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On the Hyperbolicity of Some Systems of Nonlinear First-Order Partial Differential Equations*

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Abstract. In this paper we study the hyperbolicity of some normal systems of first-order nonlinear partial differential equations, to which some multidimensional Monge-Ampère equations have been reduced in [8]. We prove that when the dimension $n \leqslant 5$ all these systems are weakly hyperbolic.

1. Introduction

We consider the following normal system of $2n+1$ first-order nonlinear partial differential equations with respect to $2n+1$ unknown functions $X(\alpha), Z(\alpha), P(\alpha)$

$$\begin{cases} \dfrac{\partial X_i}{\partial \alpha_n} &= -\sum\limits_{k=1}^{n-1} \frac{\partial X_i}{\partial \alpha_k} + g_i(\alpha), \quad i = 1, 2, \dots, n, \\ \dfrac{\partial Z}{\partial \alpha_n} &= -\sum\limits_{k=1}^{n-1} \frac{\partial Z}{\partial \alpha_k} + \sum\limits_{=1}^{n} g\ (\alpha) P\ (\alpha), \\ \dfrac{\partial P_i}{\partial \alpha_n} &= -\sum\limits_{k=1}^{n-1} \frac{\partial P_i}{\partial \alpha_k} - \sum\limits_{=1}^{n} a_i\ (X(\alpha), Z(\alpha), P(\alpha)) g\ (\alpha), \ \ i = 1, 2, \dots, n, \end{cases} \tag{1.1}$$

where $\alpha \equiv (\alpha_1, \alpha_2, \dots, \alpha_n)$ are independent variables, $X(\alpha) \equiv \big(X_1(\alpha), X_2(\alpha), \dots,$

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$X_n(\alpha)\big), P(\alpha) \equiv \big(P_1(\alpha), P_2(\alpha), \ldots, P_n(\alpha)\big)$ and $a_{ij}(X, Z, P)$ are given smooth functions defined in R^{2n+1},

$$g(\alpha) = (g_1(\alpha), g_2(\alpha), \ldots, g_n(\alpha))^T = v_1(\alpha) \times v_2(\alpha) \times \cdots \times v_{n-1}(\alpha) \in R^n, \tag{1.2}$$

$$\begin{aligned} v_j(\alpha) &= \frac{\partial P}{\partial \alpha_j} + \frac{\partial X}{\partial \alpha_j} A(X(\alpha), Z(\alpha), P(\alpha)) \\ &= (v_{j1}(\alpha), v_{j2}(\alpha), \ldots, v_{jn}(\alpha)) \in R^n, j = 1, 2, \ldots, n-1. \end{aligned} \tag{1.3}$$

where $A(X, Z, P) \equiv [a_{ij}(X, Z, P)]_{n\times n}$, $a_{ij}(X, Z, P)$ are the same as in (1.1),

$$\begin{aligned} \frac{\partial X}{\partial \alpha_j} &= \Big(\frac{\partial X_1}{\partial \alpha_j}, \frac{\partial X_2}{\partial \alpha_j}, \ldots, \frac{\partial X_n}{\partial \alpha_j}\Big) \in R^n, \quad j = 1, 2, \ldots, n. \\ \frac{\partial P}{\partial \alpha_j} &= \Big(\frac{\partial P_1}{\partial \alpha_j}, \frac{\partial P_2}{\partial \alpha_j}, \ldots, \frac{\partial P_n}{\partial \alpha_j}\Big) \in R^n, \quad j = 1, 2, \ldots, n \end{aligned}$$

$$v_1 \times v_2 \times \cdots \times v_{n-1} = \begin{vmatrix} e_1 & e_2 & \ldots & e_{n-1} & e_n \\ v_{11} & v_{12} & \ldots & v_{1,n-1} & v_{1,n} \\ v_{21} & v_{22} & \ldots & v_{2,n-1} & v_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-1,1} & v_{n-2,2} & \ldots & v_{n-1,n} & v_{n-1,n} \end{vmatrix} \in R^n, \tag{1.4}$$

$e_1, e_2, \ldots, e_n$ are unit column-vectors on coordinate axes $Ox_1, Ox_2, \ldots, Ox_n$, respectively.

We note from (1.4) that $g_i(\alpha)$ will be determined in (2.7) by a determinant of order (n-1), whose elements v_{jk} by (2.8), (2.1) and (2.2) are homogenous polynomials of degree 1 with respect to the same derivatives $\frac{\partial X(\alpha)}{\partial \alpha_k}, \frac{\partial P(\alpha)}{\partial \alpha_k}, k = 1, 2, \ldots, n-1$. So all $g_i(\alpha)$ are homogenous polynomials of degree $(n-1)$ with respect to the derivatives $\frac{\partial X(\alpha)}{\partial \alpha_k}, \frac{\partial P(\alpha)}{\partial \alpha_k}, k = 1, 2, \ldots, n-1$ with coefficients depending on $a_{ij}(X(\alpha), Z(\alpha), P(\alpha))$. Therefore the system (1.1) is normal, because all derivatives of the unknowns X, Z, P with respect to the α_n are expressed in terms of their derivatives with respect to the rest variables $\alpha_1, \alpha_2, ..., \alpha_{n-1}$.

In [1-7] the classical hyperbolic Monge-Ampère equations ($n = 2$) has been studied by reducing them to some first-order quasilinear hyperbolic systems (1.1) with 5 equations and 5 unknowns. The Cauchy problem for some hyperbolic or weakly hyperbolic systems had been studied in [11-12].

In [8] we have reduced the following multidimensional Monge-Ampère equation

$$\det\,[z_{x_i x_j} + a_{ij}(x, z, p)]_{n\times n} = 0, \tag{1.5}$$

to the system (1.1), where $x = (x_1, x_2, \ldots, x_n) \in R^n, z = z(x)$ is an unknown function, $p = (p_1, p_2, \ldots, p_n) = (z_{x_1}, z_{x_2}, \ldots, z_{x_n})$. The functions $a_{ij}(x, z, p)$ are the same ones as in (1.1). We have shown in [8] that a solution $(X(\alpha), Z(\alpha), P(\alpha))$ to the system (1.1) with $\left|\frac{DX(\alpha)}{D\alpha}\right| \neq 0$ gives a solution $z(x)$ to the equation (1.5).

The solvability of the Cauchy problem for the equations (1.5) strongly depends on the hyperbolicity of the system (1.1). So, it is important to study the hyperbolicity of the system (1.1).

In the present paper we study the hyperbolicity for the system (1.1). Our main result is Theorem 2.8 which states that when dimension $n \leqslant 5$, the system (1.1) is weakly hyperbolic. Due to a lot of calculations needed, in the case $n \geqslant 6$ we get only particular results. The outline of the paper is the following. In Sec. 2 we recall the notions of weak hyperbolicity and hyperbolicity for (1.1). In the following Secs. 3 - 6 we study the hyperbolicity for the dimensions between 2 and 5. We would like to emphasize that the hyperbolicity takes place only in the case $n = 2$, provided that the matrix $A(x, z, p) = [a_{ij}(x, z, p)]_{2\times 2}$ is not symmetric.

In the paper we use the Maple 7 for symbolic calculations to calculate the products of matrices, determinants, eigenvalues and to simplify algebraic expressions.

2. Hyperbolicity

2.1. Definitions

We introduce the following notations. For $k = 1, 2, \ldots, n-1$, set

$$V_k = (V_{1k}, V_{2k}, \ldots, V_{nk}) \equiv \frac{\partial X}{\partial \alpha_k} = (\frac{\partial X_1}{\partial \alpha_k}, \frac{\partial X_2}{\partial \alpha_k}, \ldots, \frac{\partial X_n}{\partial \alpha_k}), \tag{2.1}$$

$$W_k = (W_{1k}, W_{2k}, \ldots, W_{nk}) \equiv \frac{\partial P}{\partial \alpha_k} = (\frac{\partial P_1}{\partial \alpha_k}, \frac{\partial P_2}{\partial \alpha_k}, \ldots, \frac{\partial P_n}{\partial \alpha_k}), \tag{2.2}$$

and

$$U(\alpha) = (X(\alpha), Z(\alpha), P(\alpha))^T = \begin{bmatrix} X^T(\alpha) \\ Z(\alpha) \\ P^T(\alpha) \end{bmatrix}$$

$$F(\alpha) = -\sum_{=1}^{n-1} \frac{\partial U}{\partial \alpha} + \begin{bmatrix} g(\alpha) \\ \langle g(\alpha), P(\alpha)\rangle \\ -Ag(\alpha) \end{bmatrix}$$

where $\langle .,. \rangle$ stands for the scalar product in R^n.

We can now write system (1.1) in the matrix form

$$\frac{\partial U}{\partial \alpha_n} = F. \tag{2.3}$$

For $j = 1, 2, \ldots, n-1$, we introduce

$$Q_j = \frac{\partial U}{\partial \alpha_j}$$

and

$$\mathcal{A}_j \equiv \frac{DF}{DQ_j} = \begin{bmatrix} \frac{Dg}{DV_j} - E & 0 & \frac{Dg}{DW_j} \\ P\frac{Dg}{DV_j} & -1 & P\frac{Dg}{DW_j} \\ -A\frac{Dg}{DV_j} & 0 & -A\frac{Dg}{DW_j} - E \end{bmatrix}, \tag{2.4}$$

where E is the identity matrix of order n and

$$\frac{Dg}{DV_k} \equiv \left[\frac{\partial g_i}{\partial V_{jk}}\right]_{n\times n} = \begin{bmatrix} \frac{\partial g_1}{\partial V_{1k}} & \frac{\partial g_1}{\partial V_{2k}} & \cdots & \frac{\partial g_1}{\partial V_{nk}} \\ \frac{\partial g_2}{\partial V_{1k}} & \frac{\partial g_2}{\partial V_{2k}} & \cdots & \frac{\partial g_2}{\partial V_{nk}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_n}{\partial V_{1k}} & \frac{\partial g_n}{\partial V_{2k}} & \cdots & \frac{\partial g_n}{\partial V_{nk}} \end{bmatrix},$$

$$\frac{Dg}{DW_k} \equiv \left[\frac{\partial g_i}{\partial W_{jk}}\right]_{n\times n} = \begin{bmatrix} \frac{\partial g_1}{\partial W_{1k}} & \frac{\partial g_1}{\partial W_{2k}} & \cdots & \frac{\partial g_1}{\partial W_{nk}} \\ \frac{\partial g_2}{\partial W_{1k}} & \frac{\partial g_2}{\partial W_{2k}} & \cdots & \frac{\partial g_2}{\partial W_{nk}} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial g_n}{\partial W_{1k}} & \frac{\partial g_n}{\partial W_{2k}} & \cdots & \frac{\partial g_n}{\partial W_{nk}} \end{bmatrix}.$$

Note that each of the matrices A_j depends on $X(\alpha), Z(\alpha), P(\alpha), \frac{\partial X}{\partial \alpha_k}, \frac{\partial P}{\partial \alpha_k}, k = 1, 2, \ldots, n-1$. We recall now the notion of hyperbolicity for the system (2.3).

