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Some Results on the Properties $D_3(f)$ and $D_4(f)$

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Abstract. The aim of this paper is to give characterizations of subspaces and quotients of $\ell^\infty(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty)$ and $\ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty)$-spaces which are an extension of results of Apiola [1] for the non-nuclear case.

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1. Introduction

In a series of important papers (see [1- 5, 9]) Vogt and Wagner studied characterizations of subspaces and quotients of nuclear power series spaces. Later Apiola in [1] has given a characterization of subspaces and quotients of nuclear $L_f(\alpha,\infty)$-spaces. Namely, he proved that a Frechet space E is isomorphic to a subspace (resp. quotient) of a stable nuclear $L_f(\alpha,\infty)$-space if and only if E is $\Lambda(f,\alpha,\mathbb{N})$-nuclear in the sense of Ramanujan and Rosenberger (see [3]) and $E\in D_3(f)$ (see Theorem 3.2 in [1]) (resp $E\in D_4(f)$, see Theorem 3.4 in [1]). In this paper we investigate the Apiola's results for the non-nuclear case. Namely we prove the following result.

Main theorem. *Let E be a Frechet space. Then*

(i) *E has $D_3(f)$ property if and only if there exists an index set I such that E is isomorphic to a subspace of $\ell^\infty(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty)$-space for every stable nuclear exponent sequence $\alpha=(\alpha_j)$.*

(ii) *E has $D_4(f)$ property if and only if there exists an index set I such that*

E is a quotient of $\ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty)$-space for every stable nuclear exponent sequence $\alpha = (\alpha_j)$.

Notice that when $f(t) = t$ for $t \geqslant 0$ and $\alpha = (\log(j+1))_j$ the above theorem has been proved by Vogt [5]. This paper is organized as follows. Beside the introduction the paper contains three sections. In the second section we recall some backgrounds concerning $L_f(\alpha,\infty)$-spaces and $D_3(f)$ and $D_4(f)$ properties. Some results of Apiola in [1] are presented also in this section. The third one is devoted to prove some auxiliary results which are used for the proof of Main Theorem. The proof of Main Theorem is in the fourth section.

2. Backgrounds

2.1. Recall that a real function f on $[0,+\infty)$ is called a Dragilev function if f is rapidly increasing and logarithmically convex. This means that

$$\lim_{t\to+\infty}\frac{f(at)}{f(t)} = \infty \text{ for all } a > 1 \text{ and } t \mapsto \log f(e^t)$$

is convex.
Since f is rapidly increasing then there exists $R > 0$ such that

$$f^{-1}(Mt) \leqslant RMf^{-1}(t) \quad \forall t \geqslant 0; \forall M \geqslant 1$$

(see [1]).
For each Dragilev function f and each exponent sequence $\alpha = (\alpha_j)$, i.e $0 < \alpha_j \leqslant \alpha_{j+1}$ for $j \geqslant 1$ and $\lim\limits_{t\to+\infty}\alpha_j = +\infty$ we define

$$L_f(\alpha,\infty) = \{\xi = (\xi_j) \subset \mathbb{C} : \|\xi\|_k = \sum_{j\geqslant 1}|\xi_j|e^{f(k\alpha_j)}\} < \infty \quad \forall k \geqslant 1.$$

2.2. Let E be a Frechet space with a fundamental system of semi-norms $\|.\|_1 \leqslant \|.\|_2 \leqslant \dots$ and f a Dragilev function.

We say that E has the property $D_3(f)$ if there exists p such that for every $M \geqslant 1$ and every $q \geqslant p$, there exists $k \geqslant q$ such that

$$Mf^{-1}\big(\log(\|x\|_q \diagup \|x\|_p)\big) \leqslant f^{-1}\big(\log(\|x\|_k \diagup \|x\|_q)\big)$$

for all $x \in E \setminus \{0\}$.

We say that E has the property $D_4(f)$ if for every p there exists $q \geqslant p$, and for every $k \geqslant q$ there exists $M \geqslant 1$ such that

$$f^{-1}\big(\log(\|u\|_q^* \diagup \|u\|_k^*)\big) \leqslant Mf^{-1}\big(\log(\|u\|_p^* \diagup \|u\|_q^*)\big)$$

for all $u^* \in E' \setminus \{0\}$, where

$$\|u\|_q^* = \sup\big\{|u(x)| : \|x\|_q \leqslant 1\big\}.$$

2.3. Let E, F be Frechet spaces. We say that (E,F) has the property S and write $(E,F) \in S$ if there exists p such that for every j there exists k for every ℓ for every q there exists r such that

$$\|u\|_k^*.\|x\|_q \leqslant \|u\|_j^*.\|x\|_p + \|u\|_\ell^*.\|x\|_r$$

for all $u \in E^*$ and for all $x \in F$.

