the case of meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$. This problem continued to be studied by Smiley [9], Ji [5] and others.

Let f be a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and H be a hyperplane in $\mathbb{C}P^n$ such that $\mathrm{im} f \not\subseteq H$. Denote by $v_{(f,H)}$ the map of $\mathbb{C}^m$ into $\mathbb{N}_0$ such that $v_{(f,H)}(a)$ $(a \in \mathbb{C}^m)$ is the intersection multiplicity of the image of f and H at $f(a)$. Let k be a positive interger or $+\infty$. We set

$$v^{k)}_{(f,H)}(a) = \begin{cases} 0 & \text{if } v_{(f,H)}(a) > k, \\ v_{(f,H)}(a) & \text{if } v_{(f,H)}(a) \leqslant k. \end{cases}$$

Let f be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and $\{H_j\}_{j=1}^q$ be q hyperplanes in general position with

(a) $dim\Big\{z : v^{k)}_{(f,H_i)}(z) > 0 \text{ and } v^{k)}_{(f,H_j)}(z) > 0\Big\} \leqslant m-2$ for all $1 \leqslant i < j \leqslant q$.

For each positive integer p, denote by $F_k(\{H_j\}_{j=1}^q, f, p)$ the set of all linearly nondegenerate meromorphic mappings g of $\mathbb{C}^m$ into $\mathbb{C}P^n$ such that:

(b) $\min\{v^{k)}_{(g,H_j)}, p\} = \min\{v^{k)}_{(f,H_j)}, p\}$,

(c) $g = f$ on $\bigcup_{j=1}^{q}\{z : v^{k)}_{(f,H_j)}(z) > 0\}$.

In [5], Ji proved the following

Theorem J. [5] *If $q = 3n+1$ and $k = +\infty$, then for three mappings $f_1, f_2, f_3 \in F_k\big(\{H_j\}_{j=1}^q, f, 1\big)$, the mapping $f_1 \times f_2 \times f_3 : \mathbb{C}^m \longrightarrow \mathbb{C}P^n \times \mathbb{C}P^n \times \mathbb{C}P^n$ is algebraically degenerate, namely, $\{(f_1(z), f_2(z), f_3(z)), z \in \mathbb{C}^m\}$ is contained in a proper algebraic subset of $\mathbb{C}P^n \times \mathbb{C}P^n \times \mathbb{C}P^n$.*

In 1929, Cartan declared that there are at most two meromorphic functions on $\mathbb{C}$ which have the same inverse images (ignoring multiplicities) for four distinct values. However in 1988, Steinmetz [10] gave examples which showed that Cartan's declaration is false. On the other hand, in 1998, Fujimoto [4] showed that Cartan's declaration is true if we assume that meromorphic functions on $\mathbb{C}$ share four distinct values counted with multiplicities truncated by 2. He gave the following theorem

Theorem F. [4] *If $q = 3n+1$ and $k = +\infty$ then $F_k\big(\{H_j\}_{j=1}^q, f, 2\big)$ contains at most two mappings.*

He also proposed an open problem asking if the number $q = 3n+1$ in Theorem F can be replaced by a smaller one. Inspired by this question, in this paper we will generalize the above results to the case where the number $q = 3n+1$ is in fact replaced by a smaller one. We also obtain an improvement concerning truncating multiplicities.

Denote by Ψ the Segre embedding of $\mathbb{C}P^n \times \mathbb{C}P^n$ into $\mathbb{C}P^{n^2+2n}$ which is defined by sending the ordered pair $((w_0, ..., w_n), (v_0, ..., v_n))$ to $(..., w_i v_j, ...)$ (in lexicographic order).

Let $h : \mathbb{C}^m \longrightarrow \mathbb{C}P^n \times \mathbb{C}P^n$ be a meromorphic mapping. Let $(h_0 : ... : h_{n^2+2n})$ be a representation of $\Psi \circ h$. We say that h is linearly degenerate (with the algebraic structure in $\mathbb{C}P^n \times \mathbb{C}P^n$ given by the Segre embedding) if $h_0, ..., h_{n^2+2n}$ are linearly dependent over $\mathbb{C}$.

Our main results are stated as follows:

Theorem 1. *There are at most two distinct mappings in $F_k\big(\{H_j\}_{j=1}^q, f, p\big)$ in each of the following cases*:

i) $1 \leqslant n \leqslant 3, q = 3n+1, p = 2$ *and* $23n \leqslant k \leqslant +\infty$

ii) $4 \leqslant n \leqslant 6, q = 3n, p = 2$ *and* $\frac{(6n-1)n}{n-3} \leqslant k \leqslant +\infty$

iii) $n \geq 7, q = 3n-1, p = 1$ *and* $\frac{(6n-4)n}{n-6} \leqslant k \leqslant +\infty$

Theorem 2. *Assume that* $q = \left[\frac{5(n+1)}{2}\right], (65n+171)n \leqslant k \leqslant +\infty$, *where* $[x] := \max\{d \in \mathbb{N} : d \leqslant x\}$ *for a positive constant* x. *Then one of the following assertions holds :*

i) $\# F_k\big(\{H_j\}_{j=1}^q, f, 1\big) \leqslant 2.$

ii) *For any* $f_1, f_2 \in F_k\big(\{H_j\}_{j=1}^q, f, 1\big)$, *the mapping* $f_1 \times f_2 : \mathbb{C}^m \longrightarrow \mathbb{C}P^n \times \mathbb{C}P^n$ *is linearly degenerate (with the algebraic structure in* $\mathbb{C}P^n \times \mathbb{C}P^n$ *given by the Segre embedding).*

We finally remark that we obtained similar uniqueness theorems with moving targets in [11], but only with a bigger number of targets and with much bigger truncations.

2. Preliminaries

We set $\|z\| := (|z_1|^2 + \cdots + |z_m|^2)^{1/2}$ for $z = (z_1, \dots, z_m) \in \mathbb{C}^m$, $B(r) := \big\{z : \|z\| < r\big\}$, $S(r) := \big\{z : \|z\| = r\big\}$, $d^c := \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial)$, $\upsilon := (dd^c\|z\|^2)^{m-1}$ and $\sigma := d^c \log\|z\|^2 \wedge (dd^c \log\|z\|^2)^{m-1}$.

Let F be a nonzero holomorphic function on $\mathbb{C}^m$. For an m-tuple $\alpha := (\alpha_1, \dots, \alpha_m)$ of nonnegative integers, set $|\alpha| := \alpha_1 + \cdots + \alpha_m$ and $D^\alpha F := \frac{\partial^{|\alpha|}F}{\partial z_1^{\alpha_1} \dots \partial z_m^{\alpha_m}}$. We define the map $\nu_F : \mathbb{C}^m \to \mathbb{N}_0$ by $\nu_F(z) := \max\big\{p : D^\alpha F(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < p\big\}$. Let k be a positive integer or $+\infty$. Define the map $\nu_F^{k)}$ of $\mathbb{C}^m$ into $\mathbb{N}_0$ by

$$\nu_F^{k)}(z) := \begin{cases} 0 & \text{if } \nu_F(z) > k, \\ \nu_F(z) & \text{if } \nu_F(z) \leqslant k. \end{cases}$$

Let ϕ be a nonzero meromorphic function on $\mathbb{C}^m$. We define the map $\nu_\phi^{k)}$ as follows. For each $z \in \mathbb{C}^m$, choose nonzero holomorphic functions F and G on a neighborhood U of z such that $\phi = \frac{F}{G}$ on U and $\dim\big(F^{-1}(0) \cap G^{-1}(0)\big) \leqslant m-2$. Then put $\nu_\phi^{k)}(z) := \nu_F^{k)}(z)$. Set

$$\big|\nu_\phi^{k)}\big| := \overline{\big\{z : \nu_\phi^{k)}(z) > 0\big\}}.$$

Define

$$N^{k)}(r, \nu_\phi) := \int_1^r \frac{n^{k)}(t)}{t^{2m-1}} dt, \quad (1 < r < +\infty)$$

where

$$n^{k)}(t) := \int\limits_{\left|v_\phi^{k)}\right| \cap B(t)} v_\phi^{k)} \upsilon \quad \text{for} \quad m \geq 2,$$

and

$$n^{k)}(t) := \sum_{|z| \leqslant t} v_\phi^{k)}(z) \quad \text{for} \quad m = 1.$$

Set $N(r, v_\phi) := N^{+\infty)}(r, v_\phi)$. For l a positive integer or $+\infty$, set

$$N_l^{k)}(r, v_\phi) := \int\limits_1^r \frac{n_l^{k)}(t)}{t^{2m-1}} dt, \quad (1 < r < +\infty)$$

where $n_l^{k)}(t) := \int\limits_{\left|v_\phi^{k)}\right| \cap B(t)} \min\left\{v_\phi^{k)}, l\right\} \upsilon$ for $m \geq 2$ and $n_l^{k)}(t) := \sum_{|z| \leqslant t}$ $\min\left\{v_\phi^{k)}(z), l\right\}$ for $m = 1$. Set $\overline{N}(r, v_\phi) := N_1^{+\infty)}(r, v_\phi)$ and $\overline{N}^{k)}(r, v_\phi) := N_1^{k)}(r, v_\phi)$. For a closed subset A of a purely $(m-1)$-dimensional analytic subset of $\mathbb{C}^m$, we define

$$\bar{\bar{N}}(r, A) := \int\limits_1^r \frac{\bar{n}(t)}{t^{2m-1}} dt, \quad (1 < r < +\infty),$$

where

$$\bar{n}(t) := \begin{cases} \int\limits_{A \cap B(t)} \upsilon & \text{for } m \geq 2, \\ \#(A \cap B(t)) & \text{for } m = 1. \end{cases}$$

Let $f : \mathbb{C}^m \to \mathbb{C}P^n$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \cdots : w_n)$ on $\mathbb{C}P^n$, we take a reduced representation $f = (f_0 : \cdots : f_n)$, which means that each f_i is a holomorphic function on $\mathbb{C}^m$ and $f(z) = (f_0(z) : \cdots : f_n(z))$ outside the analytic set $\{f_0 = \cdots = f_n = 0\}$ of codimension ≥ 2.

