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# On Score Sets in Tournaments

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**Abstract.** In this paper, we give a new proof for the set of non-negative integers $\left\{s_1, \sum_{i=1}^{2} s_i, \dots, \sum_{i=1}^{p} s_i\right\}$ with $s_1 < s_2 < \cdots < s_p$ to be the score set of some tournament. The proof is constructive, using tournament construction.

## 1. Introduction

The score set $S$ of a tournament $T$, a complete oriented graph, is the set of scores (outdegrees) of the vertices of $T$. In [5], Reid conjectured that each finite, nonempty set $S$ of non-negative integers is the score set of some tournament and proved it for the cases $|S| = 1, 2, 3$, or if $S$ is an arithmetic or geometric progression. As can be seen in [2 – 4] that non-negative integers $s_1 \leqslant \ldots \leqslant s_n$ are the scores of a tournament with $n$ vertices if and only if

$$\sum_{i=1}^{k} s_i \geq \binom{k}{2}, \quad 1 \leqslant k \leqslant n-1, \quad \text{and} \quad \sum_{i=1}^{n} s_i = \begin{bmatrix} n \\ 2 \end{bmatrix}.$$

Let $S = \{t_1, \dots, t_p\}$ be a nonempty set of non-negative integers with $t_1 < \ldots < t_p$, then $S$ is a score set if and only if there exist $p$ positive integers $m_1, \dots, m_p$ such that

$$\sum_{i=1}^{k} m_i t_i \geq \begin{bmatrix} M(k) \\ 2 \end{bmatrix}, \quad 1 \leqslant k \leqslant p-1, \quad \sum_{i=1}^{p} m_i t_i = \begin{bmatrix} M(p) \\ 2 \end{bmatrix},$$

where

$$M(k) = \sum_{i=1}^{k} m_i, \quad 1 \leqslant k \leqslant p,$$

because only the inequalities in the above mentioned formula for those values of $k$, for which $s_k < s_{k+1}$ hold, need to be checked [5, p.608].

The following results can be seen in [5].

**Theorem 1.** *Every singleton or doubleton set of positive integers is the score set of some tournament.*

**Theorem 2.** *Let* $S = \{s, sd, sd^2, \ldots, sd^p\}$*, where* $s$ *and* $d$ *are positive integers,* $d > 1$*. Then, there exists a tournament* $T$ *such that* $S_T = S$*.*

**Theorem 3.** *Let* $S = \{s, s+d, s+2d, \ldots, s+pd\}$*, where* $s$ *and* $d$ *are non-negative integers,* $d > 0$*. Then, there is a tournament* $T$ *such that* $S_T = S$*.*

**Theorem 4.** *Let* $S = \{s, s+d, s+d+e\}$*, where* $s, d,$ *and* $e$ *are non-negative integers and* $de > 0$*. Let* $(d, e) = g$*. If* $d \leqslant s$ *and* $e \leqslant s + d - \binom{d}{2g} + \binom{1}{2}$*, then* $S$ *is a score set.*

**Theorem 5.** *Let* $S = \{s, s+d, s+d+e\}$*, where* $s$*,* $d$*, and* $e$ *are non-negative integers and* $de > 0$*. Let* $(d, e) = g$*. If* $d \leqslant s$ *and* $s-d+\binom{d}{2g}+\binom{1}{2} < e \leqslant s+d-1$*, then* $S$ *is a score set.*

**Theorem 6.** *Every set of three non-negative integers is a score set.*

Also the following results can be found in [1].