2.4. It is proved in [1] (see Proposition 2.9) that if E has $D_4(f)$ property and F has $D_3(f)$ property then $(E, F) \in S$.

From now on, to be brief, whenever E has $D_3(f)$ property (resp. $D_4(f)$) we write $E \in D_3(f)$ (resp. $E \in D_4(f)$).

3. Some Auxiliary Results

Proposition 3.1. *Let*

$$0 \longrightarrow \ell^\infty(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty) \longrightarrow E \xrightarrow{\text{T}} F \longrightarrow 0$$

be an exact sequence of Frechet spaces and continuous linear maps. If $F \in D_3(f)$ then the sequence splits.

Proof. Since $L_f(\alpha,\infty)$ is nuclear we have

$$\ell^\infty(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty) = \Big\{\xi = (\xi_{i,n})_{i\in I} \subset \mathbb{C} : \sup_{\substack{i\in I\\ n\geqslant 1\\ \forall k\geqslant 1}} |\xi_{i,n}|e^{f(k\alpha_n)} < \infty\Big\}.$$

Moreover, $L_f(\alpha,\infty)$ has $D_4(f)$ property (see [1, Prop. 2.11]). Proposition 2.9 in [1] implies that $(L_f(\alpha,\infty), F) \in S$.
Then, by [1, Lemma 1.5] without loss of generality we may assume that $\exists p\ \forall q\ \forall k\ \exists r = r(k,q)$:

$$\frac{1}{a_{n,k}}V_q^0 \subset \frac{1}{a_{n,k-1}}V_p^0 + \frac{1}{a_{n,k+1}}V_r^0 \text{ with } \forall n \geqslant 1 \tag{1}$$

where

$$a_{n,k} = e^{f(k\alpha_n)}$$

and $\{V_p\}_{p\geqslant 1}$ is a neighborhood basis of $0 \in F$. Let

$$\begin{aligned}\rho_k : \ell^\infty(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty) &\longrightarrow \big(\ell^\infty(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty)\big)_k\\ =&\big\{\xi = (\xi_{i,n}) \subset \mathbb{C} : \|\xi\|_k = \sup_{\substack{i\in I\\ n\geqslant 1}} |\xi_{i,n}|e^{f(k\alpha_n)} < \infty\big\}\end{aligned}$$

be the canonical map. Then ρ_k can be extended to a continuous linear map $A_k : E \to (\ell^\infty(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty))_k$. Put

$$B_k = \rho_{k+1,k}A_{k+1} - A_k \in \mathcal{L}\big(E, (\ell^\infty(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty))_k\big).$$

Since $B_k|_{kerT} = 0$ then there exists $C_k \in \mathcal{L}\big(F, (\ell^\infty(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty))_k\big)$ such that $C_k \circ T = B_k$.
Set $e_{i,n}(\xi) = \xi_{i,n}$ for $\xi = (\xi_{i,n}) \in (\ell^\infty(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty))_k$.
Then it is easy to see that

$$\|e_{i,n}\|_k^* = \frac{1}{a_{n,k}}$$

for all $i \in I, n, k \geqslant 1$. Hence we infer that $\{a_{n,k} e_{i,n} \circ C_k\}_{i\in I;n\geqslant 1}$ belongs to F'.

Put $C_{i,n}^k = e_{i,n} \circ C_k$ for $i \in I, n, k \geqslant 1$. Next we shall construct a neighborhood basis $\{W_k\}$ on F such that we have $\{a_{n,k} C_{i,n}^k\} \subset V_k^0$ for all $n, k \geqslant 1, i \in I$ and

$$\frac{2^{k+1}}{a_{n,k}} W_k^0 \subset \frac{1}{a_{n,k-1}} W_0^0 + \frac{1}{a_{n,k+1}} W_{k+1}^0 \quad \forall k \geqslant 1, \forall n \geqslant 1. \tag{2}$$