Set $\|f\| := (|f_0|^2 + \cdots + |f_n|^2)^{1/2}$. The characteristic function of f is defined by

$$T_f(r) := \int\limits_{S(r)} \log \|f\| \sigma - \int\limits_{S(1)} \log \|f\| \sigma, \quad r > 1.$$

For a nonzero meromorphic function ϕ on C^m, the characteristic function $T_\phi(r)$ of ϕ is defined by considering ϕ as a meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^1$.

Let $H = \{a_0 w_0 + \cdots + a_n w_n = 0\}$ be a hyperplane in $\mathbb{C}P^n$ such that $\operatorname{im} f \not\subseteq H$. Set $(f, H) := a_0 f_0 + \cdots + a_n f_n$. We define

$$N_f^{k)}(r, H) := N^{k)}(r, v_{(f,H)}) \quad \text{and} \quad N_{l,f}^{k)}(r, H) := N_l^{k)}(r, v_{(f,H)}).$$

Sometimes we write $\overline{N}_f^{k)}(r, H)$ for $N_{1,f}^{k)}(r, H)$, $N_{l,f}(r, H)$ for $N_{l,f}^{+\infty)}(r, H)$ and $N_f(r, H)$ for $N_{+\infty,f}^{+\infty)}(r, H)$.

Set $\psi_f(H) := \dfrac{\|f\|\big(|a_0|^2 + \cdots + |a_n|^2\big)^{1/2}}{(f, H)}$. We define the proximity function by

$$m_f(r, H) := \int\limits_{S(r)} \log |\psi_f(H)|\sigma - \int\limits_{S(1)} \log |\psi_f(H)|\sigma.$$

For a nonzero meromorphic function ϕ, the proximity function is defined by

$$m(r, \phi) := \int\limits_{S(r)} \log^+ |\phi|\sigma.$$

We note that $m(r, \phi) = m_\phi(r, +\infty) + O(1)$ ([4], p. 135).

We state First and Second Main Theorems of Value Distribution Theory.

First Main Theorem. *Let $f : \mathbb{C}^m \to \mathbb{C}P^n$ be a meromorphic mapping and H a hyperplane in $\mathbb{C}P^n$ such that* im $f \not\subseteq H$. *Then*

$$N_f(r, H) + m_f(r, H) = T_f(r).$$

For a nonzero meromorphic function ϕ we have

$$N(r, v_{\frac{1}{\phi}}) + m(r, \phi) = T_\phi(r) + O(1).$$

Second Main Theorem. *Let $f : \mathbb{C}^m \to \mathbb{C}P^n$ be a linearly nondegenerate meromorphic mapping and $H_1, ..., H_q$ be hyperplanes in general position in $\mathbb{C}P^n$. Then*

$$(q - n - 1)T_f(r) \leqslant \sum_{j=1}^{q} N_{n,f}(r, H_j) + o(T_f(r))$$

except for a set $E \subset (1, +\infty)$ of finite Lebesgue measure.

The following so-called logarithmic derivative lemma plays an essential role in Nevanlinna theory.

Theorem 2.1. ([5], Lemma 3.1) *Let ϕ be a non-constant meromorphic function on $\mathbb{C}^m$. Then for any i, $1 \leqslant i \leqslant m$, we have*

$$m\Big(r, \frac{\frac{\partial}{\partial z_i}\phi}{\phi}\Big) = o(T_\phi(r)) \quad \text{as} \quad r \to \infty, \quad r \notin E,$$

where $E \subset (1, +\infty)$ of finite Lebesgue measure.

Let F, G and H be nonzero meromorphic functions on $\mathbb{C}^m$. For each l, $1 \leqslant l \leqslant m$, we define the Cartan auxiliary function by

$$\Phi^l(F, G, H) := F \cdot G \cdot H \cdot \begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{F} & \frac{1}{G} & \frac{1}{H} \\ \frac{\partial}{\partial z_l}\big(\frac{1}{F}\big) & \frac{\partial}{\partial z_l}\big(\frac{1}{G}\big) & \frac{\partial}{\partial z_l}\big(\frac{1}{H}\big) \end{vmatrix}.$$

By [4] (Proposition 3.4) we have the following

Theorem 2.2. *Let F, G, H be nonzero meromorphic functions on $\mathbb{C}^m$. Assume that $\Phi^l(F, G, H) \equiv 0$ and $\Phi^l\Big(\frac{1}{F}, \frac{1}{G}, \frac{1}{H}\Big) \equiv 0$ for all l, $1 \leqslant l \leqslant m$. Then one of the following assertions holds*

i) $F = G$ or $G = H$ or $H = F$.

ii) $\frac{F}{G}, \frac{G}{H}, \frac{H}{F}$ *are all constant.*

3. Proof of the Theorems

First of all, we need the following lemmas:

Lemma 1. *Let $f_1, ..., f_d$ be linearly nondegenerate meromorphic mappings of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and $\{H_j\}_{j=1}^q$ be hyperplanes in $\mathbb{C}P^n$. Then there exists a dense subset $\mathcal{C} \subset \mathbb{C}^{n+1} \setminus \{0\}$ such that for any $c = (c_0, ..., c_n) \in \mathcal{C}$, the hyperplane H_c defined by $c_0\omega_0 + \cdots + c_n\omega_n = 0$ satisfies*

$$dim\ (f_i^{-1}(H_j) \cap f_i^{-1}(H_c)) \leqslant m - 2 \text{ for all } i \in \{1, \ldots, d\} \text{ and } j \in \{1, ..., q\}.$$

Proof. We refer to [5], Lemma 5.1. ∎

Let $f_1, f_2, f_3 \in F_k\big(\{H_j\}_{j=1}^q, f, 1\big)$, for $q \geq n+1$. Set

$$T(r) := T_{f_1}(r) + T_{f_2}(r) + T_{f_3}(r).$$

For each $c \in \mathcal{C}$, set $F_{ic}^j := \frac{(f_i, H_j)}{(f_i, H_c)}$ for $i \in \{1, 2, 3\}$ and $j \in \{1, \ldots, q\}$.

Lemma 2. *Assume that there exist $j_0 \in \{1, ..., q\}, c \in \mathcal{C}, l \in \{1, ..., m\}$ and a closed subset A of a purely $(m-1)$-dimensional analytic subset of $\mathbb{C}^m$ satisfying*

1) $\Phi_c^l := \Phi^l\big(F_{1c}^{j_0}, F_{2c}^{j_0}, F_{3c}^{j_0}\big) \not\equiv 0$, *and*

2) $\min\big\{v_{(f_1, H_{j_0})}^{k)}, p\big\} = \min\big\{v_{(f_2, H_{j_0})}^{k)}, p\big\} = \min\big\{v_{(f_3, H_{j_0})}^{k)}, p\big\}$ *on $\mathbb{C}^m \setminus A$, where p is a positive integer. Then*

$$2\sum_{j=1, j\neq j_0}^{q} \overline{N}_{f_i}^{k)}(r, H_j) + N_{p-1, f_i}^{k)}(r, H_{j_0}) \leqslant N(r, v_{\Phi_c^l}) + (p-1)\overline{N}(r, A)$$
$$\leqslant \frac{k+2}{k+1}T(r) + (p+2)\overline{N}(r, A) + o(T(r))$$

for all $i \in \{1, 2, 3\}$.

Proof. Without loss of generality, we may assume that $l = 1$. For an arbitrary point $a \in \mathbb{C}^m \setminus A$ satisfying $v_{(f_1, H_{j_0})}^{k)}(a) > 0$, we have $v_{(f_i, H_{j_0})}^{k)}(a) > 0$ for all $i \in \{1, 2, 3\}$. We choose a such that $a \notin \bigcup_{i=1}^{3} f_i^{-1}(H_c)$. We distinguish two cases, which lead to equations (1) and (2).

Case 1. If $v_{(f_1,H_{j_0})}(a) \geq p$, then $v_{(f_i,H_{j_0})}(a) \geq p$, $i \in \{1,2,3\}$. This means that a is a zero point of $F_{ic}^{j_0}$ with multiplicity $\geq p$ for $i \in \{1,2,3\}$. We have

$$\begin{aligned}\Phi_c^1 = F_{1c}^{j_0} F_{3c}^{j_0} \frac{\partial}{\partial z_1}\Big(\frac{1}{F_{3c}^{j_0}}\Big) - F_{1c}^{j_0} F_{2c}^{j_0} \frac{\partial}{\partial z_1}\Big(\frac{1}{F_{2c}^{j_0}}\Big) \\ + F_{2c}^{j_0} F_{1c}^{j_0} \frac{\partial}{\partial z_1}\Big(\frac{1}{F_{1c}^{j_0}}\Big) - F_{2c}^{j_0} F_{3c}^{j_0} \frac{\partial}{\partial z_1}\Big(\frac{1}{F_{3c}^{j_0}}\Big) \\ + F_{3c}^{j_0} F_{2c}^{j_0} \frac{\partial}{\partial z_1}\Big(\frac{1}{F_{2c}^{j_0}}\Big) - F_{3c}^{j_0} F_{1c}^{j_0} \frac{\partial}{\partial z_1}\Big(\frac{1}{F_{1c}^{j_0}}\Big).\end{aligned}$$

On the other hand $F_{1c}^{j_0} F_{3c}^{j_0} \frac{\partial}{\partial z_1}\Big(\frac{1}{F_{3c}^{j_0}}\Big) = \frac{-F_{1c}^{j_0}\frac{\partial}{\partial z_1}F_{3c}^{j_0}}{F_{3c}^{j_0}}$, so a is a zero point of $F_{1c}^{j_0} F_{3c}^{j_0} \frac{\partial}{\partial z_1}\Big(\frac{1}{F_{3c}^{j_0}}\Big)$ with multiplicity $\geq p-1$. By applying the same argument also to all other combinations of indices, we see that

a is a zero point of Φ_c^1 with multiplicity $\geq p-1$. (1)

Case 2. If $v_{(f_1,H_{j_0})}(a) \leqslant p$, then $p_0 := v_{(f_1,H_{j_0})}(a) = v_{(f_2,H_{j_0})}(a) = v_{(f_3,H_{j_0})}(a) \leqslant p$. There exists a neighborhood U of a such that $v_{(f_1,H_{j_0})} \leqslant p$ on U. Indeed, there exists otherwise a sequence $\{a_s\}_{s=1}^{\infty} \subset \mathbb{C}^m$, with $\lim\limits_{s\to\infty} a_s = a$ and $v_{(f_1,H_{j_0})}(a_s) \geq p+1$ for all s. By the definition, we have $D^{\beta}(f_1,H_{j_0})(a_s) = 0$ for all $|\beta| < p+1$. So $D^{\beta}(f_1,H_{j_0})(a) = \lim\limits_{s\to\infty} D^{\beta}(f_1,H_{j_0})(a_s) = 0$ for all $|\beta| < p+1$. Thus $v_{(f_1,H_{j_0})}(a) \geq p+1$. This is a contradiction. Hence $v_{(f_1,H_{j_0})} \leqslant p$ on U.