Definition 2.1. [9, 10]

1) *System (2.3) is said to be weakly hyperbolic if for any given $(X(\alpha), Z(\alpha), P(\alpha)) \in C^1$ and for any $\xi' = (\xi_1, ..., \xi_{n-1}) \in R^{n-1}$, all eigenvalues of the matrix*

$$\mathcal{A} = \sum_{i=1}^{n-1} \xi_i \mathcal{A}_i \tag{2.5}$$

are real.

2) *System (2.3) is said to be hyperbolic if it is weakly hyperbolic and if for any given $(X(\alpha), Z(\alpha), P(\alpha)) \in C^1$ and for any $\xi' = (\xi_1, ..., \xi_{n-1}) \in R^{n-1}$, there exists a basis in R^{2n+1}, consisting of its corresponding smooth left eigenvectors of the matrix $\mathcal{A}$.*

Proposition 2.2. *For each $k = 1, 2, \cdots, n-1$ the matrix $\frac{Dg}{DW_k}$ is anti-symmetrix, i.e.*

$$\left[\frac{Dg}{DW_k}\right]^T = -\frac{Dg}{DW_k}. \tag{2.6}$$

Proof. From (1.2), (1.4) we have

$$g_i = (-1)^{1+i} \times \begin{vmatrix} v_{11} & \ldots & v_{1,i-1} & v_{1,i+1} & \ldots & v_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ v_{k-1,1} & \ldots & v_{k-1,i-1} & v_{k-1,i+1} & \ldots & v_{k-1,n} \\ v_{k1} & \ldots & v_{k,i-1} & v_{k,i+1} & \ldots & v_{k,n} \\ v_{k+1,1} & \ldots & v_{k+1,i-1} & v_{k+1,i+1} & \ldots & v_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{n-1,1} & \ldots & v_{n-1,i-1} & v_{n-1,i+1} & \ldots & v_{n-1,n} \end{vmatrix} \tag{2.7}$$

From (1.3), (2.1) and (2.2) it follows that

$$v_{jm} = W_{mj} + \sum_{h=1}^{n} a_{hm} V_{hj},\ j = 1, \ldots, n-1,\ m = 1, \ldots, n. \tag{2.8}$$

We note that W_{ik}, $k = 1, 2, \ldots, n-1$ do not appear in the expression of each g_i. Therefore,

$$\frac{\partial g_i}{\partial W_{ik}} = 0,\ i = 1, \cdots, n, \quad k = 1, \cdots, n-1. \tag{2.9}$$

If $j < i$, then (2.7) yields

$$\frac{\partial g_i}{\partial W_{jk}} = (-1)^{1+i}$$
$$\times \begin{vmatrix} v_{11} & \ldots & v_{1,j-1} & 0 & v_{1,j+1} & \ldots & v_{1,i-1} & v_{1,i+1} & \ldots & v_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{k-1,1} & \ldots & v_{k-1,j-1} & 0 & v_{k-1,j+1} & \ldots & v_{k-1,i-1} & v_{k-1,i+1} & \ldots & v_{k-1,n} \\ v_{k,1} & \vdots & v_{k,j-1} & 1 & v_{k,j+1} & \vdots & v_{k,i-1} & v_{k,i} & \vdots & v_{k,n} \\ v_{k+1,1} & \ldots & v_{k+1,j-1} & 0 & v_{k+1,j+1} & \ldots & v_{k+1,i-1} & v_{k+1,i+1} & \ldots & v_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ v_{n-1,1} & \ldots & v_{n-1,j-1} & 0 & v_{n-1,j+1} & \ldots & v_{n-1,i-1} & v_{n-1,i+1} & \ldots & v_{n-1,n} \end{vmatrix} \tag{2.10}$$

On the other hand, if $j < i$, then we can rewrite (2.7) as follows

$$g_j = (-1)^{1+j}$$
$$\times \begin{vmatrix} v_{11} & \ldots & v_{1,j-1} & v_{1,j+1} & \ldots & v_{1,i-1} & v_{1,i} & v_{1,i+1} & \ldots & v_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{k-1,1} & \ldots & v_{k-1,j-1} & v_{k-1,j+1} & \ldots & v_{k-1,i-1} & v_{k-1,i} & v_{k-1,i+1} & \ldots & v_{i-1n} \\ v_{k1} & \vdots & v_{k,j-1} & v_{k,j+1} & \vdots & v_{k,i-1} & v_{k,i} & v_{k,i+1} & \vdots & v_{kn} \\ v_{k+1,1} & \ldots & v_{k+1,j-1} & v_{k+1,j+1} & \ldots & v_{k+1,i-1} & v_{j+1,i} & v_{k+1,i+1} & \ldots & v_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{n-1,1} & \ldots & v_{n-1,j-1} & v_{n-1,j+1} & \ldots & v_{n-1,i-1} & v_{n-1,i} & v_{n-1,i+1} & \ldots & v_{n-1,n} \end{vmatrix} \tag{2.11}$$

So from (2.11) we have

$$\frac{\partial g_j}{\partial W_{ik}} = (-1)^{1+j}$$
$$\times \begin{vmatrix} v_{11} & \dots & v_{1,j-1} & v_{1,j+1} & \dots & v_{1,i-1} & 0 & v_{1,i+1} & \dots & v_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{k-1,1} & \dots & v_{k-1,j-1} & v_{k-1,j+1} & \dots & v_{k-1,i-1} & 0 & v_{k-1,i+1} & \dots & v_{k-1,n} \\ v_{k1} & \vdots & v_{k,j-1} & v_{k,j+1} & \vdots & v_{k,i-1} & 1 & v_{k,i+1} & \vdots & v_{kn} \\ v_{k+1,1} & \dots & v_{k+1,j-1} & v_{k+1,j+1} & \dots & v_{k+1,i-1} & 0 & v_{k+1,i+1} & \dots & v_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{n-1,1} & \dots & v_{n-1,j-1} & v_{n-1,j+1} & \dots & v_{n-1,i-1} & 0 & v_{n-1,i+1} & \dots & v_{n-1,n} \end{vmatrix} \tag{2.12}$$

From (2.10) and (2.12) we see that the formula (2.10) is true for $i \neq j$. Moreover, from (2.12), (2.10) it follows that

$$\frac{\partial g_j}{\partial W_{ik}} = (-1)^{1+j}(-1)^{i-j-1}$$
$$\times \begin{vmatrix} v_{11} & \dots & v_{1,j-1} & 0 & v_{1,j+1} & \dots & v_{1,i-1} & v_{1,i+1} & \dots & v_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{k-1,1} & \dots & v_{k-1,j-1} & 0 & v_{k-1,j+1} & \dots & v_{k-1,i-1} & v_{k-1,i+1} & \dots & v_{k-1,n} \\ v_{k,1} & \vdots & v_{k,j-1} & 1 & v_{k,j+1} & \vdots & v_{k,i-1} & v_{k,i} & \vdots & v_{k,n} \\ v_{k+1,1} & \dots & v_{k+1,j-1} & 0 & v_{k+1,j+1} & \dots & v_{k+1,i-1} & v_{j+1,i+1} & \dots & v_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ v_{n-1,1} & \dots & v_{n-1,j-1} & 0 & v_{n-1,j+1} & \dots & v_{n-1,i-1} & v_{n-1,i+1} & \dots & v_{n-1,n} \end{vmatrix}$$
$$= -\frac{\partial g_i}{\partial W_{jk}}.$$

The proposition is proved. ∎

Proposition 2.3. *For $k = 1, 2, \dots, n-1$ we have*

$$\frac{Dg}{DV_k} = \frac{Dg}{DW_k} A^T, \tag{2.13}$$

where $A = [a_{ij}]_{n\times n}$.

Proof. From (2.7) - (2.10) it follows that

$$\frac{\partial g_i}{\partial V_{jk}} = (-1)^{1+i} \times \begin{vmatrix} v_{11} & \dots & v_{1,i-1} & v_{1,i+1} & \dots & v_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ v_{k-1,1} & \dots & v_{k-1,i-1} & v_{k-1,i+1} & \dots & v_{k-1,n} \\ a_{j,1} & \vdots & a_{j,i-1} & a_{j,i+1} & \vdots & a_{j,n} \\ v_{k+1,1} & \dots & v_{k+1,i-1} & v_{k+1,i+1} & \dots & v_{k+1,n} \\ \vdots & & \vdots & \vdots & \vdots & \vdots \\ v_{n-1,1} & \dots & v_{n-1,i-1} & v_{n-1,i+1} & \dots & v_{n-1,n} \end{vmatrix} \tag{2.14}$$

$$= \sum_{h=1}^{n} \frac{\partial g_i}{\partial W_{hk}} a_{jh}.$$

The proposition is proved. ■

Set

$$M \equiv \sum_{k=1}^{n} \xi_k \frac{Dg}{DW_k} = [m_{ij}]_{n\times n}, \tag{2.15}$$

$$M_i = \begin{vmatrix} v_{11} & \dots & v_{1,i-1} & v_{1,i+1} & \dots & v_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \\ v_{k-1,1} & \dots & v_{k-1,i-1} & v_{k-1,i+1} & \dots & v_{k-1,n} \\ v_{k1} & \dots & v_{k,i-1} & v_{k,i+1} & \dots & v_{k,n} \\ v_{k+1,1} & \dots & v_{k+1,i-1} & v_{k+1,i+1} & \dots & v_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{n-1,1} & \dots & v_{n-1,i-1} & v_{n-1,i+1} & \dots & v_{n-1,n} \end{vmatrix}, \tag{2.16}$$

and for $i < j$ denote by M_{ij} the matrix obtained from the matrix M_i by replacing its $(j-1)$-column by the column $[\xi_1\ \xi_2 ...\ \xi_{n-1}]^T$.