**Theorem 7.** *Let* $s_1, s_2, s_3, s_4$ *be four non-negative integers with* $s_2 s_3 s_4 > 0$*. Then, there exists a tournament* $T$ *with score set* $S = \{s_1, s_1 + s_2, s_1 + s_2 + s_3, s_1 + s_2 + s_3 + s_4\}$*.*

**Theorem 8.** *Let* $s_1, s_2, s_3, s_4, s_5$ *be five non-negative integers with* $s_2 s_3 s_4 s_5 > 0$*. Then, there exists a tournament* $T$ *with score set* $S = \{s_1, s_1+s_2, s_1+s_2+s_3, s_1+s_2+s_3+s_4, s_1+s_2+s_3+s_4+s_5\}$*.*

In 1986, Yao announced a proof of Reid's conjecture by pure arithmetical analysis which appeared in Chinese [6] in 1988 and in English [7] in 1989.

In the following result, we prove that any set of $p$ non-negative integers $s_1, s_2, \ldots, s_p$ with $s_1 < s_2 < \ldots < s_p$, there exists a tournament with score set $\left\{s_1, \sum_{i=1}^{2} s_i, \ldots, \sum_{i=1}^{p} s_i\right\}$. The proof is by induction using graph theoretical technique of constructing tournament.

**Theorem 9.** *If* $s_1, s_2, \ldots, s_p$ *are* $p$ *non-negative integers with* $s_1 < s_2 < \ldots < s_p$*, then there exists a tournament* $T$ *with score set* $S = \left\{s_1, \sum_{i=1}^{2} s_i, \ldots, \sum_{i=1}^{p} s_i\right\}$*.*

*Proof.* Let $s_1, s_2, \ldots, s_p$ be $p$ non-negative integers with $s_1 < s_2 < \ldots < s_p$. We induct on $p$. First assume $p$ to be odd. For $p = 1$, we have the non-negative integer $s_1$, and now, let $T$ be a regular tournament having $2s_1+1$ vertices. Then, each vertex of $T$ has score $\frac{2s_1+1-1}{2}=s_1$, so that score set of $T$ is $S=\{s_1\}$. This shows that the result is true for $p=1$.

If $p=3$, then there are three non-negative integers $s_1, s_2, s_3$ with $s_1<s_2<s_3$.

Now, $s_3>s_2$, therefore $s_3-s_2>0$, so that $s_3-s_2+s_1>0$ as $s_1\geq 0$. Let $T_1$ be a regular tournament having $2(s_3-s_2+s_1)+1$ vertices. Then, each vertex of $T_1$ has score $\frac{2(s_3-s_2+s_1)+1-1}{2}=s_3-s_2+s_1$.

Again, since $s_2>s_1$, therefore $s_2-s_1>0$, so that $s_2-s_1-1\geq 0$. Let $T_2$ be a regular tournament having $2(s_2-s_1-1)+1$ vertices. Then, each vertex of $T_2$ has score $\frac{2(s_2-s_1-1)+1-1}{2}=s_2-s_1-1$. Also, $s_1\geq 0$, let $T_3$ be a regular tournament having $2s_1+1$ vertices. Then, each vertex of $T_3$ has score $\frac{2s_1+1-1}{2}=s_1$.

Let every vertex of $T_2$ dominate each vertex of $T_3$, and every vertex of $T_1$ dominate each vertex of $T_2$ and $T_3$, so that we get a tournament $T$ having $2s_1+1+2(s_2-s_1-1)+1+2(s_3-s_2+s_1)+1=2(s_1+s_3)+1$ vertices with score set

$$\begin{aligned}S&=\big\{s_1, s_2-s_1-1+2s_1+1, s_3-s_2+s_1+2(s_2-s_1-1)+1+2s_1+1\big\}\\&=\big\{s_1,\sum_{i=1}^{2}s_i,\sum_{i=1}^{3}s_i\big\}.\end{aligned}$$

This shows that the result is true for $p=3$ also.

Assume, the result to be true for all odd $p$. That is, if $s_1, s_2, \ldots, s_p$ be $p$ non-negative integers with $s_1<s_2<\ldots<s_p$, then there exists a tournament having $2(s_1+s_3+\ldots+s_p)+1$ vertices with score set $\big\{s_1,\sum\limits_{i=1}^{2}s_i,...,\sum\limits_{i=1}^{p}s_i\big\}$. We show the result is true for $p+2$.