We can choose U such that $U \cap A = \emptyset$, $v_{(f_i,H_{j_0})} \leqslant p$ on U and (f_i, H_c) has no zero point on U for all $i \in \{1,2,3\}$. Then $v_{F_{1c}^{j_0}} = v_{F_{2c}^{j_0}} = v_{F_{3c}^{j_0}} \leqslant p$ on U. So $U \cap \{F_{1c}^{j_0} = 0\} = U \cap \{F_{2c}^{j_0} = 0\} = U \cap \{F_{3c}^{j_0} = 0\}$. Choose a such that a is regular point of $U \cap \{F_{1c}^{j_0} = 0\}$. By shrinking U we may assume that there exists a holomorphic function h on U such that dh has no zero point and $F_{ic}^{j_0} = h^{p_0} u_i$ on U, where $u_i (i = 1,2,3)$ are nowhere vanishing holomorphic functions on U (note that $v_{F_{1c}^{j_0}}(a) = v_{F_{2c}^{j_0}}(a) = v_{F_{3c}^{j_0}}(a) = p_0$). We have

$$\begin{aligned}\Phi_c^1 = u_1 \frac{\big(u_3\frac{\partial}{\partial z_1}u_2 - u_2\frac{\partial}{\partial z_1}u_3\big)h^{p_0}}{u_2u_3} + u_2 \frac{\big(u_1\frac{\partial}{\partial z_1}u_3 - u_3\frac{\partial}{\partial z_1}u_1\big)h^{p_0}}{u_3u_1} \\ + u_3 \frac{\big(u_2\frac{\partial}{\partial z_1}u_1 - u_1\frac{\partial}{\partial z_1}u_2\big)h^{p_0}}{u_1u_2}.\end{aligned}$$

So, we have

a is a zero point of Φ_c^1 with mulitplicity $\geq p_0$. (2)

By (1), (2) and our choice of a, there exists an analytic set $M \subset \mathbb{C}^m$ with codimension ≥ 2 such that $v_{\Phi_c^1} \geq \min\{v_{(f_1,H_{j_0})} , p-1\}$ on

$$\big\{z : v^{k)}_{(f_1,H_{j_0})}(z) > 0\big\} \setminus (M \cup A). \tag{3}$$

For each $j \in \{1,\ldots,q\} \setminus \{j_0\}$, let a (depending on j) be an arbitrary point in $\mathbb{C}^m$ such that $v^{k)}_{(f_1,H_j)}(a) > 0$ (if there exist any). Then $v^{k)}_{(f_i,H_j)}(a) > 0$

for all $i \in \{1,2,3\}$, since $f_1, f_2, f_3 \in \mathcal{F}_k\big(\{H_j\}_{j=1}^q, f, 1\big)$. We can choose $a \notin f_i^{-1}(H_c) \cup f_i^{-1}(H_{j_0})$, $i = 1,2,3$. Then there exists a neighborhood U of a such that $v_{(f_i,H_j)} \leqslant k$ on U and (f_i, H_{j_0}), (f_i, H_c) ($i = 1,2,3$) have no zero point on U. We have $B := f_1^{-1}(H_j) \cap U = f_2^{-1}(H_j) \cap U = f_3^{-1}(H_j) \cap U$ and $\dfrac{1}{F_{1c}^{j_0}} = \dfrac{1}{F_{2c}^{j_0}} = \dfrac{1}{F_{3c}^{j_0}}$ on B. Choose a such that a is a regular point of B. By shrinking U, we may assume that there exists a holomorphic function h on U such that dh has no zero point and $U \cap \{h = 0\} = B$. Then $\dfrac{1}{F_{2c}^{j_0}} - \dfrac{1}{F_{1c}^{j_0}} = h\phi_2$ and $\dfrac{1}{F_{3c}^{j_0}} - \dfrac{1}{F_{1c}^{j_0}} = h\phi_3$ on U where ϕ_2, ϕ_3 are holomorphic functions on U. Hence, we get

$$\begin{aligned}\Phi_c^1 &= F_{1c}^{j_0}F_{2c}^{j_0}F_{3c}^{j_0}\begin{vmatrix} 1 & 0 & 0\\ \frac{1}{F_{1c}^{j_0}} & h\phi_2 & h\phi_3\\ \frac{\partial}{\partial z_1}\Big(\frac{1}{F_{1c}^{j_0}}\Big) & \phi_2\frac{\partial}{\partial z_1}h + h\frac{\partial}{\partial z_1}\phi_2 & \phi_3\frac{\partial}{\partial z_1}h + h\frac{\partial}{\partial z_1}\phi_3\end{vmatrix}\\ &= F_{1c}^{j_0}F_{2c}^{j_0}F_{3c}^{j_0}h^2\begin{vmatrix}\phi_2 & \phi_3\\ \frac{\partial}{\partial z_1}\phi_2 & \frac{\partial}{\partial z_1}\phi_3\end{vmatrix}.\end{aligned}$$

Therefore, a is a zero point of Φ_c^1 with multiplicity ≥ 2. Thus, for each $j \in \{1,\dots,q\} \setminus \{j_0\}$, there exists an analytic set $N \subset \mathbb{C}^m$ with codimension ≥ 2 such that $v_{\Phi_c^1} \geq 2$ on

$$\big\{z : v^{k)}_{(f_1,H_j)}(z) > 0\big\} \setminus N. \tag{4}$$

By (3) and (4), we have

$$2\sum_{j=1, j\neq j_0}^{q} \overline{N}^{k)}_{f_1}(r, H_j) + N^{k)}_{p-1,f_1}(r, H_{j_0}) \leqslant N(r, v_{\Phi_c^1}) + (p-1)\overline{N}(r, A).$$

Similarly, we have

$$2\sum_{j=1, j\neq j_0}^{q} \overline{N}^{k)}_{f_i}(r, H_j) + N^{k)}_{p-1,f_i}(r, H_{j_0}) \leqslant N(r, v_{\Phi_c^1}) + (p-1)\overline{N}(r, A), \quad i = 1,2,3. \tag{5}$$

Let a be an arbitrary zero point of some $F_{ic}^{j_0}, a \notin f_i^{-1}(H_c)$, say $i = 1$. We have

$$\begin{aligned}\Phi_c^1 &= \big(F_{2c}^{j_0} - F_{3c}^{j_0}\big)F_{1c}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{1c}^{j_0}}\Big) + \big(F_{3c}^{j_0} - F_{1c}^{j_0}\big)F_{2c}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{2c}^{j_0}}\Big)\\ &\quad + \big(F_{1c}^{j_0} - F_{2c}^{j_0}\big)F_{3c}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{3c}^{j_0}}\Big).\end{aligned} \tag{6}$$

So we have

$$v_{\frac{1}{\Phi_c^1}}(a) \leqslant 1 + \max\{v_{\frac{1}{F_{ic}^{j_0}}}(a), i = 2, 3\} \leqslant 1 + v_{\frac{1}{F_{2c}^{j_0}}}(a) + v_{\frac{1}{F_{3c}^{j_0}}}(a).$$

Furthermore, if $0 < v_{F_{1c}^{j_0}}(a) \leqslant k$ $\big($and, hence, $v^{k)}_{(f_1, H_{j_0})}(a) > 0\big)$ and $a \notin A$, then by (3) we may assume that $v_{\frac{1}{\Phi_c^1}}(a) = 0$ (outside an analytic set of codimension ≥ 2). (7)

Let a be an arbitrary pole of all $F_{ic}^{j_0}$, $i = 1, 2, 3$. By (6) we have

$$v_{\frac{1}{\Phi_c^1}}(a) \leqslant \max\{v_{\frac{1}{F_{ic}^{j_0}}}(a), i = 1, 2, 3\} + 1 < \sum_{i=1}^{3} v_{\frac{1}{F_{ic}^{j_0}}}(a) \tag{8}$$

It follows from (6) that a pole of Φ_c^1 is a zero or a pole of some $F_{ic}^{j_0}$. Thus, by (6), (7) and (8), we have

$$\begin{aligned} N\Big(r, v_{\frac{1}{\Phi_c^1}}\Big) &\leqslant \sum_{i=1}^{3} N\Big(r, v_{\frac{1}{F_{ic}^{j_0}}}\Big) + \sum_{i=1}^{3}\Big(\overline{N}\big(r, v_{F_{ic}^{j_0}}\big) - \overline{N}^{k)}\big(r, v_{F_{ic}^{j_0}}\big)\Big) + 3\overline{N}(r, A) \\ &\leqslant \sum_{i=1}^{3} N\Big(r, v_{\frac{1}{F_{ic}^{j_0}}}\Big) + \frac{1}{k+1}\sum_{i=1}^{3} N\big(r, v_{F_{ic}^{j_0}}\big) + 3\overline{N}(r, A) \\ &\leqslant \sum_{i=1}^{3} N\big(r, v_{\frac{1}{F_{ic}^{j_0}}}\big) + \frac{1}{k+1}\sum_{i=1}^{3} T_{F_{ic}^{j_0}}(r) + 3\overline{N}(r, A) \\ &\leqslant \sum_{i=1}^{3} N\big(r, v_{\frac{1}{F_{ic}^{j_0}}}\big) + \frac{1}{k+1} T(r) + 3\overline{N}(r, A) + O(1). \end{aligned} \tag{9}$$