Proposition 2.4. *For $i < j$ we have*

$$m_{ij} = (-1)^{1+i} det\ M_{ij}.$$

Proof. From (2.15),

$$m_{ij} = \sum_{k=1}^{n-1} \xi_k \frac{\partial g_i}{\partial W_{jk}}. \tag{2.17}$$

From (2.7) we get

$$\frac{\partial g_i}{\partial W_{jk}} = (-1)^{1+i}$$
$$\times \begin{vmatrix} v_{11} & \dots & v_{1,i-1} & v_{1,i+1} & \dots & v_{1,j-1} & 0 & v_{1,j+1} & \dots & v_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{k-1,1} & \dots & v_{k-1,i-1} & v_{k-1,i+1} & \dots & v_{k-1,j-1} & 0 & v_{k-1,j+1} & \dots & v_{k-1,n} \\ v_{k1} & \vdots & v_{k,i-1} & v_{k,i+1} & \vdots & v_{k,j-1} & 1 & v_{k,i+1} & \vdots & v_{kn} \\ v_{k+1,1} & \dots & v_{k+1,i-1} & v_{k+1,i+1} & \dots & v_{k+1,j-1} & 0 & v_{k+1,j+1} & \dots & v_{k+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{n-1,1} & \dots & v_{n-1,i-1} & v_{n-1,i+1} & \dots & v_{n-1,j-1} & 0 & v_{n-1,j+1} & \dots & v_{n-1,n} \end{vmatrix} \tag{2.18}$$

The proposition follows from (2.17) and (2.18). ■

2.2. Transformation of the Matrix $\mathcal{A}$

Set

$$B = A^T - A,$$
$$C = BM,$$

where M is given by (2.15)

$$\nu = \sum_{i=1}^{n-1} \xi_i.$$

From (2.4) and Proposition 2.3 we have

$$\mathcal{A} = \sum_{i=1}^{n-1} \xi_i \mathcal{A}_i = \begin{bmatrix} \sum_{i=1}^{n-1} \xi_i \frac{Dg}{DW_i} A^T - \Big(\sum_{i=1}^{n-1} \xi_i\Big) E & 0 & \sum_{i=1}^{n-1} \xi_i \frac{Dg}{DW_i} \\ P \sum_{i=1}^{n-1} \xi_i \frac{Dg}{DW_i} A^T & -\sum_{i=1}^{n-1} \xi_i & P \sum_{i=1}^{n-1} \xi_i \frac{Dg}{DW_i} \\ -A \sum_{i=1}^{n-1} \xi_i \frac{Dg}{DW_i} A^T & 0 & -A \sum_{i=1}^{n-1} \xi_i \frac{Dg}{DW_i} - E \sum_{i=1}^{n-1} \xi_i \end{bmatrix}$$
$$= \begin{bmatrix} MA^T - \nu E & 0 & M \\ PMA^T & -\nu & PM \\ -AMA^T & 0 & -AM - \nu E \end{bmatrix}. \tag{2.19}$$

Theorem 2.5. *The matrix $\mathcal{A}$ is similar to the following one*

$$\tilde{\mathcal{A}} = \begin{bmatrix} -E\sum_{i=1}^{n-1}\xi_i & 0 & \sum_{n=1}^{n-1}\xi_i \frac{Dg}{DW_i} \\ 0 & -\sum_{i=1}^{n-1}\xi_i & P\sum_{i=1}^{n-1}\xi_i \frac{Dg}{DW_i} \\ 0 & 0 & (A^T - A)\sum_{i=1}^{n-1}\xi_i \frac{Dg}{DW_i} - E\sum_{n=1}^{n-1}\xi_i \end{bmatrix} \tag{2.20}$$

$$= \begin{bmatrix} -E\nu & 0 & M \\ 0 & -\nu & PM \\ 0 & 0 & C - E\nu \end{bmatrix}$$

Proof. Setting the block matrix

$$\mathcal{D} = \begin{bmatrix} E & 0 & 0 \\ 0 & 1 & 0 \\ -A^T & 0 & E \end{bmatrix},$$

we have

$$\mathcal{D}^{-1} = \begin{bmatrix} E & 0 & 0 \\ 0 & 1 & 0 \\ A^T & 0 & E \end{bmatrix}.$$

It is easy to see that

$$\tilde{\mathcal{A}} = \mathcal{D}^{-1}\mathcal{A}\mathcal{D}.$$

The theorem is proved. ∎

Corollary 2.6. *If $A^T = A$, ($i.e. B = 0$) then the system* (1.1) *is weakly hyperbolic.*

Corollary 2.7. *If all eigenvalues of the matrix $C = BM$ are real, then the system* (1.1) *is weakly hyperbolic.*

We formulate now the main result of the paper.

Theorem 2.8. *For $n = 2$, if $A^T \neq A^T$, i.e. if $a_{12} \neq a_{21}$, then the system (1.1) is hyperbolic. For $n = 3, 4, 5$, it is weakly hyperbolic.*

All the following sections are devoted to the proof of the theorem when $n = 2, n = 3, n = 4$ and $n = 5$. For the last three cases we will prove that all the eigenvalues of the matrices C are real, and therefore, the systems (1.1) in this cases are weakly hyperbolic.

3. Proof of the Theorem 2.8 for the Case $n = 2$

Suppose that $A^T \neq A$, i.e. $a_{12} \neq a_{21}$. We prove that the system (1.1) is hyperbolic.

$1^c irc$) First we prove that all eigenvalues of the matrix $\tilde{\mathcal{A}}$ are real.

From (2.20) we have

$$|\tilde{\mathcal{A}} - \lambda E| = -(\xi_1 + \lambda)^3 \left| (A^T - A)\xi_1 \frac{Dg}{DW_1} - (\xi_1 + \lambda)E \right| \tag{3.1}$$

where

$$\frac{Dg}{DW_1} = \begin{bmatrix} \frac{\partial g_1}{\partial W_{11}} & \frac{\partial g_1}{\partial W_{21}} \\ \frac{\partial g_2}{\partial W_{11}} & \frac{\partial g_2}{\partial W_{21}} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{3.2}$$

$$(A^T - A) = \begin{bmatrix} 0 & a_{21} - a_{12} \\ a_{12} - a_{21} & 0 \end{bmatrix}.$$

$$(A^T - A)\xi_1 \frac{Dg}{DW_1} = \begin{bmatrix} \xi_1(a_{12} - a_{21}) & 0 \\ 0 & \xi_1(a_{12} - a_{21}) \end{bmatrix} = \xi_1(a_{12} - a_{21})E. \tag{3.3}$$

From (3.1), (3.2) we obtain

$$\lambda_1 = \lambda_2 = \lambda_3 = -\xi_1,$$
$$\lambda_4 = \lambda_5 = \xi_1(a_{12} - a_{21} - 1).$$

This means that in the case $n = 2$ the system (1.1) is always weakly hyperbolic.

2°) Suppose that $\xi_1 \neq 0$. Since the martix $\mathcal{A}$ is simmilar to $\tilde{\mathcal{A}}$, to prove the theorem we have to show that there exists a basis of R^5 generated by left eigenvectors ^{1}, $^2, \ldots$ ^{5} of the matrix $\tilde{\mathcal{A}}$.

Lemma 3.1. *Let X_1 be the space of left eigenvectors of the matrix $\tilde{\mathcal{A}}$ corresponding to the eigenvalue $\lambda = -\xi_1$. Then* $\dim X_1 = 3$.

Proof. From (2.20) with $n = 2$ and $\lambda = -\xi_1$ we have

$$\tilde{\mathcal{A}} - \lambda E = \begin{bmatrix} 0 & 0 & \xi_1 \frac{Dg}{DW_1} \\ 0 & 0 & P\xi_1 \frac{Dg}{DW_1} \\ 0 & 0 & (A^T - A)\xi_1 \frac{Dg}{DW_1} \end{bmatrix}$$

Because det $\left[\xi_1 \frac{Dg}{DW}\right] = \xi_1^2 \neq 0$ we have rank$(\tilde{\mathcal{A}} - \lambda E) = 2$. Therefore, $\dim X_1 = 5 - 2 = 3$. ∎

Lemma 3.2. *Let X_2 be the space of left eigenvectors of the matrix $\tilde{\mathcal{A}}$ corresponding to the eigenvalue $\lambda = -\xi_1(a_{12} - a_{21} - 1)$. Then* dim $X_2 = 2$.

Proof. From (2.20) with $n = 2$ and $\lambda = -\xi_1(a_{12} - a_{21} - 1)$ we have

$$\tilde{\mathcal{A}} - \lambda E$$
$$= \begin{bmatrix} -\xi_1(a_{12} - a_{21})E & 0 & \xi \dfrac{Dg}{DW_1} \\ 0 & -\xi_1(a_{12} - a_{21}) & P\xi \dfrac{Dg}{DW_1} \\ 0 & 0 & (A^T - A)\xi_1 \dfrac{Dg}{DW_1} - \xi_1(a_{12} - a_{21})E \end{bmatrix} \tag{3.4}$$

From (3.2), (3.3) and (3.4) we have

$$\tilde{\mathcal{A}} - \lambda E = \begin{bmatrix} -\xi_1(a_{12} - a_{21}) & 0 & 0 & 0 & \xi_1 \\ 0 & -\xi_1(a_{12} - a_{21}) & 0 & -\xi_1 & 0 \\ 0 & 0 & -\xi_1(a_{12} - a_{21}) & -P_2\xi_1 & P_1\xi_1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.5}$$

It is clear that, if $a_{12} \neq a_{21}$, then rank $(\tilde{\mathcal{A}} - \lambda E) = 3$. Therefore, $\dim X_2 = 5 - 3 = 2$. ■

Lemma 3.3. *Suppose that* $\lambda_1 \neq \lambda_4$, $^1 \in X_1$, $^4 \in X_2$. *If* $^1 + {}^4 = 0$, *then* $^1 = 0$, $^4 = 0$.