Let $s_1, s_2, \ldots, s_{p+2}$ be $p+2$ non-negative integers with $s_1<s_2<\ldots<s_{p+2}$. This implies that $s_1<s_2<\ldots<s_p$. Therefore, by induction hypothesis, there exists a tournament $T_1$ having $2(s_1+s_3+\ldots+s_p)+1$ vertices with score set $\big\{s_1,\sum\limits_{i=1}^{2}s_i,...,\sum\limits_{i=1}^{p}s_i\big\}$.

Since, $s_2>s_1, s_4>s_3, \ldots, s_{p-1}>s_{p-2}, s_{p+1}>s_p$, therefore $s_2-s_1>0, s_4-s_3>0, \ldots, s_{p-1}-s_{p-2}>0, s_{p+1}-s_p>0$, so that $s_{p+1}-s_p+s_{p-1}-s_{p-2}+\ldots+s_4-s_3+s_2-s_1>0$, that is, $s_{p+1}-s_p+s_{p-1}-s_{p-2}+\ldots+s_4-s_3+s_2-s_1-1\geq 0$. Let $T_2$ be a regular tournament having $2(s_{p+1}-s_p+s_{p-1}-s_{p-2}+\ldots+s_4-s_3+s_2-s_1-1)+1$ vertices. Then, each vertex of $T_2$ has score

$$\begin{aligned}&\frac{2(s_{p+1}-s_p+s_{p-1}-s_{p-2}+\cdots+s_4-s_3+s_2-s_1-1)+1-1}{2}\\&=s_{p+1}-s_p+s_{p-1}-s_{p-2}+\cdots+s_4-s_3+s_2-s_1-1.\end{aligned}$$

Again, $s_3>s_2, \ldots, s_p>s_{p-1}, s_{p+2}>s_{p+1}$, therefore $s_3-s_2>0, \ldots, s_p-s_{p-1}>0, s_{p+2}-s_{p+1}>0$, so that

$$s_{p+2}-s_{p+1}+s_p-s_{p-1}+\ldots+s_3-s_2+s_1>0 \quad \text{as} \quad s_1\geq 0.$$

Let $T_3$ be a regular tournament having

$$2(s_{p+2} - s_{p+1} + s_p - s_{p-1} + \ldots + s_3 - s_2 + s_1) + 1$$

vertices. Then, each vertex of $T_3$ has score

$$\frac{2(s_{p+2} - s_{p+1} + s_p - s_{p-1} + \cdots + s_3 - s_2 + s_1) + 1 - 1}{2}$$
$$= s_{p+2} - s_{p+1} + s_p - s_{p-1} + \cdots + s_3 - s_2 + s_1.$$

Let every vertex of $T_2$ dominate each vertex of $T_1$, and every vertex of $T_3$ dominate each vertex of $T_1$ and $T_2$, so that we get a tournament $T$ having

$$2(s_1 + s_3 + \cdots + s_p) + 1 + 2(s_{p+1} - s_p + s_{p-1} - s_{p-2} + \cdots + s_4 - s_3 + s_2 - s_1 - 1) + 1$$
$$+2(s_{p+2} - s_{p+1} + s_p - s_{p-1} + \cdots + s_3 - s_2 + s_1) + 1 = 2(s_1 + s_3 + \cdots + s_{p+2}) + 1$$

vertices with score set

$$S = \Big\{s_1, \sum_{i=1}^{2} s_i, ..., \sum_{i=1}^{p} s_{i,} s_{p+1} - s_p + s_{p-1} - \cdots + s_4 - s_3 + s_2 - s_1 - 1$$
$$+ 2(s_1 + s_3 + \cdots + s_{p-2} + s_p) + 1, s_{p+2} - s_{p+1} + s_p - s_{p-1} + \cdots + s_3 - s_2 + s_1$$
$$+ 2(s_1 + s_3 + \cdots + s_{p-2} + s_p) + 1 + 2(s_{p+1} - s_p + s_{p-1} - \cdots + s_4 - s_3 + s_2 - s_1 - 1) + 1\Big\}$$
$$= \Big\{s_1, \sum_{i=1}^{2} s_i, \ldots, \sum_{i=1}^{p} s_i, \sum_{i=1}^{p+1}\Big\}.$$

This shows that the result is true for $p+2$ also. Hence, by induction, the result is true for all odd $p$.