We have

$$\begin{aligned} \Phi_c^1 = F_{1c}^{j_0}\Big[F_{3c}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{3c}^{j_0}}\Big) - F_{2c}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{2c}^{j_0}}\Big)\Big] \\ + F_{2c}^{j_0}\Big[F_{1c}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{1c}^{j_0}}\Big) - F_{3c}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{3c}^{j_0}}\Big)\Big] \\ + F_{3c}^{j_0}\Big[F_{2c}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{2c}^{j_0}}\Big) - F_{1c}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{1c}^{j_0}}\Big)\Big] \end{aligned}$$

so $m(r, \Phi_c^1) \leqslant \sum_{i=1}^{3} m(r, F_{ic}^{j_0}) + 2\sum_{i=1}^{3} m\Big(r, F_{ic}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{ic}^{j_0}}\Big)\Big) + 0(1)$. By Theorem 2.1, we have

$$m\Big(r, F_{ic}^{j_0}\frac{\partial}{\partial z_1}\Big(\frac{1}{F_{ic}^{j_0}}\Big)\Big) = o\big(T_{F_{ic}^{j_0}}(r)\big).$$

Thus, we get

$$m(r, \Phi_c^1) \leqslant \sum_{i=1}^{3} m(r, F_{ic}^{j_0}) + o(T(r)), \tag{10}$$

(note that $T_{F_{ic}^{j_0}}(r) \leqslant T_{f_i}(r) + O(1)$).

By (9), (10) and by First Main Theorem, we have

$$
\begin{aligned}
N(r, v_{\Phi_c^1}) &\leqslant T_{\Phi_c^1}(r) + O(1) = N\big(r, v_{\frac{1}{\Phi_c^1}}\big) + m(r, \Phi_c^1) + O(1) \\
&\leqslant \sum_{i=1}^{3}\Big(N\big(r, v_{\frac{1}{F_{ic}^{j_0}}}\big) + m(r, F_{ic}^{j_0})\Big) + \frac{1}{k+1}T(r) + 3\overline{\overline{N}}(r, A) + o(T(r)) \\
&\leqslant \sum_{i=1}^{3} T_{F_{ic}^{j_0}}(r) + \frac{1}{k+1}T(r) + 3\overline{\overline{N}}(r, A) + o(T(r)) \\
&\leqslant \sum_{i=1}^{3} T_{f_i}(r) + \frac{1}{k+1}T(r) + 3\overline{\overline{N}}(r, A) + o(T(r)) \\
&= \frac{k+2}{k+1}T(r) + 3\overline{\overline{N}}(r, A) + o(T(r)). \qquad (11)
\end{aligned}
$$

By (5) and (11) we get Lemma 2. ■

The following lemma is a version of Second Main Theorem without taking account of multiplicities of order $> k$ in the counting functions.

Lemma 3. *Let f be a linearly nondegenerate meromorphic mapping of $\mathbb{C}^m$ into $\mathbb{C}P^n$ and $\{H_j\}_{j=1}^q (q \geq n+2)$ be hyperplanes in $\mathbb{C}P^n$ in general position. Take a positive integer k with $\frac{qn}{q-n-1} \leqslant k \leqslant +\infty$. Then*

$$
\begin{aligned}
T_f(r) &\leqslant \frac{k}{(q-n-1)(k+1)-qn}\sum_{j=1}^{q} N_{n,f}^{k)}(r, H_j) + o(T_f(r)) \\
&\leqslant \frac{nk}{(q-n-1)(k+1)-qn}\sum_{j=1}^{q} \overline{N}_f^{k)}(r, H_j) + o(T_f(r))
\end{aligned}
$$

for all $r > 1$ except a set E of finite Lebesgue measure.

Proof. By First and Second Main Theorems, we have

$$
\begin{aligned}
(q-n-1)T_f(r) &\leqslant \sum_{j=1}^{q} N_{n,f}(r, H_j) + o\big(T_f(r)\big) \\
&\leqslant \frac{k}{k+1}\sum_{j=1}^{q} N_{n,f}^{k)}(r, H_j) + \frac{n}{k+1}\sum_{j=1}^{q} N_f(r, H_j) + o\big(T_f(r)\big) \\
&\leq \frac{k}{k+1}\sum_{j=1}^{q} N_{n,f}^{k)}(r, H_j) + \frac{qn}{k+1}T_f(r) + o\big(T_f(r)\big), \quad r \notin E,
\end{aligned}
$$

which impies that

$$
\Big(q - n - 1 - \frac{qn}{k+1}\Big)T_f(r) \leqslant \frac{k}{k+1}\sum_{j=1}^{q} N_{n,f}^{k)}(r, H_j) + o\big(T_f(r)\big).
$$

Thus, we have

$$\begin{aligned} T_f(r) &\leqslant \frac{k}{(q-n-1)(k+1)-qn}\sum_{j=1}^{q} N^{k)}_{n,f}(r,H_j)+o(T_f(r)) \\ &\leqslant \frac{nk}{(q-n-1)(k+1)-qn}\sum_{j=1}^{q} \overline{N}^{k)}_{f}(r,H_j)+o(T_f(r)) \end{aligned}$$

■

Proof of Theorem 1. Assume that there exist three distinct mappings $f_1, f_2, f_3 \in F_k(\{H_j\}_{j=1}^q, f, p)$. Denote by Q the set which contains all indices $j \in \{1, \dots, q\}$ satisfying $\Phi^l(F^j_{1c}, F^j_{2c}, F^j_{3c}) \not\equiv 0$ for some $c \in \mathcal{C}$ and some $l \in \{1, ..., m\}$. We now prove that

$$\#(\{1, ..., q\} \setminus Q) \geq 3n - 1. \tag{12}$$

For the proof of (12) we distinguish three cases:

Case 1. $1 \leqslant n \leqslant 3, q = 3n + 1, p = 2, k \geq 23n$. Suppose that (12) does not hold, then $\#Q \geq 3$. For each $j_0 \in Q$, by Lemma 2 (with $A = \emptyset$, $p = 2$) we have

$$2\sum_{j=1, j\neq i_0}^{q} \overline{N}^{k)}_{f_i}(r,H_j)+\overline{N}^{k)}_{f_i}(r,H_{j_0}) \leqslant \frac{k+2}{k+1}T(r)+o(T(r)), \quad i=1,2,3. \tag{13}$$

By (13) and Lemma 3 we have

$$\begin{aligned} &\Big(q-n-1-\frac{qn}{k+1}\Big)T_{f_i}(r) \leqslant \frac{nk}{k+1}\sum_{j=1}^{q}\overline{N}^{k)}_{f_i}(r,H_j)+o\big(T_{f_i}(r)\big) \\ &\leqslant \frac{nk(k+2)}{2(k+1)^2}T(r)+\frac{nk}{2(k+1)}\overline{N}^{k)}_{f_i}(r,H_{j_0})+o\big(T(r)\big), \quad i=1,2,3. \end{aligned}$$

Thus, we obtain

$$\Big(q-n-1-\frac{qn}{k+1}\Big)T(r) \leqslant \frac{3nk(k+2)}{2(k+1)^2}T(r)+\frac{nk}{2(k+1)}\sum_{i=1}^{3}\overline{N}^{k)}_{f_i}(r,H_{j_0})+o\big(T(r)\big),$$

which implies

$$\begin{aligned} &\big[2(q-n-1)(k+1)^2-2qn(k+1)-3nk(k+2)\big]T(r) \\ &\leqslant nk(k+1)\sum_{i=1}^{3}\overline{N}^{k)}_{f_i}(r,H_{j_0})+o(T(r)) = 3nk(k+1)\overline{N}^{k)}_{f_i}(r,H_{j_0})+o(T(r)). \end{aligned}$$

Hence, we have

$$\begin{aligned} \liminf_{r\to\infty\ r\notin E}\frac{\overline{N}^{k)}_{f_i}(r,H_{j_0})}{T(r)} &\geq \frac{2(q-n-1)(k+1)^2-2qn(k+1)-3nk(k+2)}{3nk(k+1)} \\ &= \frac{k^2-6nk-6n+2}{3k(k+1)}, \quad i \in \{1,2,3\}. \end{aligned} \tag{14}$$

Set

$$A_i := \{r > 1 : T_{f_i}(r) = \min\{T_{f_1}(r), T_{f_2}(r), T_{f_3}(r)\}\}, \quad i \in \{1, 2, 3\}.$$

Then $A_1 \cup A_2 \cup A_3 = (1, +\infty)$. Without loss of generality, we may assume that the Lebesgue measure of A_1 is infinite. By (14) we have

$$\liminf_{r\to\infty \ r\in A_1\setminus E} \frac{\overline{N}^{k)}_{f_1}(r, H_{j_0})}{T_{f_1}(r)} \geq \frac{k^2 - 6nk - 6n + 2}{k(k+1)}, \; j_0 \in Q.$$

Take three distinct indices $j_1, j_2, j_3 \in Q$ (note that $\#Q \geq 3$). Then we have

$$\liminf_{r\to\infty \ r\in A_1\setminus E} \frac{\overline{N}^{k)}_{f_1}(r, H_{j_1}) + \overline{N}^{k)}_{f_1}(r, H_{j_2}) + \overline{N}^{k)}_{f_1}(r, H_{j_3})}{T_{f_1}(r)} \geq \frac{3(k^2 - 6nk - 6n + 2)}{k(k+1)},$$

which implies that

$$\liminf_{r\to\infty \ r\in A_1\setminus E} \frac{\sum_{j=1}^{q} \overline{N}^{k)}_{f_1}(r, H_j)}{T_{f_1}(r)} \geq \frac{3(k^2 - 6nk - 6n + 2)}{k(k+1)}. \tag{15}$$