Proof. Since $^1 \in X_1$,

$$^1\mathcal{A} = \lambda_1 {}^1. \tag{3.6}$$

Analogously,

$$^4 \in X_2 \Rightarrow {}^4\mathcal{A} = \lambda_4 {}^4. \tag{3.7}$$

On the other hand,

$$\lambda_4 {}^4 = \lambda_1 {}^4 + \lambda_4 {}^4 - \lambda_1 {}^4 = \lambda_1 {}^4 + (\lambda_4 - \lambda_1) {}^4. \tag{3.8}$$

From (3.6), (3.7), (3.8) we get

$$({}^1 + {}^4)\mathcal{A} = \lambda_1({}^1 + {}^4) + (\lambda_4 - \lambda_1) {}^4. \tag{3.9}$$

From (3.9) and $^1 + {}^4 = 0$ we have $^4 = 0$ and $^1 = 0$. ■

Continuation of the proof of Theorem 2.8 for $n = 2$

Since $\dim X_1 = 3$, we can choose ^{1}, ^{2}, ^{3} as a basis of X_1. Similarly, since $\dim X_2 = 2$, we can chose ^{4}, ^{5} as a basis of X_2. We prove that the vectors ^{1}, ^{2}, ^{3}, ^{4}, ^{5} are linearly independent. Indeed, suppose that

$$(c_1 {}^1 + c_2 {}^2 + c_3 {}^3) + (c_4 {}^4 + c_5 {}^5) = 0.$$

From Lemma 3.4 it follows that

$$\begin{cases} c_1 {}^1 + c_2 {}^2 + c_3 {}^3 = 0 \\ c_4 {}^4 + c_5 {}^5 = 0 \end{cases}$$

Hence,

$$\begin{cases} c_1 = c_2 = c_3 = 0 \\ c_4 = c_5 = 0 \end{cases}$$

So the vectors ^{1}, ^{2}, ^{3}, ^{4}, ^{5} form a basis of the space $\mathbb{R}^5$. Therefore Theorem 2.8 is proved in the case $n = 2$.

4. Proof of Theorem 2.8 for the Case $n = 3$

Put

$$(A^T - A) = B = [b_{ij}] = \begin{bmatrix} 0 & b_{12} & b_{13} \\ -b_{12} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{bmatrix} \tag{4.1}$$

$$M = \sum_{i=1}^{2} \xi_i \frac{Dg}{DW_i} = \begin{bmatrix} 0 & m_{12} & m_{13} \\ -m_{12} & 0 & m_{23} \\ -m_{13} & -m_{23} & 0 \end{bmatrix} \tag{4.2}$$

$$C = BM = \begin{bmatrix} -b_{12}m_{12} - b_{13}m_{13} & -b_{13}m_{23} & -b_{12}m_{23} \\ -b_{23}m_{13} & -b_{12}m_{12} - b_{23}m_{23} & -b_{12}m_{13} \\ -b_{23}m_{12} & -b_{13}m_{12} & -b_{13}m_{13} - b_{23}m_{23} \end{bmatrix}$$

From (4.3),

$$\begin{aligned} |C - \mu E| = & -\mu\Big[\mu^2 + 2(b_{23}m_{23} + b_{13}m_{13} + b_{12}m_{12})\mu \\ & + [(b_{12}m_{12})^2 + 2(b_{12}m_{12}b_{23}m_{23} + b_{13}m_{13}b_{12}m_{12} + b_{13}m_{13}b_{23}m_{23}) \\ & + (b_{13}m_{13})^2 + (b_{23}m_{23})^2\Big] \\ = & -\mu(\mu + b_{12}m_{12} + b_{23}m_{23} + b_{13}m_{13})^2. \end{aligned}$$

So

$$|C - \mu E| = 0$$

if and only if $-\mu(\mu + b_{12}m_{12} + b_{23}m_{23} + b_{13}m_{13})^2 = 0$. So eigenvalues of the matrix C are the following

$$\begin{cases} \mu_1 = 0, \\ \mu_2 = \mu_3 = -b_{12}m_{12} - b_{23}m_{23} - b_{13}m_{13}. \end{cases}$$

Theorem 2.8 in the case $n = 3$ follows from the Corollary 2.7.

5. Proof of the Theorem 3.8 for the Case $n = 4$

We put

$$B = (A^T - A) = [b_{ij}] = \begin{bmatrix} 0 & b_{12} & b_{13} & b_{14} \\ -b_{12} & 0 & b_{23} & b_{24} \\ -b_{13} & -b_{23} & 0 & b_{34} \\ -b_{14} & -b_{24} & -b_{34} & 0 \end{bmatrix}$$

$$M = \sum_{i=1}^{3} \xi_i \frac{Dg}{DW_i} = [m_{ij}] = \begin{bmatrix} 0 & m_{12} & m_{13} & m_{14} \\ -m_{12} & 0 & m_{23} & m_{24} \\ -m_{13} & -m_{23} & 0 & m_{34} \\ -m_{14} & -m_{24} & -m_{34} & 0 \end{bmatrix}$$

$$C = B \times M = C_1 + C_2,$$

where

$$C_1 = \begin{bmatrix} -b_{12}m_{12} - b_{13}m_{13} - b_{14}m_{14} & -b_{13}m_{23} - b_{14}m_{24} & b_{12}m_{23} - b_{14}m_{34} & 0 \\ -b_{23}m_{13} - b_{24}m_{14} & -b_{12}m_{12} - b_{23}m_{23} - b_{24}m_{24} & -b_{12}m_{13} - b_{24}m_{34} & 0 \\ b_{23}m_{12} - b_{34}m_{14} & -b_{13}m_{12} - b_{34}m_{24} & -b_{13}m_{13} - b_{23}m_{23} - b_{34}m_{34} & 0 \\ b_{24}m_{12} + b_{34}m_{13} & -b_{14}m_{12} + b_{34}m_{23} & -b_{14}m_{13} - b_{24}m_{23} & 0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 0 & 0 & b_{12}m_{24} + b_{13}m_{34} \\ 0 & 0 & 0 & -b_{12}m_{14} + b_{23}m_{34} \\ 0 & 0 & 0 & -b_{13}m_{14} - b_{23}m_{24} \\ 0 & 0 & 0 & -b_{14}m_{14} - b_{24}m_{24} - b_{34}m_{34} \end{bmatrix}$$

We prove that all eigenvalues of the matrix C are real. With the aid of the Maple 7, the eigenvalues of the matrix C are calculated as following

$$\begin{aligned} \mu_1 = \mu_2 = & -\frac{1}{2} b_{14}\, m_{14} - \frac{1}{2} b_{13}\, m_{13} - \frac{1}{2} b_{23}\, m_{23} - \frac{1}{2} b_{34}\, m_{34} \\ & - \frac{1}{2} b_{12}\, m_{12} - \frac{1}{2} b_{24}\, m_{24} + \frac{1}{2} \Delta^{\frac{1}{2}}, \\ \mu_3 = \mu_4 = & -\frac{1}{2} b_{14}\, m_{14} - \frac{1}{2} b_{13}\, m_{13} - \frac{1}{2} b_{23}\, m_{23} - \frac{1}{2} b_{34}\, m_{34} \\ & - \frac{1}{2} b_{12}\, m_{12} - \frac{1}{2} b_{24}\, m_{24} - \frac{1}{2} \Delta^{\frac{1}{2}}, \end{aligned}$$

where

$$\begin{aligned} \Delta := {} & 2\, b_{14}\, m_{14}\, b_{13}\, m_{13} + 2\, b_{14}\, m_{14}\, b_{12}\, m_{12} + {b_{34}}^2\, {m_{34}}^2 + {b_{13}}^2\, {m_{13}}^2 + {b_{14}}^2\, {m_{14}}^2 \\ & + 2\, b_{13}\, m_{13}\, b_{23}\, m_{23} + 2\, b_{14}\, m_{14}\, b_{24}\, m_{24} + {b_{24}}^2\, {m_{24}}^2 + {b_{12}}^2\, {m_{12}}^2 + {b_{23}}^2\, {m_{23}}^2 \\ & + 2\, b_{13}\, m_{13}\, b_{12}\, m_{12} + 2\, b_{23}\, m_{23}\, b_{24}\, m_{24} + 2\, b_{23}\, m_{23}\, b_{12}\, m_{12} - 2\, b_{12}\, b_{34}\, m_{34}\, m_{12} \\ & + 4\, b_{13}\, b_{24}\, m_{34}\, m_{12} + 2\, b_{12}\, m_{12}\, b_{24}\, m_{24} - 4\, m_{14}\, b_{34}\, b_{12}\, m_{23} + 4\, b_{12}\, b_{34}\, m_{24}\, m_{13} \\ & - 4\, b_{23}\, m_{34}\, m_{12}\, b_{14} + 2\, b_{34}\, m_{34}\, b_{24}\, m_{24} + 2\, b_{34}\, m_{34}\, b_{14}\, m_{14} + 2\, b_{23}\, m_{23}\, b_{34}\, m_{34} \\ & - 2\, b_{23}\, m_{23}\, b_{14}\, m_{14} + 2\, b_{13}\, m_{13}\, b_{34}\, m_{34} - 2\, b_{13}\, m_{13}\, b_{24}\, m_{24} + 4\, b_{23}\, m_{24}\, b_{14}\, m_{13} \\ & + 4\, b_{13}\, m_{14}\, b_{24}\, m_{23}. \end{aligned}$$

