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A Fixed Point Theorem for Nonexpansive Mappings in Locally Convex Spaces

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Abstract. In this note, first we establish a fixed point theorem for a nonexpansive mapping in a locally convex space, then we apply it to get a fixed point theorem in probabilistic normed spaces.

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1. Introduction

After the work [2] a lot of fixed point theorems for semigroups of mappings in Banach spaces were obtained. However, for such results in locally convex spaces up to now there is only one paper [4] with a restrictive condition : compactness of the domain. In Sec. 2 slightly modifying the method in [3] we get a fixed point theorem for a nonexpansive mapping in a locally convex space and apply it to get an analogous result for probabilistic normed spaces.

2. Fixed Point Theorems

2.1. A Fixed Point Theorem for Nonexpansive Mappings in Locally Convex Spaces

Let us first give some definitions.

Definition 1. [4] *Let E be a Hausdorff locally convex topological vector space and P a family of continuous seminorms which generates the topology of E.*

For any $p \in P$ and $A \subset E$, let $\delta_p(A)$ denote the p-diameter of A,i.e.,

$$\delta_p(A) = \sup\{p(x-y) : x, y \in A\}.$$

A convex subset K of E is said to have normal structure with respect to P if for each nonempty bounded convex subset H of K and for each $p \in P$ with $\delta_p(H) > 0$, there is a point x_p in H such that

$$\sup\{p(x_p - y) : y \in H\} < \delta_p(H).$$

Definition 2. [4] *Let E and P be as in Definition 1, and $K \subset E$. A mapping $T : K \to K$ is said to be P-nonexpansive if for all $x, y \in K$ and $p \in P$,*

$$p(Tx - Ty) \leq p(x - y).$$

Definition 3. *Let E and P be as in Definition 1. E is said to be strictly convex if the following implication holds for all $x, y \in E$ and $p \in P$:*

$$p(x) = 1, p(y) = 1, x \neq y \Longrightarrow p(\frac{x+y}{2}) < 1.$$

Proposition 1. *Let (E, P) be a strictly convex space and $p(x+y) = p(x) + p(y), p(x) \neq 0, p(y) \neq 0$. Then $x = \lambda_p y$ for some $\lambda_p > 0$.*

Proof. Suppose $p(x) \leq p(y)$. Put $x' = \frac{x}{p(x)}$, $y' = \frac{y}{p(y)}$, then $p(x') = p(y') = 1$. We have

$$\begin{aligned} 2 &= p(x') + p(y') \geq p(x' + y') = p(\frac{x}{p(x)} + \frac{y}{p(y)}) \\ &= p(\frac{x}{p(x)} + \frac{y}{p(x)} - \frac{y}{p(x)} + \frac{y}{p(y)}) \geq p(\frac{x+y}{p(x)}) - (\frac{1}{p(x)} - \frac{1}{p(y)})p(y) \\ &= \frac{p(x)+p(y)}{p(x)} - \frac{p(y)}{p(x)} + \frac{p(y)}{p(y)} = 2. \end{aligned}$$

So $p(x'+y') = 2$, hence $p(\frac{x'+y'}{2}) = 1$. Since E is strictly convex, we have $x' = y'$. From this it follows that $x = \frac{p(x)}{p(y)} y$ and the proof is complete.

Theorem 1. *Let C be a nonempty weakly compact convex subset of a Hausdorff locally convex space (E, P) which has normal structure, and $T : C \to C$ a P-nonexpansive mapping. Then T has a fixed point.*

Moreover, if E is a strictly convex space, then the set $FixT$ of fixed points of T is nonempty and convex.

Proof. We first prove that T has a fixed point.

Denote by $\mathcal{F}$ the family of all nonempty closed convex subsets of C and invariant under T, i.e.,

$$\mathcal{F} = \{K \subset C : K \text{ is a nonempty closed convex set and } T(K) \subset K\}.$$

Clearly $\mathcal{F}$ is a nonempty family, since $C \in \mathcal{F}$. By weakly compactnees of C and Zorn's Lemma, $\mathcal{F}$ has a minimal element H.

Now we shall show that H consists of a single point. Assume on the contrary that there exists $p_o \in P$ such that $\delta_{p_o}(H) = d > 0$. Since C has normal structure, there exists $z_o \in H$ such that $r = \sup_{x\in H} p_o(z_o - x) < d$.