Since $f_1 \not\equiv f_2$ there exists $c \in \mathcal{C}$ such that $\frac{(f_1, H_1)}{(f_1, H_c)} \not\equiv \frac{(f_2, H_1)}{(f_2, H_c)}$. Indeed, otherwise by Lemma 1 we have that $\frac{(f_1, H_1)}{(f_1, H)} \equiv \frac{(f_2, H_1)}{(f_2, H)}$ for all hyperplanes H in $\mathbb{C}P^n$. In particular $\frac{(f_1, H_1)}{(f_1, H_j)} \equiv \frac{(f_2, H_1)}{(f_2, H_j)}$ for all $j = 2, ..., n+1$. We choose homogeneous coordinates $(\omega_0 : \cdots : \omega_n)$ on $\mathbb{C}P^n$ with $H_j = \{\omega_j = 0\}$ $(1 \leqslant j \leqslant n+1)$ and take reduced representations:

$$f_1 = (f_{1_1} : \cdots : f_{1_{n+1}}),$$
$$f_2 = (f_{2_1} : \cdots : f_{2_{n+1}}).$$

Then

$$\begin{cases} \dfrac{f_{1_j}}{f_{1_1}} = \dfrac{f_{2_j}}{f_{2_1}} \\ (j = 2, \ldots, n+1) \end{cases} \Rightarrow \quad \frac{f_{1_1}}{f_{2_1}} = \cdots = \frac{f_{1_{n+1}}}{f_{2_{n+1}}}, \quad \text{so} \quad f_1 \equiv f_2.$$

This is a contradiction.

Since $\dim(f_i^{-1}(H_1) \cap f_i^{-1}(H_c)) \leqslant m - 2$ we have

$$\begin{aligned} T_{\frac{(f_i, H_1)}{(f_i, H_c)}}(r) &= \int_{S(r)} \log\left(|(f_i, H_1)|^2 + |(f_i, H_c)|^2\right)^{\frac{1}{2}}\sigma \quad + O(1) \\ &\leqslant \int_{S(r)} \log \|f_i\|\sigma + O(1) = T_{f_i}(r) + O(1), \quad i = 1, 2, 3. \end{aligned}$$

Since $f_1 = f_2$ on $\bigcup_{j=1}^{q} \{z : \nu^{k)}_{(f_1, H_j)}(z) > 0\}$ and

$$\dim\left\{z : v^{k)}_{(f_1,H_i)}(z) > 0 \quad \text{and} \quad v^{k)}_{(f_1,H_j)}(z) > 0\right\} \leqslant m-2 \quad \text{for all } \ i \neq j,$$

we have

$$\sum_{j=1}^{q} \overline{N}^{k)}_{f_1}(r, H_j) \leqslant N\left(r, v_{\frac{(f_1,H_1)}{(f_1,H_c)} - \frac{(f_2,H_1)}{(f_2,H_c)}}\right) \leqslant T_{\frac{(f_1,H_1)}{(f_1,H_c)} - \frac{(f_2,H_1)}{(f_2,H_c)}}(r) \ + 0(1)$$
$$\leqslant T_{\frac{(f_1,H_1)}{(f_1,H_c)}}(r) + T_{\frac{(f_2,H_1)}{(f_2,H_c)}}(r) + 0(1) \leqslant T_{f_1}(r) + T_{f_2}(r) + 0(1),$$

which implies

$$\liminf_{r\to\infty} \frac{T_{f_1}(r) + T_{f_2}(r)}{\sum_{j=1}^{q} \overline{N}^{k)}_{f_1}(r, H_j)} \geq 1.$$

On the other hand, by Lemma 3, we have

$$\left(q - n - 1 - \frac{qn}{k+1}\right) T_{f_i}(r) \leqslant \frac{nk}{k+1} \sum_{j=1}^{q} \overline{N}^{k)}_{f_i}(r, H_j) + o\big(T_{f_i}(r)\big)$$
$$= \frac{nk}{k+1} \sum_{j=1}^{q} \overline{N}^{k)}_{f_1}(r, H_j) + o\big(T_{f_i}(r)\big),$$

which implies

$$\limsup_{r\to\infty \ r\notin E} \frac{T_{f_i}(r)}{\sum_{j=1}^{q} \overline{N}^{k)}_{f_1}(r, H_j)} \leqslant \frac{nk}{(q-n-1)(k+1) - qn}, \quad i = 1, 2, 3.$$

Hence, we obtain

$$\limsup_{r\to\infty \ r\in A_1\setminus E} \frac{T_{f_1}(r)}{\sum_{j=1}^{q} \overline{N}^{k)}_{f_1}(r, H_j)} = \limsup_{r\to\infty \ r\in A_1\setminus E} \left(\frac{T_{f_1}(r) + T_{f_2}(r)}{\sum_{j=1}^{q} \overline{N}^{k)}_{f_1}(r, H_j)} - \frac{T_{f_2}(r)}{\sum_{j=1}^{q} \overline{N}^{k)}_{f_1}(r, H_j)}\right)$$
$$\geq \liminf_{r\to\infty \ r\in A_1\setminus E} \frac{T_{f_1}(r) + T_{f_2}(r)}{\sum_{j=1}^{q} \overline{N}^{k)}_{f_1}(r, H_j)} - \limsup_{r\to\infty \ r\in A_1\setminus E} \frac{T_{f_2}(r)}{\sum_{j=1}^{q} \overline{N}^{k)}_{f_1}(r, H_j)}$$
$$\geq 1 - \frac{nk}{(q-n-1)(k+1) - qn}.$$

Consequently, we get

$$\liminf_{r\to\infty \ r\in A_1\setminus E} \frac{\sum_{j=1}^{q} \overline{N}^{k)}_{f_1}(r, H_j)}{T_{f_1}(r)} \leqslant \frac{(q-n-1)(k+1) - qn}{(q-n-1)(k+1) - qn - nk}$$
$$= \frac{2k + 1 - 3n}{k + 1 - 3n}. \tag{16}$$

By (15) and (16) we have

$$\frac{3(k^2-6nk-6n+2)}{k(k+1)} \leqslant \frac{2k+1-3n}{k+1-3n}.$$

This contradicts $k \geq 23n$. Thus, we get (12) in this case.

Case 2. $4 \leqslant n \leqslant 6$, $q = 3n$, $p = 2$, $k \geq \frac{(6n-1)n}{n-3}$.

Suppose that (12) does not hold, then there exists $j_0 \in Q$. By Lemma 2 (with $A = \emptyset, p = 2$) we have

$$2\sum_{j=1, j\neq j_0}^{3n} \overline{N}_{f_i}^{k)}(r, H_j) + \overline{N}_{f_i}^{k)}(r, H_{j_0}) \leqslant \frac{k+2}{k+1}T(r) + o(T(r)), \quad i = 1, 2, 3.$$

On the other hand, by Lemma 3 we have

$$\sum_{j=1, j\neq j_0}^{3n} \overline{N}_{f_i}^{k)}(r, H_j) + o(T_{f_i}(r)) \geq \frac{(2n-2)(k+1)-(3n-1)n}{nk}T_{f_i}(r),$$

and

$$\sum_{j=1}^{3n} \overline{N}_{f_i}^{k)}(r, H_j) + o(T_{f_i}(r)) \geq \frac{(2n-1)(k+1)-3n^2}{nk}T_{f_i}(r),$$

which implies that

$$2\sum_{j=1, j\neq j_0}^{3n} \overline{N}_{f_i}^{k)}(r, H_j) + \overline{N}_{f_i}^{k)}(r, H_{j_0}) + o(T_{f_i}(r)) \geq \frac{(4n-3)(k+1)-(6n-1)n}{nk}T_{f_i}(r).$$

Hence, we have

$$\frac{(4n-3)(k+1)-(6n-1)n}{nk}T_{f_i}(r) \leqslant \frac{k+2}{k+1}T(r) + o(T(r)), \quad i = 1, 2, 3.$$

Consequently, we get

$$\frac{(4n-3)(k+1)-(6n-1)n}{nk}T(r) \leqslant \frac{3(k+2)}{k+1}T(r) + o(T(r)),$$

which implies that

$$\begin{aligned}\big((4n-3)(k+1)-(6n-1)n\big)T(r) &\leqslant \frac{3nk(k+2)}{k+1}T(r) + o(T(r))\\ &\leqslant 3n(k+1)T(r) + o(T(r).\end{aligned}$$

Hence, we obtain $k+1 \leqslant \frac{(6n-1)n}{n-3}$. This is a contradiction. Thus, we get (12) in this case.

Case 3. $n \geq 7$, $q = 3n-1$, $p = 1$, $k \geq \dfrac{(6n-4)n}{n-6}$.

Suppose that (12) does not hold, then there exists $j_0 \in Q$. By Lemma 2 (with $A = \emptyset, p = 1$) we have

$$2 \sum_{j=1, j \neq j_0}^{3n-1} \overline{N}_{f_i}^{k)}(r, H_j) \leqslant \frac{k+2}{k+1} T(r) + o(T(r)), \quad i = 1, 2, 3,$$

(note that $N_{0,f_i}^{k)}(r, H_{j_0}) = 0$). On the other hand, by Lemma 3, we have

$$2 \sum_{j=1, j \neq j_0}^{3n-1} \overline{N}_{f_i}^{k)}(r, H_j) + o(T_{f_i}(r)) \geq 2 \frac{(2n-3)(k+1) - (3n-2)n}{nk} T_{f_i}(r).$$

Hence, we get

$$\frac{2[(2n-3)(k+1) - (3n-2)n)]}{nk} T_{f_i}(r) \leqslant \frac{k+2}{k+1} T(r) + o(T(r)),$$

which implies

$$((4n-6)(k+1) - (6n-4)n) T_{f_i}(r) \leqslant \frac{nk(k+2)}{k+1} T(r) + o(T(r)), \quad i = 1, 2, 3.$$

Hence, we have

$$\begin{aligned} ((4n-6)(k+1) - (6n-4)n) T(r) &\leqslant \frac{3nk(k+2)}{k+1} T(r) + o(T(r)) \\ &\leqslant 3n(k+1) T(r) + o(T(r)). \end{aligned}$$

Thus, we obtain

$$(4n-6)(k+1) - (6n-4)n \leqslant 3n(k+1)$$

implying

$$k + 1 \leqslant \frac{(6n-4)n}{n-6},$$

which is a contradiction. Thus, we get (12) in this case.