To prove the theorem we show that $\Delta \geqslant 0$. In fact, we can write Δ as

$$\begin{aligned}\Delta = &-4\,b_{14}\,m_{14}\,b_{23}\,m_{23} - 4\,b_{13}\,m_{13}\,b_{24}\,m_{24} - 4\,b_{34}\,m_{34}\,b_{12}\,m_{12} + 4\,b_{13}\,b_{24}\,m_{34}\,m_{12}\\ &- 4\,m_{14}\,b_{34}\,b_{12}\,m_{23} + 4\,b_{12}\,b_{34}\,m_{24}\,m_{13} - 4\,b_{23}\,m_{34}\,m_{12}\,b_{14} + 4\,b_{23}\,m_{24}\,b_{14}\,m_{13}\\ &+ 4\,b_{13}\,m_{14}\,b_{24}\,m_{23} \qquad (5.1)\\ &+ [b_{14}\,m_{14} + b_{13}\,m_{13} + b_{23}\,m_{23} + b_{34}\,m_{34} + b_{12}\,m_{12} + b_{24}\,m_{24}]^2.\end{aligned}$$

and then transform Δ to the following form

$$\begin{aligned}\Delta = &-4\,(m_{34}\,m_{12} + m_{14}\,m_{23} - m_{24}\,m_{13})\,(b_{12}\,b_{34} + b_{23}\,b_{14} - b_{13}\,b_{24})\\ &+ [b_{14}\,m_{14} + b_{13}\,m_{13} + b_{23}\,m_{23} + b_{34}\,m_{34} + b_{12}\,m_{12} + b_{24}\,m_{24}]^2\end{aligned}$$

To prove $\Delta \geq 0$ we show that

$$m_{34}\,m_{12} + m_{14}\,m_{23} - m_{24}\,m_{13} \equiv 0.$$

From Proposition 2.4 we obtain

$$\begin{aligned}m_{12} &= \xi_1\,\frac{\partial g_1}{\partial W_{21}} + \xi_2\,\frac{\partial g_1}{\partial W_{22}} + \xi_3\,\frac{\partial g_1}{\partial W_{23}} = \det\ M_{12}\\ &= \xi_1\,v_{23}\,v_{34} - \xi_1\,v_{24}\,v_{33} - \xi_2\,v_{13}\,v_{34} + \xi_2\,v_{14}\,v_{33} + \xi_3\,v_{13}\,v_{24} - \xi_3\,v_{14}\,v_{23}. \qquad (5.2)\end{aligned}$$

Analogously, we have

$$m_{34} = v_{11}\,v_{22}\,\xi_3 - v_{11}\,\xi_2\,v_{32} - v_{21}\,v_{12}\,\xi_3 + v_{21}\,\xi_1\,v_{32} + v_{31}\,v_{12}\,\xi_2 - v_{31}\,\xi_1\,v_{22}, \qquad (5.3)$$

$$m_{23} = -v_{11}\,\xi_2\,v_{34} + v_{11}\,v_{24}\,\xi_3 + v_{21}\,\xi_1\,v_{34} - v_{21}\,v_{14}\,\xi_3 - v_{31}\,\xi_1\,v_{24} + v_{31}\,v_{14}\,\xi_2, \qquad (5.4)$$

$$m_{14} = v_{12}\,v_{23}\,\xi_3 - v_{12}\,\xi_2\,v_{33} - v_{22}\,v_{13}\,\xi_3 + v_{22}\,\xi_1\,v_{33} + v_{32}\,v_{13}\,\xi_2 - v_{32}\,\xi_1\,v_{23}, \qquad (5.5)$$

$$m_{13} = v_{12}\,\xi_2\,v_{34} - v_{12}\,v_{24}\,\xi_3 - v_{22}\,\xi_1\,v_{34} + v_{22}\,v_{14}\,\xi_3 + v_{32}\,\xi_1\,v_{24} - v_{32}\,v_{14}\,\xi_2, \qquad (5.6)$$

$$m_{24} = -v_{11}\,v_{23}\,\xi_3 + v_{11}\,\xi_2\,v_{33} + v_{21}\,v_{13}\,\xi_3 - v_{21}\,\xi_1\,v_{33} - v_{31}\,v_{13}\,\xi_2 + v_{31}\,\xi_1\,v_{23}. \qquad (5.7)$$

From (5.2) - (5.7) we obtain

$$m_{12}\,m_{34} + m_{14}\,m_{23} - m_{13}\,m_{24} = 0.$$

So we have proved $\Delta \geq 0$. The Theorem 2.8 in the case $n = 4$ follows from Corollary 2.7.

6. Proof of Theorem 2.8 for the Case $n = 5$

We put

$$B = (A^T - A) = [b_{ij}] = \begin{bmatrix} 0 & b_{12} & b_{13} & b_{14} & b_{15} \\ -b_{12} & 0 & b_{23} & b_{24} & b_{25} \\ -b_{13} & -b_{23} & 0 & b_{34} & b_{35} \\ -b_{14} & -b_{24} & -b_{34} & 0 & b_{45} \\ -b_{15} & -b_{25} & -b_{35} & -b_{45} & 0 \end{bmatrix},$$

$$M = \sum_{i=1}^{4} \xi_i \frac{Dg}{DW_i} = [m_{ij}] = \begin{bmatrix} 0 & m_{12} & m_{13} & m_{14} & m_{15} \\ -m_{12} & 0 & m_{23} & m_{24} & m_{25} \\ -m_{13} & -m_{23} & 0 & m_{34} & m_{35} \\ -m_{14} & -m_{24} & -m_{34} & 0 & m_{45} \\ -m_{15} & -m_{25} & -m_{35} & -m_{45} & 0 \end{bmatrix},$$

$$C := BM = \sum_{i=1}^{5} C_i^T,$$

where

$$C_1 = \begin{bmatrix} -b_{12}m_{12} - b_{13}m_{13} - b_{14}m_{14} - b_{15}m_{15} & 0 & 0 & 0 & 0 \\ -b_{13}m_{23} - b_{14}m_{24} - b_{15}m_{25} & 0 & 0 & 0 & 0 \\ b_{12}m_{23} - b_{14}m_{34} - b_{15}m_{35} & 0 & 0 & 0 & 0 \\ b_{12}m_{24} + b_{13}m_{34} - b_{15}m_{45} & 0 & 0 & 0 & 0 \\ b_{12}m_{25} + b_{13}m_{35} + b_{14}m_{45} & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{6.1}$$

$$C_2 = \begin{bmatrix} 0 & -b_{23}m_{13} - b_{24}m_{14} - b_{25}m_{15} & 0 & 0 & 0 \\ 0 & -b_{12}m_{12} - b_{23}m_{23} - b_{24}m_{24} - b_{25}m_{25} & 0 & 0 & 0 \\ 0 & -b_{12}m_{13} - b_{24}m_{34} - b_{25}m_{35} & 0 & 0 & 0 \\ 0 & -b_{12}m_{14} + b_{23}m_{34} - b_{25}m_{45} & 0 & 0 & 0 \\ 0 & -b_{12}m_{15} + b_{23}m_{35} + b_{24}m_{45} & 0 & 0 & 0 \end{bmatrix}, \tag{6.2}$$

$$C_3 = \begin{bmatrix} 0 & 0 & b_{23}m_{12} - b_{34}m_{14} - b_{35}m_{15} & 0 & 0 \\ 0 & 0 & -b_{13}m_{12} - b_{34}m_{24} - b_{35}m_{25} & 0 & 0 \\ 0 & 0 & -b_{13}m_{13} - b_{23}m_{23} - b_{34}m_{34} - b_{35}m_{35} & 0 & 0 \\ 0 & 0 & -b_{13}m_{14} - b_{23}m_{24} - b_{35}m_{45} & 0 & 0 \\ 0 & 0 & -b_{13}m_{15} - b_{23}m_{25} + b_{34}m_{45} & 0 & 0 \end{bmatrix}, \tag{6.3}$$

$$C_4 = \begin{bmatrix} 0 & 0 & 0 & b_{24}m_{12} + b_{34}m_{13} - b_{45}m_{15} & 0 \\ 0 & 0 & 0 & -b_{14}m_{12} + b_{34}m_{23} - b_{45}m_{25} & 0 \\ 0 & 0 & 0 & -b_{14}m_{13} - b_{24}m_{23} - b_{45}m_{35} & 0 \\ 0 & 0 & 0 & -b_{14}m_{14} - b_{24}m_{24} - b_{34}m_{34} - b_{45}m_{45} & 0 \\ 0 & 0 & 0 & -b_{14}m_{15} - b_{24}m_{25} - b_{34}m_{35} & 0 \end{bmatrix}, \tag{6.4}$$

$$C_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & b_{25}m_{12} + b_{35}m_{13} + b_{45}m_{14} \\ 0 & 0 & 0 & 0 & -b_{15}m_{12} + b_{35}m_{23} + b_{45}m_{24} \\ 0 & 0 & 0 & 0 & -b_{15}m_{13} - b_{25}m_{23} + b_{45}m_{34} \\ 0 & 0 & 0 & 0 & -b_{15}m_{14} - b_{25}m_{24} - b_{35}m_{34} \\ 0 & 0 & 0 & 0 & -b_{15}m_{15} - b_{25}m_{25} - b_{35}m_{35} - b_{45}m_{45} \end{bmatrix}. \tag{6.5}$$