Put $W_0 = V_p$. By the equicontinuity of $\{a_{n,1} C_{i,n}^1\}_{i\in I;n\geqslant 1}$ we can pick a neighborhood W_1 such that $\{a_{n1} C_{in}^1\} \subset W_1^0$. Assume that the neighborhoods $W_1, W_2, \ldots, W_k$ are chosen. Take $q \geqslant 1$ such that $V_q \subset 2^{-k-1} W_k$. Applying (1) to V_q we can find $W_{k+1} = V_{r(k,q)}$ satisfying (2). This completes the construction. Since $C_{i,n}^k \in \left(\frac{1}{a_{n,k}}\right)$ together with (2) enables us to define, for fixed n, inductively a sequence $\{D_{i,n}^k\} \subset F'$ such that

$$C_{i,n}^k + D_{i,n}^k - D_{i,n}^{k+1} \in \frac{2^{-k}}{a_{n,k-1}} W_0^0. \tag{3}$$

Now define the continuous linear maps $D_k : F \to (\ell^\infty(I) \widehat{\otimes}_\Pi L_f(\alpha, \infty))_k$ by

$$D_k x = \left(D_{i,n}^k x\right)_{i\in I;n\geqslant 1}.$$

Let $\widehat{D}_k = D_k \circ T$ and $\Pi_k = A_k - \widehat{D}_k$. From (3) we infer that for all $m \geqslant 1$ and $x \in E$ there exists $\lim\limits_{k\to+\infty} \rho_{k,m} \circ \Pi_k(x)$ which will be denoted by $\widetilde{\Pi}_m(x)$. It is easy to check that the map $x \mapsto \{\widetilde{\Pi}_m(x)\}_{m\geqslant 1}$ is a continuous linear projection of E onto $\ell^\infty(I) \widehat{\otimes}_\Pi L_f(\alpha, \infty)$. Hence, T has a right inverse.

Next we need the following.

Proposition 3.2. *Let*

$$0 \longrightarrow E \longrightarrow H \xrightarrow{q} \ell^1(I) \widehat{\otimes}_\Pi L_f(\alpha, \infty) \longrightarrow 0$$

be an exact sequence of Frechet spaces and continuous linear maps. If $E \in D_4(f)$ then the sequence splits.

Proof. Since $E \in D_4(f)$ and $L_f(\alpha, \infty) \in D_3(f)$ (see Proposition 2.11 in [1]) then $(E, L_f(\alpha, \infty)) \in S$ (see Proposition 2.9 in [1]). Then by Lemma 1.7 in [1] there exists a neighborhood basis $\{U_k\}$ of $0 \in E$ such that

$$\forall k \; \forall j \; \exists \ell(k,j) : 2a_{n,j} U_k \subset a_{n,e(k,j)} U_{k+1} + 2^{-k} a_{n,0} U_{k-1} \tag{4}$$

for all $n \geqslant 1$.

Without loss of generality we may assume that $U_k = W_k \cap E$ where $\{W_k\}$ is a neighborhood basis of $0 \in H$. Put $V_k = q(W_k)$. Then $\{V_k\}_{k\geqslant 1}$ is also a neighborhood basis of $0 \in \ell^1(I) \widehat{\otimes}_\Pi L_f(\alpha, \infty)$. We may assume that

$$(\ell^1(I) \widehat{\otimes}_\Pi L_f(\alpha, \infty))_{V_k} = (\ell^1(I) \widehat{\otimes}_\Pi L_f(\alpha, \infty))_k$$

for $k \geqslant 1$.

Thus for each $k \geqslant 1$ we have an exact sequence

$$0 \longrightarrow E_k \longrightarrow H_k \xrightarrow{q_k} (\ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty))_k \longrightarrow 0$$

Since

$$(\ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty))_k = \big\{\xi = (\xi_{i,n}) \subset \mathbb{C} : \|\xi\|_k = \sup_{\substack{i\in I\\ n\geqslant 1}} |\xi_{i,n}| a_{n,k} < \infty\big\},$$

we can find $R_k \in \mathcal{L}\big((\ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty)), H_k\big)$ such that

$$q_k.R_k = \omega_k$$

where $\omega_k : \ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty) \to (\ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty))_k$ is the canonical map. Let $S_k = \rho_{k+1,k}R_{k+1} - R_k$. Then $q_k S_k = 0$.

Hence S_k can be considered as a continuous linear map from $\ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty)$ into E_k. Put $x_{i,n,k} = S_k(e_{i,n})$ where $\{e_{i,n}\}$ denotes the coordinate basis of $\ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty)$. By the continuity of S_k there exists a function $k \mapsto m(k)$ such that

$$\|S_k z\|_k \leqslant \|z\|_{m(k)}$$

for $z \in \ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty)$.