Denoting $D = \{z \in H : p_o(z - x) \le r \text{ for all } x \in H\}$, it is easy to prove that D is a nonempty closed convex subset in C, since $z_0 \in D$ and p_o is a convex continuous function.

Now we show that D is invariant under T. For any z in D, we have $p_o(z-x) \le r$ for all $x \in H$. Since T is a nonexpansive mapping, we get

$$p_o(Tz - Tx) \le p_o(z - x) \le r,$$

for all $x \in H$. Hence $p_o(Tz-x) \le r, \forall x \in T(H)$. So we have $p_o(Tz-x) \le r, \forall x \in \overline{co}T(H)$, because p_o is a convex continuous function, where $\overline{co}T(H)$ denotes the closed convex hull of $T(H)$. Since $T(H) \subset H$, this implies $\overline{co}(T(H)) \subset \overline{co}(H) = H$. Hence $T(\overline{co}(T(H))) \subset T(H) \subset \overline{co}(T(H))$. Thus $\overline{co}T(H) \in \mathcal{F}$. From this and the minimality of H we get $\overline{co}T(H) = H$ and hence

$$p_o(Tz - x) \le r, \forall x \in H.$$

So $Tz \in D$, and $T(D) \subset D$. Hence $D \in \mathcal{F}$. By the minimality of H in $\mathcal{F}$, we get $H = D$. Thus for every u, v in H, we have $p_o(u - v) \le r$. It follows that $d = \delta_{p_o}(H) = \delta_{p_o}(D) = \sup_{u,v\in D} p_o(u - v) \le r$. This is a contradiction, so $\delta_p(H) = 0, \forall p \in P$; thus $H = \{z\}$ and $Tz = z$.

Lastly we prove that $FixT$ is a convex set.

For any $u, v \in FixT$, i.e., $u = Tu, v = Tv$, we put $z = \lambda u + (1 - \lambda)v$ with any $\lambda \in (0, 1)$. We have $u - z = (1 - \lambda)(u - v)$ and $v - z = \lambda(v - u)$.

Since T is a P-nonexpansive mapping, we have

$$p(u - Tz) + p(Tz - v) \le p(u - z) + p(z - v) = p(u - v).$$

On the other hand, since $u - v = (u - Tz) + (Tz - v)$, we get

$$p(u - v) \le p(u - Tz) + p(Tz - v).$$

From these we get

$$p(u - v) = p(u - Tz) + p(Tz - v).$$

We claim that $p(u - Tz) \ne 0$ and $p(v - Tz) \ne 0$. Indeed, if $p(u - Tz) = 0$ then we get

$$p(u - v) = p(v - Tz) = p(Tv - Tz).$$

On the other hand,

$$p(Tv - Tz) \le p(v - z) = \lambda p(v - u) < p(v - u).$$

We have a contradiction, so $p(u - Tz) \ne 0$. Similarly, we have $p(v - Tz) \ne 0$.

Putting $x = u - Tz, y = Tz - v$, we have

$$p(x) + p(y) = p(x + y).$$

Since E is strictly convex Proposition 1 implies that $\exists \alpha_p > 0$ such that $x = \alpha_p y$, i.e.,

$$u - Tz = \alpha_p(Tz - v)$$

from this

$$Tz = \frac{1}{1+\alpha_p}u + \frac{\alpha_p}{1+\alpha_p}v.$$

We claim that $\lambda = \frac{1}{1+\alpha_p}$. Indeed, supposing $\lambda < \frac{1}{1+\alpha_p}$, we have

$$p(v - Tz) = p(Tv - Tz) = p(u - v) - p(u - Tz) = p(u - v) - \alpha_p p(Tz - v).$$

It follows that $p(u - v) = (1 + \alpha_p)p(Tz - v)$. Hence

$$p(Tz - Tv) = p(Tz - v) = \frac{1}{1+\alpha_p}p(u - v) > \lambda p(u - v) = p(z - v).$$

This is a contradiction, because T is a P-nonexpansive mapping. In the same way, if $\lambda > \frac{1}{1+\alpha_p}$ then we also have a contradiction. Thus we get $Tz = z$, hence $z \in FixT$ and the proof is complete.