Using Maple 7 we obtain eigenvalues of the matrix C as follows

$$
\begin{aligned}
\mu_1 &= 0,\\
\mu_2 &= \mu_3 = -\frac{1}{2} b_{13} m_{13} - \frac{1}{2} b_{25} m_{25} - \frac{1}{2} b_{15} m_{15} - \frac{1}{2} b_{35} m_{35} - \frac{1}{2} b_{24} m_{24}\\
&\quad - \frac{1}{2} b_{23} m_{23} - \frac{1}{2} b_{12} m_{12} - \frac{1}{2} b_{14} m_{14} - \frac{1}{2} b_{34} m_{34} - \frac{1}{2} b_{45} m_{45} + \frac{1}{2} \Delta^{\frac{1}{2}},\\
\mu_4 &= \mu_5 - \frac{1}{2} b_{13} m_{13} - \frac{1}{2} b_{25} m_{25} - \frac{1}{2} b_{15} m_{15} - \frac{1}{2} b_{35} m_{35} - \frac{1}{2} b_{24} m_{24}\\
&\quad - \frac{1}{2} b_{23} m_{23} - \frac{1}{2} b_{12} m_{12} - \frac{1}{2} b_{14} m_{14} - \frac{1}{2} b_{34} m_{34} - \frac{1}{2} b_{45} m_{45} - \frac{1}{2} \Delta^{\frac{1}{2}},
\end{aligned}
$$

where

$$
\begin{aligned}
\Delta :=\; & 2 b_{35} m_{35} b_{23} m_{23} + {b_{35}}^2 {m_{35}}^2 - 2 b_{34} m_{34} b_{15} m_{15} + 2 b_{24} m_{24} b_{45} m_{45}\\
& - 2 b_{24} m_{24} b_{35} m_{35} - 2 b_{24} m_{24} b_{15} m_{15} + 2 b_{14} m_{14} b_{45} m_{45} - 2 b_{14} m_{14} b_{35} m_{35}\\
& - 2 b_{14} m_{14} b_{25} m_{25} + 2 b_{15} m_{15} b_{12} m_{12} + 2 b_{15} m_{15} b_{14} m_{14} + 2 b_{15} m_{15} b_{35} m_{35}\\
& + {b_{13}}^2 {m_{13}}^2 + 2 b_{34} m_{34} b_{45} m_{45} - 2 b_{34} m_{34} b_{25} m_{25} + 4 b_{34} m_{35} b_{25} m_{24}\\
& + 4 b_{34} m_{35} b_{15} m_{14} + 4 b_{24} m_{25} b_{35} m_{34} + 4 b_{24} m_{25} b_{15} m_{14} + 4 b_{14} m_{15} b_{35} m_{34}\\
& + 4 b_{14} m_{15} b_{25} m_{24} + 2 b_{45} m_{45} b_{35} m_{35} + 2 b_{45} m_{45} b_{25} m_{25} + 2 b_{45} m_{45} b_{15} m_{15}\\
& + 2 b_{13} m_{13} b_{23} m_{23} + 2 b_{13} m_{13} b_{12} m_{12} + 2 b_{13} m_{13} b_{14} m_{14} + 2 b_{13} m_{13} b_{34} m_{34}\\
& + 2 b_{25} m_{25} b_{35} m_{35} + 2 b_{25} m_{25} b_{24} m_{24} + 2 b_{25} m_{25} b_{15} m_{15} + 2 b_{13} m_{13} b_{35} m_{35}\\
& + {b_{24}}^2 {m_{24}}^2 + 2 b_{12} m_{12} b_{14} m_{14} + 2 b_{14} m_{14} b_{34} m_{34} + {b_{23}}^2 {m_{23}}^2 + 2 b_{35} m_{35} b_{34} m_{34}\\
& + 2 b_{24} m_{24} b_{23} m_{23} + 2 b_{24} m_{24} b_{12} m_{12} + 2 b_{24} m_{24} b_{14} m_{14} + 2 b_{24} m_{24} b_{34} m_{34}\\
& + 2 b_{23} m_{23} b_{12} m_{12} + 2 b_{23} m_{23} b_{34} m_{34} + {b_{12}}^2 {m_{12}}^2 + 4 m_{25} m_{14} b_{12} b_{45}\\
& - 4 m_{24} m_{15} b_{12} b_{45} + 4 m_{12} m_{45} b_{14} b_{25} - 4 m_{23} m_{14} b_{12} b_{34} - 2 m_{12} m_{34} b_{12} b_{34}\\
& + 4 b_{14} m_{45} b_{35} m_{13} + 4 b_{12} m_{25} b_{35} m_{13} - 2 m_{23} m_{15} b_{15} b_{23} - 4 m_{12} m_{35} b_{15} b_{23}\\
& - 2 m_{23} m_{14} b_{14} b_{23} - 4 m_{12} m_{34} b_{14} b_{23} + 4 b_{15} m_{25} b_{23} m_{13} + 4 b_{14} m_{24} b_{23} m_{13}\\
& - 2 b_{24} m_{24} m_{13} b_{13} - 4 b_{15} m_{45} b_{34} m_{13} + 4 b_{12} m_{24} b_{34} m_{13} - 2 b_{45} m_{45} m_{13} b_{13}\\
& + 4 m_{35} m_{14} b_{45} b_{13} - 2 b_{25} m_{25} m_{13} b_{13} - 4 m_{34} m_{15} b_{45} b_{13} + {b_{45}}^2 {m_{45}}^2\\
& + 4 m_{23} m_{45} b_{24} b_{35} + 4 m_{12} m_{34} b_{24} b_{13} + 4 m_{12} m_{35} b_{25} b_{13} + 4 m_{23} m_{14} b_{24} b_{13}\\
& - 4 m_{23} m_{15} b_{12} b_{35} - 2 m_{12} m_{35} b_{12} b_{35} + 4 m_{23} m_{15} b_{25} b_{13} + 2 b_{25} m_{25} b_{23} m_{23}\\
& + 2 b_{25} m_{25} b_{12} m_{12} + {b_{15}}^2 {m_{15}}^2 + {b_{14}}^2 {m_{14}}^2 + {b_{34}}^2 {m_{34}}^2 - 2 m_{12} m_{45} b_{12} b_{45}\\
& - 4 m_{34} m_{25} b_{23} b_{45} + 4 m_{35} m_{24} b_{23} b_{45} - 2 m_{23} m_{45} b_{23} b_{45} - 4 m_{23} m_{45} b_{34} b_{25}\\
& - 4 m_{12} m_{45} b_{15} b_{24} + 2 b_{13} m_{13} b_{15} m_{15} + {b_{25}}^2 {m_{25}}^2.
\end{aligned}
$$

We prove that $\Delta \geq 0$. To this end we write Δ in the form

$$\begin{aligned}\Delta = &-4\,b_{34}\,m_{34}\,b_{15}\,m_{15}-4\,b_{24}\,m_{24}\,b_{35}\,m_{35}-4\,b_{24}\,m_{24}\,b_{15}\,m_{15}-4\,b_{14}\,m_{14}\,b_{35}\,m_{35}\\
&-4\,b_{14}\,m_{14}\,b_{25}\,m_{25}-4\,b_{34}\,m_{34}\,b_{25}\,m_{25}+4\,b_{34}\,m_{35}\,b_{25}\,m_{24}+4\,b_{34}\,m_{35}\,b_{15}\,m_{14}\\
&+4\,b_{24}\,m_{25}\,b_{35}\,m_{34}+4\,b_{24}\,m_{25}\,b_{15}\,m_{14}+4\,b_{14}\,m_{15}\,b_{35}\,m_{34}+4\,b_{14}\,m_{15}\,b_{25}\,m_{24}\\
&+4\,m_{25}\,m_{14}\,b_{12}\,b_{45}-4\,m_{24}\,m_{15}\,b_{12}\,b_{45}+4\,m_{12}\,m_{45}\,b_{14}\,b_{25}-4\,m_{23}\,m_{14}\,b_{12}\,b_{34}\\
&-4\,m_{12}\,m_{34}\,b_{12}\,b_{34}+4\,b_{14}\,m_{45}\,b_{35}\,m_{13}+4\,b_{12}\,m_{25}\,b_{35}\,m_{13}-4\,m_{23}\,m_{15}\,b_{15}\,b_{23}\\
&-4\,m_{12}\,m_{35}\,b_{15}\,b_{23}-4\,m_{23}\,m_{14}\,b_{14}\,b_{23}-4\,m_{12}\,m_{34}\,b_{14}\,b_{23}+4\,b_{15}\,m_{25}\,b_{23}\,m_{13}\\
&+4\,b_{14}\,m_{24}\,b_{23}\,m_{13}-4\,b_{24}\,m_{24}\,m_{13}\,b_{13}-4\,b_{15}\,m_{45}\,b_{34}\,m_{13}+4\,b_{12}\,m_{24}\,b_{34}\,m_{13}\\
&-4\,b_{45}\,m_{45}\,m_{13}\,b_{13}+4\,m_{35}\,m_{14}\,b_{45}\,b_{13}-4\,b_{25}\,m_{25}\,m_{13}\,b_{13}-4\,m_{34}\,m_{15}\,b_{45}\,b_{13}\\
&+4\,m_{23}\,m_{45}\,b_{24}\,b_{35}+4\,m_{12}\,m_{34}\,b_{24}\,b_{13}+4\,m_{12}\,m_{35}\,b_{25}\,b_{13}+4\,m_{23}\,m_{14}\,b_{24}\,b_{13}\\
&-4\,m_{23}\,m_{15}\,b_{12}\,b_{35}-4\,m_{12}\,m_{35}\,b_{12}\,b_{35}+4\,m_{23}\,m_{15}\,b_{25}\,b_{13}-4\,m_{12}\,m_{45}\,b_{12}\,b_{45}\\
&-4\,m_{34}\,m_{25}\,b_{23}\,b_{45}+4\,m_{35}\,m_{24}\,b_{23}\,b_{45}-4\,m_{23}\,m_{45}\,b_{23}\,b_{45}-4\,m_{23}\,m_{45}\,b_{34}\,b_{25}\\
&-4\,m_{12}\,m_{45}\,b_{15}\,b_{24}+(b_{13}\,m_{13}+b_{25}\,m_{25}+b_{15}\,m_{15}+b_{35}\,m_{35}+b_{24}\,m_{24}\\
&+b_{23}\,m_{23}+b_{12}\,m_{12}+b_{14}\,m_{14}+b_{34}\,m_{34}+b_{45}\,m_{45})^2.\end{aligned}$$