Applying (4) to $k = 1$ and $j = m(1)$ we can find $\ell(k,j)$ such that

$$2a_{n,j}U_1 \subset a_{n,\ell(k,j)}U_2 + 2^{-1}a_{n,0}U_0 \quad n \geqslant 1.$$

Let $\nu(2) = \max(\ell(k,j), m(2))$. Next we apply (4) to $k = 2$, $j = \nu(2)$ and choose $\nu(3) = \max(\ell(k,j), m(3))$. Continuing this way and by putting $a_{nk} = a_{n\nu(k)}$ we get the following

$$\|x_{i,n,k}\| \leqslant a_{n,k} \tag{5}$$

and

$$2a_{n,k}U_k \subset a_{n,k+1}U_{k+1} + 2^{-k}a_{n,0}U_{k-1}. \tag{6}$$

For each $(i,n,k) \in I \times \mathbb{N}^2$ choose $\widetilde{x}_{i,n,k} \in a_{n,k}U_k$ such that

$$\|x_{i,n,k} - \widetilde{x}_{i,n,k}\|_k < 2^{-k}.$$

By (5) and (6) we can find $y_{i,n,k} \in 2^{-k+1}a_{n,0}U_{k-1}$ such that

$$\widetilde{x}_{i,n,k} \in \frac{a_{n,k+1}}{2}U_k + y_{i,n,k}.$$

Then the series

$$y_{i,n} = \sum_{k=0}^{\infty} y_{i,n,k} + (x_{i,n,k} - \widetilde{x}_{i,n,k})$$

is convergent in E_q. Put

$$R([\xi_{i,n}; I \times \mathbb{N}]) = R_0([\xi_{i,n}; I \times \mathbb{N}]) + \sum_{i\in I}\sum_{n=1}^{\infty} y_{i,n}\xi_{i,n}.$$

Then $R : \ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty) \to H_0$ is continuous linear and $q \circ R = id$, Hence, R is the right inverse of q and the proposition is proved. ■

4. Proof of Main Theorem

(i) The sufficiency is obvious. Now we prove the necessity. Let E be a Frechet space with the $D_3(f)$ property. Given $\alpha = (\alpha_n)$ a stable nuclear exponent sequence. This is equivalent to

$$\sup_n \frac{\log n}{f(\alpha_n)} < \infty \text{ and } \sup \frac{\alpha_{2n}}{\alpha_n} < \infty.$$

By [9] there exists an exact sequence

$$0 \longrightarrow L_f(\alpha,\infty) \longrightarrow L_f(\alpha,\infty) \xrightarrow{q} L_f(\alpha,\infty)^{\mathbb{N}} \longrightarrow 0. \tag{7}$$

Choose arbitrary $\nu = (\nu_n) \in L_f(\alpha,\infty), \nu_n \neq 0$ for all $n \geqslant 1$. It follows that the form

$$\omega \ni (\xi_n) \longmapsto (\xi_n \nu) \in L_f(\alpha,\infty)^{\mathbb{N}}$$

defines an isomorphism from ω into $L_f(\alpha,\infty)^{\mathbb{N}}$ where ω denotes the space of all complex number sequences. Putting $\widetilde{E} = q^{-1}(\omega)$. Then we obtain the exact sequence of nuclear Frechet spaces

$$0 \longrightarrow L_f(\alpha,\infty) \longrightarrow \widetilde{E} \xrightarrow{q} \omega \longrightarrow 0. \tag{8}$$

Take an index set I such that E is embedded into $\ell^\infty(I)^{\mathbb{N}}$. By tensoring (8) with $\ell^\infty(I)$ we get the exact sequence

$$0 \longrightarrow L_f(\alpha,\infty)\widehat{\otimes}_\Pi \ell^\infty(I) \longrightarrow \widetilde{E}\widehat{\otimes}_\Pi \ell^\infty(I) \xrightarrow{\widehat{q}} \ell^\infty(I)^{\mathbb{N}} \longrightarrow 0. \tag{9}$$

By Proposition 3.1 $\widehat{q}$ has a right inverse. This yields that E is isomophic to a subspace of $\widetilde{E}\widehat{\otimes}_\Pi \ell^\infty(I)$ and, hence, of $L_f(\alpha,\infty)\widehat{\otimes}_\Pi \ell^\infty(I)$. Thus (i) is completely proved.