2.2. Application to Probabilistic Normed Spaces

Definition 4. [5] *A probabilistic normed space is a triple* $(X, \mathcal{F}, \min)$*, where* X *is a linear space,* $\mathcal{F} = \{F_x : x \in X\}$ *is a family of distribution functions satisfying:*

1) $F_x(0) = 0$ *for all* $x \in X$*,*
2) $F_x(t) = 1$ *for all* $t > 0 \Leftrightarrow x = 0$*,*
3) $F_{\alpha x}(t) = F_x\left(\frac{t}{|\alpha|}\right), \forall t \geq 0, \forall \alpha \in \mathbb{C}$ *or* $\mathbb{R}, \alpha \neq 0, \forall x \in X$.
4) $F_{x+y}(s + t) \geq \min\{F_x(s), F_y(t)\}, \forall x, y \in X, \forall t, s \geq 0.$

The topology in X is defined by the system of neighborhoods of $0 \in X$:

$$U(0, \epsilon, \lambda) = \{x \in X : F_x(\epsilon) > 1 - \lambda\}, \epsilon > 0, \lambda \in (0, 1).$$

This is a locally convex Hausdorff topology, called the (ϵ, λ)-topology. To see this we define for each $\lambda \in (0, 1)$

$$p_\lambda(x) = \sup\{t \in \mathbb{R} : F_x(t) \leq 1 - \lambda\}.$$

From properties 1) - 4) of F_x one can verify that p_λ is a seminorm on X and $p_\lambda(x) = 0, \forall \lambda \in (0, 1) \Rightarrow x = 0$, and the topology on X defined by the family of seminorms $\{p_\lambda : \lambda \in (0, 1)\}$ coincides with the (ϵ, λ)-topology. In particular, we have

$$F_x(p_\lambda(x)) \leq 1 - \lambda, \forall x \in X, \forall \lambda \in (0, 1) \tag{1}$$

and

$$p_\lambda(x) < \epsilon \Leftrightarrow F_x(\epsilon) > 1 - \lambda. \tag{2}$$

(For details, see [5]). In the sequel all topological notions (boundedness, compactness, weak compactness,...) in a probabilistic normed space are understood as those in the corresponding locally convex space.

Definition 5. *A mapping T in $(X, \mathcal{F}, \min)$ is said to be probabilistic nonexpansive if for all $x, y \in X$ and $t \in \mathbb{R}$ we have*

$$F_{Tx-Ty}(t) \geq F_{x-y}(t).$$

Definition 6. *A subset C of a probabilistic normed space $(X, \mathcal{F}, \min)$ is said to have probabilistic uniformly normal structure if for every convex closed bounded subset H of C containing more than one point, there exists $x_o \in H$ and $0 < k < 1$ such that*

$$\inf_{y \in H} F_{x_0-y}(kt) \geq \inf_{x,y \in H} F_{x-y}(t)$$

for all $t \geq 0$.

Definition 7. *A probabilistic normed space $(X, \mathcal{F}, \min)$ is said to be probabilistic strictly convex if $\forall x, y \in X, x \neq y, \exists k > 1$ such that*

$$F_{\frac{x+y}{2}}(t) \geq \min\{F_x(kt), F_y(kt)\}, \forall t \geq 0.$$

Before stating another fixed point theorem we establish three following lemmas.

Lemma 1. *Every probabilistic nonexpansive mapping in a probabilistic normed space $(X, \mathcal{F}, \min)$ is P-nonexpansive in the corresponding locally convex space $(X, \{p_\lambda\})$.*

Proof. Suppose on the contrary that there exist $\lambda \in (0, 1)$ and $x, y \in X$ such that

$$p_\lambda(Tx - Ty) > p_\lambda(x - y).$$

Putting $t_o = p_\lambda(Tx - Ty)$ we have $p_\lambda(x - y) < t_o$, and by (2), $F_{x-y}(t_o) > 1 - \lambda$.

On the other hand, it follows from (1) that

$$F_{Tx-Ty}(t_o) = F_{Tx-Ty}(p_\lambda(Tx - Ty)) \leq 1 - \lambda.$$

So we get

$$F_{x-y}(t_o) > 1 - \lambda \geq F_{Tx-Ty}(t_o),$$

a contradiction and the proof is complete.