So, in any case we have $\#(\{1, \ldots, q\} \setminus Q) \geq 3n - 1$. Without loss of generality, we may assume that $1, \ldots, 3n-1 \notin Q$. Then we have

$$\Phi^l\big(F_{1c}^j, F_{2c}^j, F_{3c}^j\big) \equiv 0 \quad \text{for all} \quad c \in \mathcal{C}, \ \ l \in \{1, ..., m\}, \ \ j \in \{1, \ldots, 3n-1\}.$$

On the other hand, $\mathcal{C}$ is dense in $\mathbb{C}^{n+1}$. Hence, $\Phi^l\big(F_{1c}^j, F_{2c}^j, F_{3c}^j\big) \equiv 0$ for all $c \in \mathbb{C}^{n+1} \setminus \{0\}$, $l \in \{1, ..., m\}$, $j \in \{1, \ldots, 3n-1\}$. In particular (for $H_c = H_i$) we have

$$\Phi^l \left(\frac{(f_1, H_j)}{(f_1, H_i)}, \ \frac{(f_2, H_j)}{(f_2, H_i)}, \ \frac{(f_3, H_j)}{(f_3, H_i)} \right) \equiv 0$$

for all $1 \leqslant i \neq j \leqslant 3n-1, \quad l \in \{1, \ldots, m\}$. (17)

In the following we distinguish the cases $n = 1$ and $n \geq 2$.

Case 1. If $n = 1$, then $a_j := H_j (j = 1, 2, 3, 4)$ are distinct points in $\mathbb{C}P^1$. We have that

$$g_1 := \frac{(f_1, a_1)}{(f_1, a_2)}, \quad g_2 := \frac{(f_2, a_1)}{(f_2, a_2)}, \quad g_3 := \frac{(f_3, a_1)}{(f_3, a_2)}$$

are distinct nonconstant meromorphic functions. By (17) and by Theorem 2.2, there exist constants α, β such that

$$g_2 = \alpha g_1, \quad g_3 = \beta g_1, (\alpha, \beta \notin \{1, \infty, 0\}, \alpha \neq \beta). \tag{18}$$

We have $v_{(f_1,a_3)} \geq k+1$ on $\{z : (f_1, a_3)(z) = 0\}$. Indeed, otherwise there exists z_0 such that $0 < v_{(f_1,a_3)}(z_0) \leqslant k$. Then $v^{k)}_{(f_i,a_3)}(z_0) > 0$, for all $i \in \{1, 2, 3\}$. We have $(f_1, a_3)(z_0) = (f_2, a_3)(z_0) = 0$ so $f_1(z_0) = f_2(z_0) = a_3^*$, where we denote $a_j^* := (a_{j_1} : -a_{j_0})$ for every point $a_j = (a_{j_0} : a_{j_1}) \in \mathbb{C}P^1$. So $g_1(z_0) = g_2(z_0) = \dfrac{(a_3^*, a_1)}{(a_3^*, a_2)} \neq 0, \infty$ (note that $a_3 \neq a_1$, $a_3 \neq a_2$). So, by (18) we have $\alpha = 1$. This is a contradiction. Thus $v_{(f_1,a_3)} \geq k+1$ on $\{z : (f_1, a_3)(z) = 0\}$. Similarly, $v_{(f_i,a_j)} \geq k+1$ on $\{z : (f_i, a_j)(z) = 0\}$ for $i \in \{1, 2, 3\}$, $j \in \{3, 4\}$.

Set $b_1 = \alpha\dfrac{(a_3^*, a_1)}{(a_3^*, a_2)}$, $b_2 = \dfrac{\alpha}{\beta}\dfrac{(a_3^*, a_1)}{(a_3^*, a_2)}$, $b_3 = \dfrac{(a_3^*, a_1)}{(a_3^*, a_2)}$. Then we have

$$v_{g_2-b_3} = v_{\frac{(f_2,a_3)(a_1^*,a_2)}{(f_2,a_2)(a_3^*,a_2)}} \geq k+1 \text{ on } \{z : (g_2 - b_3)(z) = 0\},$$

$$v_{g_2-b_1} = v_{g_1-\frac{1}{\alpha}b_1} = v_{\frac{(f_1,a_3)(a_1^*,a_2)}{(f_1,a_2)(a_3^*,a_2)}} \geq k+1 \text{ on } \{z : (g_2 - b_1)(z) = 0\}, \text{ and}$$

$$v_{g_2-b_2} = v_{g_3-\frac{\beta}{\alpha}b_2} = v_{\frac{(f_3,a_3)(a_1^*,a_2)}{(f_3,a_2)(a_3^*,a_2)}} \geq k+1 \text{ on } \{z : (g_2 - b_2)(z) = 0\}.$$

Since the points b_1, b_2, b_3 are distinct, by First and Second Main Theorems, we have

$$\begin{aligned} T_{g_2}(r) &\leqslant \sum_{j=1}^{3} \overline{N}\big(r, v_{g_2-b_j}\big) + o\big(T_{g_2}(r)\big) \\ &\leqslant \frac{1}{k+1}\sum_{j=1}^{3} N\left(r, v_{g_2-b_j}\right) + o\big(T_{g_2}(r)\big) \\ &\leqslant \frac{3}{k+1} T_{g_2}(r) + o\big(T_{g_2}(r)\big) \; . \end{aligned}$$

This contradicts $k \geq 23$.

Case 2. If $n \geq 2$, for each $1 \leqslant i \neq j \leqslant 3n - 1$, by (17) and Theorem 2.2, there exists a constant α_{ij} such that

$$\frac{(f_2, H_j)}{(f_2, H_i)} = \alpha_{ij}\frac{(f_1, H_j)}{(f_1, H_i)} \text{ or } \frac{(f_3, H_j)}{(f_3, H_i)} = \alpha_{ij}\frac{(f_1, H_j)}{(f_1, H_i)}$$

or

$$\frac{(f_3,H_j)}{(f_3,H_i)} = \alpha_{ij}\frac{(f_2,H_j)}{(f_2,H_i)}. \tag{19}$$

We now prove that $\alpha_{ij} = 1$ for all $1 \leqslant i \neq j \leqslant 3n-1$. Indeed, if there exists $\alpha_{i_0 j_0} \neq 1$, we may assume without loss of generality that $\frac{(f_2,H_{j_0})}{(f_2,H_{i_0})} = \alpha_{i_0 j_0}\frac{(f_1,H_{j_0})}{(f_1,H_{i_0})}$. On the other hand $f_1 = f_2$ on $\Omega := \bigcup_{j=1}^{q}\{z : v^{k)}_{(f_1,H_j)}(z) > 0\}$. Hence, $(f_1,H_{j_0}) = (f_2,H_{j_0}) = 0$ on $\Omega \setminus f_1^{-1}(H_{i_0})$. So we have

$$\sum_{j=1, j\neq i_0}^{q} \overline{N}^{k)}_{f_1}(r,H_j) \leqslant N\Big(r, v_{\frac{(f_1,H_{j_0})}{(f_1,H_{i_0})}}\Big) + \Big(\overline{N}\big(r, v_{(f_1,H_{i_0})}\big) - \overline{N}^{k)}\big(r, v_{(f_1,H_{i_0})}\big)\Big).$$

Thus, by First and Second Main Theorems, we have

$$\begin{aligned}
(q-n-2)T_{f_1}(r) &\leqslant \sum_{j=1, j\neq i_0}^{q} N_{n,f_1}(r,H_j) + o(T_{f_1}(r))\\
&\leqslant n\sum_{j=1, j\neq i_0}^{q} N_{1,f_1}(r,H_j) + o(T_{f_1}(r))\\
&\leqslant \frac{nk}{k+1}\sum_{j=1, j\neq i_0}^{q} \overline{N}^{k)}_{f_1}(r,H_j) + \frac{n}{k+1}\sum_{j=1, j\neq i_0}^{q} N_{f_1}(r,H_j) + o(T_{f_1}(r))\\
&\leqslant \frac{nk}{k+1}N\Big(r, v_{\frac{(f_1,H_{j_0})}{(f_1,H_{i_0})}}\Big) + \frac{nk}{k+1}\Big(\overline{N}\big(r, v_{(f_1,H_{i_0})}\big) - \overline{N}^{k)}\big(r, v_{(f_1,H_{i_0})}\big)\Big)\\
&\quad + \frac{(q-1)n}{k+1}T_{f_1}(r) + o(T_{f_1}(r))\\
&\leqslant \frac{nk}{k+1}T_{\frac{(f_1,H_{j_0})}{(f_1,H_{i_0})}}(r) + \frac{nk}{(k+1)^2}N_{f_1}(r,H_{i_0}) + \frac{(q-1)n}{k+1}T_{f_1}(r) + o\big(T_{f_1}(r)\big)\\
&\leqslant \Big(\frac{nk}{k+1} + \frac{nk}{(k+1)^2} + \frac{(q-1)n}{k+1}\Big)T_{f_1}(r) + o\big(T_{f_1}(r)\big).
\end{aligned}$$

Thus, we get $(q-n-2) \leqslant \frac{nk}{k+1} + \frac{nk}{(k+1)^2} + \frac{(q-1)n}{k+1} \leqslant n + \frac{qn}{k}$. This contradicts any of the following cases:

i) $2 \leqslant n \leqslant 3$, $q = 3n+1$ and $k \geq 23n$,

ii) $4 \leqslant n \leqslant 6$, $q = 3n$ and $k \geq \frac{(6n-1)n}{n-3}$,

iii) $n \geq 7$, $q = 3n-1$ and $k \geq \frac{(6n-4)n}{n-6}$.