(ii) It remains to prove the necessity. Assume that $E \in D_4(f)$, as in [2] there exists the canonical resolution

$$0 \longrightarrow E \longrightarrow \prod_k E_k \xrightarrow{\sigma} \prod_k E_k \longrightarrow 0. \tag{10}$$

where E_k denotes the Banach space associated to the semi-norm $\|.\|_k$. Set

$$F = \{x = (x_k) \in \prod_k E_k : \|x\| = \sum_{k\geqslant 1} ||x_k|| < \infty\}.$$

For each $k \geqslant 1$, let F_k be a topological completement of E_k in F, i.e $F = E_k \oplus F_k$. The direct sum of (9) with the exact sequence

$$0 \longrightarrow 0 \longrightarrow \prod_{k\geqslant 1} F_k \xrightarrow{id} id \prod_{k\geqslant 1} F_k \longrightarrow 0$$

gives the exact sequence

$$0 \longrightarrow E \longrightarrow F^{\mathbb{N}} \longrightarrow F^{\mathbb{N}} \longrightarrow 0.$$

Next we choose an exact sequence

$$0 \longrightarrow K \longrightarrow \ell^1(I) \longrightarrow F \longrightarrow 0$$

and consider the exact sequence

$$0 \longrightarrow L_f(\alpha,\infty) \longrightarrow \widetilde{E} \longrightarrow \omega \longrightarrow 0$$

as in (i). By tensoring this sequence with the previous exact sequence we obtain the following commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & F\widehat{\otimes}_\Pi L_f(\alpha,\infty) & \longrightarrow & F\widehat{\otimes}_\Pi \widetilde{E} & \longrightarrow & F^{\mathbb{N}} & \longrightarrow & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty) & \longrightarrow & \ell^1(I)\widehat{\otimes}_\Pi \widetilde{E} & \longrightarrow & \ell^1(I)^{\mathbb{N}} & \longrightarrow & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & K\widehat{\otimes}_\Pi L_f(\alpha,\infty) & \longrightarrow & K\widehat{\otimes}_\Pi \widetilde{E} & \longrightarrow & K^{\mathbb{N}} & \longrightarrow & 0\\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

In a natural way we lead to the exact sequence

$$(\ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty)) \oplus (K\widehat{\otimes}_\Pi L_f(\alpha,\infty)) \longrightarrow \ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty) \xrightarrow{q} F^{\mathbb{N}} \longrightarrow 0.$$

We consider the following diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & E & \longrightarrow & F^{\mathbb{N}} & \xrightarrow{q_1} & F^{\mathbb{N}} & \longrightarrow & 0\\
 & & & & \uparrow p_1 & & \uparrow q_2 & & \\
0 & \longrightarrow & E & \longrightarrow & H & \underset{p_2}{\longrightarrow} & \ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty) & \longrightarrow & 0\\
 & & & & \uparrow & & \uparrow & & \\
 & & & & N & & N & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

where $H = \{(x,y) \in F^{\mathbb{N}} \times (\ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty)) : q_1 x = q_2 y\}$. and $p_1(x,y) = x, p_2(x,y) = y$ are the canonical projections. By the Proposition 3.2 the second row splits. Thus we have the following diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & N & \longrightarrow & E\oplus(\ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty)) & \longrightarrow & F^{\mathbb{N}} & \longrightarrow & 0 \\
 & & & & & & \uparrow & & \\
0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & \ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty) & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & N & & N & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

N has $D_4(f)$ property because it is a quotient of

$$(\ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty)) \oplus (K\widehat{\otimes}_\Pi L_f(\alpha,\infty)) \cong (\ell^1(I) \oplus K)\widehat{\otimes}_\Pi L_f(\alpha,\infty).$$

By again Proposition 3.2 the second row splits and we obtain from the first column the exact sequence

$$0 \longrightarrow N \longrightarrow N \oplus (\ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty)) \longrightarrow E \oplus (\ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty)) \longrightarrow 0.$$

Hence E is a quotient of $N \oplus (\ell^1(I)\widehat{\otimes}_\Pi L_f(\alpha,\infty))$ and, hence, of

$$(\ell^1(I) \oplus K \oplus \ell^1(I))\widehat{\otimes}_\Pi L_f(\alpha,\infty).$$

The Main theorem is completely proved. ∎

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