Lemma 2. *Let a probabilistic normed space $(X, \mathcal{F}, \min)$ satisfy the following condition:*

For each fixed $t \in \mathbb{R}$, the function $F_x(t) : X \to [0, 1]$ is weakly lower semicontinuous in $x \in X$. (3)

Then every weakly compact set $C \subset X$ having probabilistic uniformly normal structure has normal structure in the corresponding locally convex space $(X, \{p_\lambda\})$.

Proof. Let D be any closed convex subset of C, then D is also weakly compact. We show that for each $\lambda \in (0, 1)$

$$\sup_{x\in D} \sup\{t : F_x(t) \le 1-\lambda\} = \sup\{t : \inf_{x\in D} F_x(t) \le 1-\lambda\}. \tag{4}$$

Since $F(t) = \inf_{x\in D} F_x(t) \le F_x(t)$ for each $x \in D$, we have

$$a = \sup\{t : F(t) \le 1-\lambda\} \ge \sup_{x\in D} \sup\{t : F_x(t) \le 1-\lambda\} = b.$$

If $a > b$, then we have $F_x(a) > 1-\lambda$ for each $x \in D$. The condition (3) shows that $F(a) > 1-\lambda$, this implies $a > a$, a contradiction. Thus $a = b$, so (4) is proved.

Now we prove the assertion of the lemma. From the inequality

$$\inf_{y\in D} F_{x_0-y}(kt) \ge \inf_{x,y\in D} F_{x-y}(t)$$

we get

$$\{t : \inf_{y\in D} F_{x_0-y}(kt) \le 1-\lambda\} \subset \{t : \inf_{x,y\in D} F_{x-y}(t) \le 1-\lambda\},$$

hence

$$\frac{1}{k}\{t : \inf_{y\in D} F_{x_0-y}(t) \le 1-\lambda\} \subset \{t : \inf_{x,y\in D} F_{x-y}(t) \le 1-\lambda\},$$

so

$$\{t : \inf_{y\in D} F_{x_0-y}(t) \le 1-\lambda\} \subset k\{t : \inf_{x,y\in D} F_{x-y}(t) \le 1-\lambda\}.$$

This implies

$$\sup\{t : \inf_{y\in D} F_{x_0-y}(t) \le 1-\lambda\} \le k \sup\{t : \inf_{x,y\in D} F_{x-y}(t) \le 1-\lambda\}.$$

From this and (4) we get

$$\sup_{y\in D} \sup\{t : F_{x_0-y}(t) \le 1-\lambda\} \le k \sup_{x,y\in D} \sup\{t : F_{x-y}(t) \le 1-\lambda\},$$

and finally

$$\sup_{y\in D} p_\lambda(x_0 - y) \le k \sup_{x,y\in D} p_\lambda(x-y) = k\delta_{p_\lambda}(D) < \delta_{p_\lambda}(D)$$

if $\delta_{p_\lambda}(D) > 0$, as desired. The proof is complete.

Lemma 3. *If $(X, \mathcal{F}, \min)$ is probabilistic strictly convex then its corresponding $(X, \{p_\lambda\})$ is strictly convex.*

Proof. Putting $t = 1$ in Definition 7 we get

$$F_{\frac{x+y}{2}}(1) \geq \min\{F_x(k), F_y(k)\}. \tag{5}$$

Let $p_\lambda(x) = p_\lambda(y) = 1$ then $p_\lambda(x) < k, p_\lambda(y) < k$. By (2) this is equivalent to $F_x(k) > 1 - \lambda$ and $F_y(k) > 1 - \lambda$, hence, by (5), $F_{\frac{x+y}{2}}(1) > 1 - \lambda$. But this is equivalent to $p_\lambda(\frac{x+y}{2}) < 1$ as desired. The proof is complete.

Now we state an analogous result to Theorem 1 for probabilistic normed spaces.

Theorem 2. *Let C be a nonempty weakly compact convex set having probabilistic uniformly normal structure in a probabilistic normed space $(X, \mathcal{F}, \min)$ satisfying condition (3). Let T be a probabilistic nonexpansive mapping from C into C. Then T has a fixed point. Moreover, if X is a probabilistic strictly convex space, then the set $FixT$ of fixed points of T is convex.*

Proof. By Lemmas 1, 2 and 3, T satisfies all conditions in Theorem 1 with $E = (X, \{p_\lambda\})$ corresponding to $(X, \mathcal{F}, \min)$, so T has a fixed point and the set $FixT$ of fixed points of T is convex and the theorem follows.

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