Thus $\alpha_{ij} = 1$ for all $1 \leqslant i \neq j \leqslant 3n-1$.

By (19), for $i = 3n-1, j \in \{1,\dots,3n-2\}$, we may asssume without loss of generality that

$$\frac{(f_1,H_j)}{(f_1,H_{3n-1})} = \frac{(f_2,H_j)}{(f_2,H_{3n-1})},\ j = 1,\dots,n. \tag{20}$$

For $1 \leqslant s < v \leqslant 3$, denote by L_{sv} the set of all $j \in \{1, ..., 3n-2\}$ such that $\frac{(f_s, H_j)}{(f_s, H_{3n-1})} = \frac{(f_v, H_j)}{(f_v, H_{3n-1})}$. By (19) we have $L_{12} \cup L_{23} \cup L_{13} = \{1, ..., 3n-2\}$. So by Dirichlet principle, one of the three sets contains at least n different indices, which are, without loss of generality, $j = 1, ..., n$, which proves (20).

We choose homogeneous coordinates $(\omega_0 : \cdots : \omega_n)$ on $\mathbb{C}P^n$ with $H_j = \{\omega_j = 0\}$ $(1 \leqslant j \leqslant n)$, $H_{3n-1} = \{\omega_0 = 0\}$ and take reduced representations: $f_1 = (f_{1_0} : \cdots : f_{1_n})$, $f_2 = (f_{2_0} : \cdots : f_{2_n})$. Then by (20) we have

$$\begin{cases} \dfrac{f_{1_j}}{f_{1_0}} = \dfrac{f_{2_j}}{f_{2_0}} \\ (j = 1, \ldots, n) \end{cases} \quad \text{so} \quad \frac{f_{1_0}}{f_{2_0}} = \cdots = \frac{f_{1_n}}{f_{2_n}}, \quad \text{hence} \quad f_1 \equiv f_2.$$

This is a contradiction. Thus, for any case we have that f_1, f_2, f_3 can not be distinct. Hence, the proof of Theorem 1 is complete. ■

Proof of Theorem 2. Assume that $\# F_k(\{H_j\}_{j=1}^q, f, 1) \geq 3$. Take arbitrarily three distinct mappings $f_1, f_2, f_3 \in F_k(\{H_j\}_{j=1}^q, f, 1)$. We have to prove that $f_s \times f_v : \mathbb{C}^m \longrightarrow \mathbb{C}P^n \times \mathbb{C}P^n$ is linearly degenerate for all $1 \leqslant s < v \leqslant 3$.

Denote by Q the set which contains all indices $j \in \{1, ..., q\}$ satisfying $\Phi^l\big(F_{1c}^j, F_{2c}^j, F_{3c}^j\big) \not\equiv 0$ for some $c \in \mathcal{C}$. We distinguish two cases n odd and n even.

Case 1. If n is odd, then $q = \frac{5(n+1)}{2}$.

We now pove that $Q = \emptyset$. (21)

Indeed, otherwise there exists $j_0 \in Q$. Then by Lemma 2 (with $A = \emptyset, p = 1$) we have

$$2 \sum_{j=1, j \neq j_0}^{q} \overline{N}_{f_i}^{k)}(r, H_j) \leqslant \frac{k+2}{k+1} T(r) + o(T(r)), \quad i = 1, 2, 3,$$

(note that $N_{0,f_i}^{k)}(r, H_{j_0}) = 0$). On the other hand, by Lemma 3 we have

$$2 \sum_{j=1, j \neq j_0}^{q} \overline{N}_{f_i}^{k)}(r, H_j) + o(T_{f_i}(r)) \geq \frac{2[(q-n-2)(k+1) - (q-1)n]}{nk} T_{f_i}(r), \quad i = 1, 2, 3.$$

Hence, we have

$$\big((2q - 2n - 4)(k+1) - 2(q-1)n\big) T_{f_i}(r) \leqslant \frac{(k+2)nk}{k+1} T(r) + o(T(r)), \quad i = 1, 2, 3,$$

which implies

$$\begin{aligned} \Big((2q - 2n - 4)(k+1) - 2(q-1)n\Big) T(r) &\leqslant \frac{3(k+2)nk}{k+1} T(r) + o(T(r)) \\ &\leqslant 3n(k+1) T(r) + o(T(r)). \end{aligned}$$

Hence, we obtain

$$(2q-2n-4)(k+1)-2(q-1)n \leqslant 3n(k+1)$$

implying

$$k+1 \leqslant (5n+3)n.$$

This is a contradiction. Thus, we get (21).

Case 2. If n is even, then $q=\frac{5n+4}{2}$.

We now prove that $\#Q \leqslant 1$. (22)

Indeed, suppose that this assertion does not hold, then there exist two distinct indices $j_0, j_1 \in Q$. By Lemma 2 (with $A=\emptyset, p=1$) we have

$$2\sum_{j=1, j\neq j_0}^{q} \overline{N}_{f_i}^{k)}(r, H_j) \leqslant \frac{k+2}{k+1}T(r)+o(T(r)), \quad i=1,2,3,$$

which implies that, for $i=1,2,3$

$$2\sum_{j=1, j\neq j_0}^{q}\Big(\overline{N}_{f_i}^{k)}(r,H_j)-\frac{1}{n}N_{n,f_i}^{k)}(r,H_j)\Big) \leqslant \frac{k+2}{k+1}T(r)+o(T(r)) \\ -\frac{2}{n}\sum_{j=1, j\neq i_0}^{q} N_{n,f_i}^{k)}(r,H_j), \quad i=1,2,3.$$

Hence, we get

$$2\sum_{j=1, j\neq j_0}^{q}\sum_{i=1}^{3}\Big(\overline{N}_{f_i}^{k)}(r,H_j)-\frac{1}{n}N_{n,f_i}^{k)}(r,H_j)\Big) \leqslant \frac{3(k+2)}{k+1}T(r)+o(T(r)) \\ -\frac{2}{n}\sum_{j=1, j\neq j_0}^{q}\sum_{i=1}^{3} N_{n,f_i}^{k)}(r,H_j), \tag{23}$$

By Lemma 3 (with $q=\frac{5n+4}{2}$), we have

$$2\sum_{j=1, j\neq j_0}^{q} N_{n,f_i}^{k)}(r,H_j)+o(T_{f_i}(r)) \geq \frac{3n(k+1)-(5n+2)n}{k}T_{f_i}(r), \quad i=1,2,3.$$

Hence, we have

$$\frac{2}{n}\sum_{j=1, j\neq j_0}^{q}\sum_{i=1}^{3} N_{n,f_i}^{k)}(r,H_j)+o(T(r)) \geq \frac{3n(k+1)-(5n+2)n}{nk}T(r). \tag{24}$$

By (23) and (24) we have

$$2\sum_{j=1, j\neq j_0}^{q}\sum_{i=1}^{3}\Big(\overline{N}^{k)}_{f_i}(r,H_j) - \frac{1}{n}N^{k)}_{n,f_i}(r,H_j)\Big) \leqslant \frac{(5n+2)n(k+1)-3n}{nk(k+1)}T(r)+o(T(r))$$
$$\leqslant \frac{5n+2}{k}T(r)+o(T(r)).$$

On the other hand, we obtain

$$\overline{N}^{k)}_{f_i}(r,H_j) - \frac{1}{n}N^{k)}_{n,f_i}(r,H_j) \geq 0 \text{ for all } i\in\{1,2,3\},\quad j\in\{1,...,q\}.$$

Hence, we get

$$\sum_{i=1}^{3}\Big(\overline{N}^{k)}_{f_i}(r,H_j) - \frac{1}{n}N^{k)}_{n,f_i}(r,H_j)\Big)$$
$$\leqslant \frac{5n+2}{k}T(r)+o(T(r)), j\in\{1,...,q\}\setminus\{j_0\}.$$

In particular, we get

$$\sum_{i=1}^{3}\bigg(\overline{N}^{k)}_{f_i}(r,H_{j_1}) - \frac{1}{n}N^{k)}_{n,f_i}(r,H_{j_1})\bigg) \leqslant \frac{5n+2}{k}T(r)+o(T(r)). \tag{25}$$

Set $A_i := \{z\in\mathbb{C}^m : v_{(f_i,H_{j_1})}(z)=1\}$ for $i=1,2,3$. For each $i\in\{1,2,3\}$, we have $\overline{A}_i\setminus A_i \subseteq$ sing $f_i^{-1}(H_{j_1})$. Indeed, otherwise there exists $a\in\big(\overline{A}_i\setminus A_i\big)\cap$ reg $f_i^{-1}(H_{j_1})$. Then $p_0 := v_{(f_i,H_{j_1})}(a)\geq 2$. Since a is a regular point of $f_i^{-1}(H_{j_1})$ we can choose nonzero holomorphic functions h and u on a neighborhood U of a such that dh and u have no zeroes and $(f_i,H_{j_1})\equiv h^{p_0}u$ on U. Since $a\in\overline{A}_i$ there exists $b\in A_i\cap U$. Then, we get $1 = v_{(f_i,H_{j_1})}(b) = v_{h^{p_0}u}(b) = p_0\geq 2$. This is a contradiction.

Thus, we see that $\overline{A}_i\setminus A_i\subseteq$ sing $f_i^{-1}(H_{j_1})$.