We can transform Δ to the form

$$\begin{aligned}\Delta = &-4(m_{15}\,m_{23}-m_{13}\,m_{25}-m_{35}\,m_{12})(b_{15}\,b_{23}-b_{13}\,b_{25}-b_{15}\,b_{12})\\
&-4(m_{14}\,m_{35}+m_{45}\,m_{13}-m_{15}\,m_{34})(b_{14}\,b_{35}+b_{45}\,b_{13}-b_{15}\,b_{34})\\
&-4(m_{34}\,m_{25}-m_{35}\,m_{24}+m_{23}\,m_{45})(b_{34}\,b_{25}-b_{35}\,b_{24}+b_{23}\,b_{45})\\
&-4(m_{15}\,m_{24}-m_{25}\,m_{14}+m_{45}\,m_{12})(b_{15}\,b_{24}-b_{25}\,b_{14}+b_{45}\,b_{12})\\
&-4(m_{14}\,m_{23}-m_{13}\,m_{24}+m_{34}\,m_{12})(b_{14}\,b_{23}-b_{13}\,b_{24}+b_{34}\,b_{12})\\
&+(b_{13}\,m_{13}+b_{25}\,m_{25}+b_{15}\,m_{15}+b_{35}\,m_{35}+b_{24}\,m_{24}+b_{23}\,m_{23}\\
&+b_{12}\,m_{12}+b_{14}\,m_{14}+b_{34}\,m_{34}+b_{45}\,m_{45})^2.\end{aligned}\tag{6.6}$$

From Proposition 2.4 we have

$$\begin{aligned}m_{12} =&\xi_1\,\frac{\partial g_1}{\partial W_{21}}+\xi_2\,\frac{\partial g_1}{\partial W_{22}}+\xi_3\,\frac{\partial g_1}{\partial W_{23}}+\xi_4\,\frac{\partial g_1}{\partial W_{24}}=\det\,M_{12}\\
&=\xi_1\,v_{23}\,v_{34}\,v_{45}-\xi_1\,v_{23}\,v_{35}\,v_{44}-\xi_1\,v_{33}\,v_{24}\,v_{45}+\xi_1\,v_{33}\,v_{25}\,v_{44}\\
&+\xi_1\,v_{43}\,v_{24}\,v_{35}-\xi_1\,v_{43}\,v_{25}\,v_{34}-\xi_2\,v_{13}\,v_{34}\,v_{45}+\xi_2\,v_{13}\,v_{35}\,v_{44}\\
&+\xi_2\,v_{33}\,v_{14}\,v_{45}-\xi_2\,v_{33}\,v_{15}\,v_{44}-\xi_2\,v_{43}\,v_{14}\,v_{35}+\xi_2\,v_{43}\,v_{15}\,v_{34}\\
&+\xi_3\,v_{13}\,v_{24}\,v_{45}-\xi_3\,v_{13}\,v_{25}\,v_{44}-\xi_3\,v_{23}\,v_{14}\,v_{45}+\xi_3\,v_{23}\,v_{15}\,v_{44}\\
&+\xi_3\,v_{43}\,v_{14}\,v_{25}-\xi_3\,v_{43}\,v_{15}\,v_{24}-\xi_4\,v_{13}\,v_{24}\,v_{35}+\xi_4\,v_{13}\,v_{25}\,v_{34}\\
&+\xi_4\,v_{23}\,v_{14}\,v_{35}-\xi_4\,v_{23}\,v_{15}\,v_{34}-\xi_4\,v_{33}\,v_{14}\,v_{25}+\xi_4\,v_{33}\,v_{15}\,v_{24},\end{aligned}\tag{6.7}$$

$$\begin{aligned}m_{34} &= v_{11}\,v_{22}\,\xi_3\,v_{45} - v_{11}\,v_{22}\,v_{35}\,\xi_4 - v_{11}\,v_{32}\,\xi_2\,v_{45} + v_{11}\,v_{32}\,v_{25}\,\xi_4\\ &+ v_{11}\,v_{42}\,\xi_2\,v_{35} - v_{11}\,v_{42}\,v_{25}\,\xi_3 - v_{21}\,v_{12}\,\xi_3\,v_{45} + v_{21}\,v_{12}\,v_{35}\,\xi_4\\ &+ v_{21}\,v_{32}\,\xi_1\,v_{45} - v_{21}\,v_{32}\,v_{15}\,\xi_4 - v_{21}\,v_{42}\,\xi_1\,v_{35} + v_{21}\,v_{42}\,v_{15}\,\xi_3\\ &+ v_{31}\,v_{12}\,\xi_2\,v_{45} - v_{31}\,v_{12}\,v_{25}\,\xi_4 - v_{31}\,v_{22}\,\xi_1\,v_{45} + v_{31}\,v_{22}\,v_{15}\,\xi_4\\ &+ v_{31}\,v_{42}\,\xi_1\,v_{25} - v_{31}\,v_{42}\,v_{15}\,\xi_2 - v_{41}\,v_{12}\,\xi_2\,v_{35} + v_{41}\,v_{12}\,v_{25}\,\xi_3\\ &+ v_{41}\,v_{22}\,\xi_1\,v_{35} - v_{41}\,v_{22}\,v_{15}\,\xi_3 - v_{41}\,v_{32}\,\xi_1\,v_{25} + v_{41}\,v_{32}\,v_{15}\,\xi_2,\end{aligned} \tag{6.8}$$

$$\begin{aligned}m_{23} &= -v_{11}\,\xi_2\,v_{34}\,v_{45} + v_{11}\,\xi_2\,v_{35}\,v_{44} + v_{11}\,\xi_3\,v_{24}\,v_{45} - v_{11}\,\xi_3\,v_{25}\,v_{44}\\ &- v_{11}\,\xi_4\,v_{24}\,v_{35} + v_{11}\,\xi_4\,v_{25}\,v_{34}\\ &+ v_{21}\,\xi_1\,v_{34}\,v_{45} - v_{21}\,\xi_1\,v_{35}\,v_{44} - v_{21}\,\xi_3\,v_{14}\,v_{45} + v_{21}\,\xi_3\,v_{15}\,v_{44}\\ &+ v_{21}\,\xi_4\,v_{14}\,v_{35} - v_{21}\,\xi_4\,v_{15}\,v_{34}\\ &- v_{31}\,\xi_1\,v_{24}\,v_{45} + v_{31}\,\xi_1\,v_{25}\,v_{44} + v_{31}\,\xi_2\,v_{14}\,v_{45} - v_{31}\,\xi_2\,v_{15}\,v_{44}\\ &- v_{31}\,\xi_4\,v_{14}\,v_{25} + v_{31}\,\xi_4\,v_{15}\,v_{24} + v_{41}\,\xi_1\,v_{24}\,v_{35} - v_{41}\,\xi_1\,v_{25}\,v_{34}\\ &- v_{41}\,\xi_2\,v_{14}\,v_{35} + v_{41}\,\xi_2\,v_{15}\,v_{34} + v_{41}\,\xi_3\,v_{14}\,v_{25} - v_{41}\,\xi_3\,v_{15}\,v_{24},\end{aligned} \tag{6.9}$$

$$\begin{aligned}m_{14} &= v_{12}\,v_{23}\,\xi_3\,v_{45} - v_{12}\,v_{23}\,v_{35}\,\xi_4 - v_{12}\,v_{33}\,\xi_2\,v_{45} + v_{12}\,v_{33}\,v_{25}\,\xi_4\\ &+ v_{12}\,v_{43}\,\xi_2\,v_{35} - v_{12}\,v_{43}\,v_{25}\,\xi_3 - v_{22}\,v_{13}\,\xi_3\,v_{45} + v_{22}\,v_{13}\,v_{35}\,\xi_4\\ &+ v_{22}\,v_{33}\,\xi_1\,v_{45} - v_{22}\,v_{33}\,v_{15}\,\xi_4 - v_{22}\,v_{43}\,\xi_1\,v_{35} + v_{22}\,v_{43}\,v_{15}\,\xi_3\\ &+ v_{32}\,v_{13}\,\xi_2\,v_{45} - v_{32}\,v_{13}\,v_{25}\,\xi_4 - v_{32}\,v_{23}\,\xi_1\,v_{45} + v_{32}\,v_{23}\,v_{15}\,\xi_4\\ &+ v_{32}\,v_{43}\,\xi_1\,v_{25} - v_{32}\,v_{43}\,v_{15}\,\xi_2 - v_{42}\,v_{13}\,\xi_2\,v_{35} + v_{42}\,v_{13}\,v_{25}\,\xi_3\\ &+ v_{42}\,v_{23}\,\xi_1\,v_{35} - v_{42}\,v_{23}\,v_{15}\,\xi_3 - v_{42}\,v_{33}\,\xi_1\,v_{25} + v_{42}\,v_{33}\,v_{15}\,\xi_2,\end{aligned} \tag{6.10}$$

$$\begin{aligned}m_{13} &= v_{12}\,\xi_2\,v_{34}\,v_{45} - v_{12}\,\xi_2\,v_{35}\,v_{44} - v_{12}\,\xi_3\,v_{24}\,v_{45} + v_{12}\,\xi_3\,v_{25}\,v_{44}\\ &+ v_{12}\,\xi_4\,v_{24}\,v_{35} - v_{12}\,\xi_4\,v_{25}\,v_{34} - v_{22}\,\xi_1\,v_{34}\,v_{45} + v_{22}\,\xi_1\,v_{35}\,v_{44}\\ &+ v_{22}\,\xi_3\,v_{14}\,v_{45} - v_{22}\,\xi_3\,v_{15}\,v_{44} - v_{22}\,\xi_4\,v_{14}\,v_{35} + v_{22}\,\xi_4\,v_{15}\,v_{34}\\ &+ v_{32}\,\xi_1\,v_{24}\,v_{45} - v_{32}\,\xi_1\,v_{25}\,v_{44} - v_{32}\,\xi_2\,v_{14}\,v_{45} + v_{32}\,\xi_2\,v_{15}\,v_{44}\\ &+ v_{32}\,\xi_4\,v_{14}\,v_{25} - v_{32}\,\xi_4\,v_{15}\,v_{24} - v_{42}\,\xi_1\,v_{24}\,v_{35} + v_{42}\,\xi_1\,v_{25}\,v_{34}\\ &+ v_{42}\,\xi_2\,v_{14}\,v_{35} - v_{42}\,\xi_2\,v_{15}\,v_{34} - v_{42}\,\xi_3\,v_{14}\,v_{25} + v_{42}\,\xi_3\,v_{15}\,v_{24},\end{aligned} \tag{6.11}$$