Set $B := A_1\cup A_2\cup A_3$. Then $\overline{B}\setminus B\subseteq \bigcup_{i=1}^{3}$ sing $f_i^{-1}(H_{j_1})$. This means that $\overline{B}\setminus B$ is included in an analytic set of codimension ≥ 2. So we have

$$(n-1)\overline{N}(r,\overline{B}) \leqslant \sum_{i=1}^{3}\big(n\overline{N}^{k)}_{f_i}(r,H_{j_1}) - N^{k)}_{n,f_i}(r,H_{j_1})\big).$$

By (25) we have

$$\overline{N}(r,\overline{B}) \leqslant \frac{(5n+2)n}{(n-1)k}T(r)+o(T(r)),$$

where we note that $n\geq 2$, since n is even. It is clear that

$$\min\{v^{k)}_{(f_1,H_{j_1})},2\} = \min\{v^{k)}_{(f_2,H_{j_1})},2\} = \min\{v^{k)}_{(f_3,H_{j_1})},2\} \text{ on } \mathbb{C}^m\setminus\overline{B}(\subseteq\mathbb{C}^m\setminus B).$$

By Lemma 2 (with $A = \overline{B}$, $p = 2$) we have

$$2\sum_{j=1, j\neq j_1}^{q} \overline{N}^{k)}_{f_i}(r, H_j) + \overline{N}^{k)}_{f_i}(r, H_{j_1}) \leqslant \frac{k+2}{k+1}T(r) + 4\overline{N}(r, \overline{B}) + o(T(r))$$
$$\leqslant \Big(\frac{k+2}{k+1} + \frac{4(5n+2)n}{(n-1)k}\Big)T(r) + o(T(r)) \tag{26}$$

(note that $j_1 \in Q$). By Lemma 3 we have

$$\sum_{j=1, j\neq j_1}^{q} \overline{N}^{k)}_{f_i}(r, H_j) + o(T_{f_i}(r)) \geq \frac{(q-n-2)(k+1)-(q-1)n}{nk}T_{f_i}(r), \text{ and}$$
$$\sum_{j=1}^{q} \overline{N}^{k)}_{f_i}(r, H_j) + o(T_{f_i}(r)) \geq \frac{(q-n-1)(k+1)-qn}{nk}T_{f_i}(r).$$

Consequently, we obtain

$$2\sum_{j=1, j\neq j_1}^{q} \overline{N}^{k)}_{f_i}(r, H_j) + \overline{N}^{k)}_{f_i}(r, H_{j_1}) + o(T_{f_i}(r)) \geq \frac{(2q-2n-3)(k+1)-(2q-1)n}{nk}T_{f_i}(r) \tag{27}$$

By (26) and (27) we have

$$\frac{(2q-2n-3)(k+1)-(2q-1)n}{nk}T_{f_i}(r) \leqslant \left(\frac{k+2}{k+1} + \frac{4(5n+2)n}{(n-1)k}\right)T(r) + o(T(r)),$$

which implies

$$\big((3n+1)(k+1)-(5n+3)n\big)T(r) \leqslant \Big(\frac{3nk(k+2)}{k+1} + \frac{12(5n+2)n^2}{(n-1)}\Big)T(r) + o(T(r))$$
$$\leqslant (3n(k+1) + \frac{12(5n+2)n^2}{(n-1)})T(r) + o(T(r)),$$

and hence,

$$k+1 \leqslant (5n+3)n + \frac{12(5n+2)n^2}{(n-1)}.$$

This contradicts $k \geq (65n+171)n$, $n \geq 2$. Hence, we have $\#Q \leqslant 1$. So we get (22).

By (21) and (22) we have $\#(\{1, ..., q\}\setminus Q) \geq q-1$. Without loss of generality we may assume that $1, ..., q-1 \notin Q$. For any $j \in \{1, \ldots, q-1\}$ we have

$\Phi^l\big(F^j_{1c}, F^j_{2c}, F^j_{3c}\big) \equiv 0$ for all $c \in \mathcal{C}$, $l \in \{1, ..., m\}$.

On the other hand, $\mathcal{C}$ is dense in $\mathbb{C}^{n+1}$. Hence, we get that $\Phi^l\big(F^j_{1c}, F^j_{2c}, F^j_{3c}\big) \equiv 0$ for all $c \in \mathbb{C}^{n+1} \setminus \{0\}$, $l \in \{1, ..., m\}$, $j \in \{1, \ldots, q-1\}$. In particular (for $H_c = H_i$), we get

$$\Phi^l\Big(\frac{(f_1,H_j)}{(f_1,H_i)},\ \frac{(f_2,H_j)}{(f_2,H_i)},\ \frac{(f_3,H_j)}{(f_3,H_i)}\Big)\equiv 0$$

for all $1\leqslant i\neq j\leqslant q-1,\quad l\in\{1,\dots,m\}$.

For each $1\leqslant i\neq j\leqslant q-1$, by Theorem 2.2, there exists a constant α_{ij} such that

$$\frac{(f_2,H_j)}{(f_2,H_i)}=\alpha_{ij}\frac{(f_1,H_j)}{(f_1,H_i)}\ \text{or}\ \frac{(f_3,H_j)}{(f_3,H_i)}=\alpha_{ij}\frac{(f_1,H_j)}{(f_1,H_i)}\ \text{or}\ \frac{(f_3,H_j)}{(f_3,H_i)}=\alpha_{ij}\frac{(f_2,H_j)}{(f_2,H_i)}.$$

We now prove that

$\alpha_{ij}=1$ for all $1\leqslant i\neq j\leqslant q-1$. (28)

Indeed, if there exists $\alpha_{i_0j_0}\neq 1$, we may assume without loss of generality that $\frac{(f_2,H_{j_0})}{(f_2,H_{i_0})}=\alpha_{i_0j_0}\frac{(f_1,H_{j_0})}{(f_1,H_{i_0})}$. On the other hand, we have $f_1=f_2$ on $D:=\bigcup_{j=1}^{q}\big\{z:v^{k)}_{(f_1,H_j)}>0\big\}$. Hence, we get $(f_1,H_{j_0})=(f_2,H_{j_0})=0$ on $D\setminus f_1^{-1}(H_{i_0})$. So we have

$$\sum_{j=1,j\neq i_0}^{q}\overline{N}^{k)}_{f_1}(r,H_j)\leqslant N\Big(r,v_{\frac{(f_1,H_{j_0})}{(f_1,H_{i_0})}}\Big)+\Big(\overline{N}\big(r,v_{(f_1,H_{i_0})}\big)-\overline{N}^{k)}\big(r,v_{(f_1,H_{i_0})}\big)\Big).$$

Thus, by First and Second Main Theorems, we have

$$\begin{aligned}
&(q-n-2)T_{f_1}(r)\leqslant\sum_{j=1,j\neq i_0}^{q}N_{n,f_1}(r,H_j)+o(T_{f_1}(r))\\
&\leqslant n\sum_{j=1,j\neq i_0}^{q}N_{1,f_1}(r,H_j)+o(T_{f_1}(r))\\
&\leqslant\frac{nk}{k+1}\sum_{j=1,j\neq i_0}^{q}\overline{N}^{k)}_{f_1}(r,H_j)+\frac{n}{k+1}\sum_{j=1,j\neq i_0}^{q}N_{f_1}(r,H_j)+o(T_{f_1}(r))\\
&\leqslant\frac{nk}{k+1}N\Big(r,v_{\frac{(f_1,H_{j_0})}{(f_1,H_{i_0})}}\Big)+\frac{nk}{k+1}\Big(\overline{N}\big(r,v_{(f_1,H_{i_0})}\big)-\overline{N}^{k)}\big(r,v_{(f_1,H_{i_0})}\big)\Big)\\
&\quad+\frac{(q-1)n}{k+1}T_{f_1}(r)+o(T_{f_1}(r))\\
&\leqslant\frac{nk}{k+1}T_{\frac{(f_1,H_{j_0})}{(f_1,H_{i_0})}}(r)+\frac{nk}{(k+1)^2}N_{f_1}(r,H_{i_0})+\frac{(q-1)n}{k+1}T_{f_1}(r)+o\big(T_{f_1}(r)\big)\\
&\leqslant\Big(\frac{nk}{k+1}+\frac{nk}{(k+1)^2}+\frac{(q-1)n}{k+1}\Big)T_{f_1}(r)+o\big(T_{f_1}(r)\big).
\end{aligned}$$

Thus, we have $(q-n-2)\leqslant\frac{nk}{k+1}+\frac{nk}{(k+1)^2}+\frac{(q-1)n}{k+1}\leqslant n+\frac{nq}{k}$.

This contradicts $q = \left[\frac{5(n+1)}{2}\right], k \geq (65n + 171)n$. Thus, we get that $\alpha_{ij} = 1$ for all $1 \leqslant i \neq j \leqslant q-1$.

For $1 \leqslant s < v \leqslant 3$, denote by L_{sv} the set of all $j \in \{1, ..., q-2\}$ such that $\frac{(f_s, H_j)}{(f_s, H_{q-1})} = \frac{(f_v, H_j)}{(f_v, H_{q-1})}$. By (28), we have $L_{12} \cup L_{23} \cup L_{13} = \{1, ..., q-2\}$.

If there exists some $L_{sv} = \emptyset$, we may assume without loss of generality that $L_{13} = \emptyset$. Then $L_{12} \cup L_{23} = \{1, ..., q-2\}$. Since $q = \left[\dfrac{5(n+1)}{2}\right]$ we have $\#L_{12} \geq n$ or $\#L_{23} \geq n$. We may assume that $\#L_{12} \geq n$, and furthermore $1, ..., n \in L_{12}$. Then $\dfrac{(f_1, H_j)}{(f_1, H_{q-1})} = \dfrac{(f_2, H_j)}{(f_2, H_{q-1})}$ for all $j \in \{1, ..., n\}$, so $f_1 \equiv f_2$ (as in the proof of Theorem 1). This is a contradiction.

Thus, we have $L_{sv} \neq \emptyset$ for all $1 \leqslant s < v \leqslant 3$. Then for any $1 \leqslant s < v \leqslant 3$, there exists $j \in \{1, ..., q-2\}$ such that $\frac{(f_s, H_j)}{(f_s, H_{q-1})} = \dfrac{(f_v, H_j)}{(f_v, H_{q-1})}$. Hence, we finally get that $f_s \times f_v : \mathbb{C}^m \longrightarrow \mathbb{C}P^n \times \mathbb{C}P^n$ is linearly degenerate. We thus have completed the proof of Theorem 2. ■

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