$$\begin{aligned}m_{24} &= -v_{11}\,v_{23}\,\xi_3\,v_{45} + v_{11}\,v_{23}\,v_{35}\,\xi_4 + v_{11}\,v_{33}\,\xi_2\,v_{45} - v_{11}\,v_{33}\,v_{25}\,\xi_4\\ &- v_{11}\,v_{43}\,\xi_2\,v_{35} + v_{11}\,v_{43}\,v_{25}\,\xi_3 + v_{21}\,v_{13}\,\xi_3\,v_{45} - v_{21}\,v_{13}\,v_{35}\,\xi_4\\ &- v_{21}\,v_{33}\,\xi_1\,v_{45} + v_{21}\,v_{33}\,v_{15}\,\xi_4 + v_{21}\,v_{43}\,\xi_1\,v_{35} - v_{21}\,v_{43}\,v_{15}\,\xi_3\\ &- v_{31}\,v_{13}\,\xi_2\,v_{45} + v_{31}\,v_{13}\,v_{25}\,\xi_4 + v_{31}\,v_{23}\,\xi_1\,v_{45} - v_{31}\,v_{23}\,v_{15}\,\xi_4\\ &- v_{31}\,v_{43}\,\xi_1\,v_{25} + v_{31}\,v_{43}\,v_{15}\,\xi_2 + v_{41}\,v_{13}\,\xi_2\,v_{35} - v_{41}\,v_{13}\,v_{25}\,\xi_3\\ &- v_{41}\,v_{23}\,\xi_1\,v_{35} + v_{41}\,v_{23}\,v_{15}\,\xi_3 + v_{41}\,v_{33}\,\xi_1\,v_{25} - v_{41}\,v_{33}\,v_{15}\,\xi_2,\end{aligned} \tag{6.12}$$

$$\begin{aligned} m_{15} = {} & v_{12}\,v_{23}\,v_{34}\,\xi_4 - \xi_{12}\,\xi_{23}\,\xi_3\,v_{44} - v_{12}\,v_{33}\,v_{24}\,\xi_4 + v_{12}\,v_{33}\,\xi_2\,v_{44} \\ & + v_{12}\,v_{43}\,v_{24}\,\xi_3 - v_{12}\,v_{43}\,\xi_2\,v_{34} - v_{22}\,v_{13}\,v_{34}\,\xi_4 + v_{22}\,v_{13}\,\xi_3\,v_{44} \\ & + v_{22}\,v_{33}\,v_{14}\,\xi_4 - v_{22}\,v_{33}\,\xi_1\,v_{44} - v_{22}\,v_{43}\,v_{14}\,\xi_3 + v_{22}\,v_{43}\,\xi_1\,v_{34} \\ & + v_{32}\,v_{13}\,v_{24}\,\xi_4 - v_{32}\,v_{13}\,\xi_2\,v_{44} - v_{32}\,v_{23}\,v_{14}\,\xi_4 + v_{32}\,v_{23}\,\xi_1\,v_{44} \\ & + v_{32}\,v_{43}\,v_{14}\,\xi_2 - v_{32}\,v_{43}\,\xi_1\,v_{24} - v_{42}\,v_{13}\,v_{24}\,\xi_3 + v_{42}\,v_{13}\,\xi_2\,v_{34} \\ & + v_{42}\,v_{23}\,v_{14}\,\xi_3 - v_{42}\,v_{23}\,\xi_1\,v_{34} - v_{42}\,v_{33}\,v_{14}\,\xi_2 + v_{42}\,v_{33}\,\xi_1\,v_{24}, \end{aligned} \tag{6.13}$$

$$\begin{aligned} m_{25} = {} & -v_{11}\,v_{23}\,v_{34}\,\xi_4 + v_{11}\,v_{23}\,\xi_3\,v_{44} + v_{11}\,v_{33}\,v_{24}\,\xi_4 - v_{11}\,v_{33}\,\xi_2\,v_{44} \\ & - v_{11}\,v_{43}\,v_{24}\,\xi_3 + v_{11}\,v_{43}\,\xi_2\,v_{34} + v_{21}\,v_{13}\,v_{34}\,\xi_4 - v_{21}\,v_{13}\,\xi_3\,v_{44} \\ & - v_{21}\,v_{33}\,v_{14}\,\xi_4 + v_{21}\,v_{33}\,\xi_1\,v_{44} + v_{21}\,v_{43}\,v_{14}\,\xi_3 - v_{21}\,v_{43}\,\xi_1\,v_{34} \\ & - v_{31}\,v_{13}\,v_{24}\,\xi_4 + v_{31}\,v_{13}\,\xi_2\,v_{44} + v_{31}\,v_{23}\,v_{14}\,\xi_4 - v_{31}\,v_{23}\,\xi_1\,v_{44} \\ & - v_{31}\,v_{43}\,v_{14}\,\xi_2 + v_{31}\,v_{43}\,\xi_1\,v_{24} + v_{41}\,v_{13}\,v_{24}\,\xi_3 - v_{41}\,v_{13}\,\xi_2\,v_{34} \\ & - v_{41}\,v_{23}\,v_{14}\,\xi_3 + v_{41}\,v_{23}\,\xi_1\,v_{34} + v_{41}\,v_{33}\,v_{14}\,\xi_2 - v_{41}\,v_{33}\,\xi_1\,v_{24}, \end{aligned} \tag{6.14}$$

$$\begin{aligned} m_{35} = {} & v_{11}\,v_{22}\,v_{34}\,\xi_4 - v_{11}\,v_{22}\,\xi_3\,v_{44} - v_{11}\,v_{32}\,v_{24}\,\xi_4 + v_{11}\,v_{32}\,\xi_2\,v_{44} \\ & + v_{11}\,v_{42}\,v_{24}\,\xi_3 - v_{11}\,v_{42}\,\xi_2\,v_{34} - v_{21}\,v_{12}\,v_{34}\,\xi_4 + v_{21}\,v_{12}\,\xi_3\,v_{44} \\ & + v_{21}\,v_{32}\,v_{14}\,\xi_4 - v_{21}\,v_{32}\,\xi_1\,v_{44} - v_{21}\,v_{42}\,v_{14}\,\xi_3 + v_{21}\,v_{42}\,\xi_1\,v_{34} \\ & + v_{31}\,v_{12}\,v_{24}\,\xi_4 - v_{31}\,v_{12}\,\xi_2\,v_{44} - v_{31}\,v_{22}\,v_{14}\,\xi_4 + v_{31}\,v_{22}\,\xi_1\,v_{44} \\ & + v_{31}\,v_{42}\,v_{14}\,\xi_2 - v_{31}\,v_{42}\,\xi_1\,v_{24} - v_{41}\,v_{12}\,v_{24}\,\xi_3 + v_{41}\,v_{12}\,\xi_2\,v_{34} \\ & + v_{41}\,v_{22}\,v_{14}\,\xi_3 - v_{41}\,v_{22}\,\xi_1\,v_{34} - v_{41}\,v_{32}\,v_{14}\,\xi_2 + v_{41}\,v_{32}\,\xi_1\,v_{24}, \end{aligned} \tag{6.15}$$

$$\begin{aligned} m_{45} = {} & -v_{11}\,v_{22}\,v_{33}\,\xi_4 + v_{11}\,v_{22}\,\xi_3\,v_{43} + v_{11}\,v_{32}\,v_{23}\,\xi_4 - v_{11}\,v_{32}\,\xi_2\,v_{43} \\ & - v_{11}\,v_{42}\,v_{23}\,\xi_3 + v_{11}\,v_{42}\,\xi_2\,v_{33} + v_{21}\,v_{12}\,v_{33}\,\xi_4 - v_{21}\,v_{12}\,\xi_3\,v_{43} \\ & - v_{21}\,v_{32}\,v_{13}\,\xi_4 + v_{21}\,v_{32}\,\xi_1\,v_{43} + v_{21}\,v_{42}\,v_{13}\,\xi_3 - v_{21}\,v_{42}\,\xi_1\,v_{33} \\ & - v_{31}\,v_{12}\,v_{23}\,\xi_4 + v_{31}\,v_{12}\,\xi_2\,v_{43} + v_{31}\,v_{22}\,v_{13}\,\xi_4 - v_{31}\,v_{22}\,\xi_1\,v_{43} \\ & - v_{31}\,v_{42}\,v_{13}\,\xi_2 + v_{31}\,v_{42}\,\xi_1\,v_{23} + v_{41}\,v_{12}\,v_{23}\,\xi_3 - v_{41}\,v_{12}\,\xi_2\,v_{33} \\ & - v_{41}\,v_{22}\,v_{13}\,\xi_3 + v_{41}\,v_{22}\,\xi_1\,v_{33} + v_{41}\,v_{32}\,v_{13}\,\xi_2 - v_{41}\,v_{32}\,\xi_1\,v_{23}. \end{aligned} \tag{6.16}$$

From (6.6)-(6.16) we obtain

$$\begin{aligned} &(m_{15}\,m_{23} - m_{13}\,m_{25} - m_{35}\,m_{12}) = 0, \\ &(m_{14}\,m_{35} + m_{45}\,m_{13} - m_{15}\,m_{34}) = 0, \\ &(m_{34}\,m_{25} - m_{35}\,m_{24} + m_{23}\,m_{45}) = 0, \\ &(m_{15}\,m_{24} - m_{25}\,m_{14} + m_{45}\,m_{12}) = 0, \\ &(m_{14}\,m_{23} - m_{13}\,m_{24} + m_{34}\,m_{12}) = 0. \end{aligned} \tag{6.17}$$

From (6.6) and (6.17) we have

$$\Delta \geq 0.$$

Therefore, we obtained that all eigenvalues of the matrix C are real. Theorem 2.8 in the case $n = 5$ follows from Corollary 2.6.

Thus Theorem 2.8 is completely proved. ∎

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