To prove the result for even case, if $p$ is odd, then $p+1$ is even.

Let $s_1, s_2, \ldots, s_{p+1}$ be $p+1$ non-negative integers with $s_1 < s_2 < \ldots < s_{p+1.}$ Therefore, $s_1 < s_2 < \ldots < s_p$, where $p$ is odd. Then, by above case, there exists a tournament $T_1$ having $2(s_1 + s_3 + \ldots + s_p) + 1$ vertices with score set $\Big\{s_1, \sum_{i=1}^{2} s_i, \ldots, \sum_{i=1}^{p} s_i\Big\}$.

Also, since $s_2 > s_1, s_4 > s_3, \ldots, s_{p-1} > s_{p-2}, s_{p+1} > s_{p,}$ then as above, we have a regular tournament $T_2$ having $2(s_{p+1} - s_p + s_{p-1} - s_{p-2} + \ldots + s_4 - s_3 + s_2 - s_1 - 1) + 1$ vertices and score for each vertex is

$$s_{p+1} - s_p + s_{p-1} - s_{p-2} + \cdots + s_4 - s_3 + s_2 - s_1 - 1.$$

Let every vertex of $T_2$ dominate each vertex of $T_1$, so that we get a tournament $T$ having

$$2(s_1 + s_3 + \cdots + s_{p-2} + s_p) + 1 + 2(s_{p+1} - s_p + s_{p-1} - s_{p-2}$$
$$+ \cdots + s_4 - s_3 + s_2 - s_1 - 1) + 1 = 2(s_2 + s_4 + \cdots + s_{p+1})$$

vertices with score set

$$S = \Big\{s_1, \sum_{i=1}^{2} s_i, ..., \sum_{i=1}^{p} s_i, s_{p+1} - s_p + s_{p-1} - s_{p-2} + \cdots + s_4 - s_3 + s_2 - s_1 - 1$$
$$+ 2\,(s_1 + s_3 + \cdots + s_{p-2} + s_p) + 1\Big\}$$
$$= \Big\{s_1, \sum_{i=1}^{2} s_i, ..., \sum_{i=1}^{p} s_i, \sum_{i=1}^{p+1} s_i\Big\}.$$

This shows that the result is true for even also. Hence, the result. ■

*Remark.* In the proof of Theorem 9, we note that when $p$ is odd, the tournament $T$ constructed with score set $S = \{s_1, \sum_{i=1}^{2} s_i, ..., \sum_{i=1}^{p} s_i\}$, where $s_1 < s_2 < \cdots < s_p$, has $2(s_1+s_3+\ldots+s_p)+1$ vertices; and when $p$ is even the tournament constructed has $2(s_2 + s_4 + \ldots + s_{p+1})$ vertices.

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## References


1. M. Hager, On score sets for tournaments, *J. Discrete Math.* **58**(1986) 25–34.
2. H. G. Landau, On dominance relations and the structure of animal societies, III, the condition for a score structure, *Bull. Math. Biophys.* **15** (1953) 143–148.
3. J. W. Moon, *Topics on Tournaments Halt*, Rinehart and Winston, New York, 1968.
4. K. B. Reid and L. W. Beineke, Tournaments, Selected Topics in Graph Theory, L.W. Beineke and R.J. Wilson, (Eds.), Academic Press, New York, 1979.
5. K. B. Reid, Score sets for tournaments, Congressus Numerantium XXI, Proceedings of the Ninth Southeastern Conference on Combinatorics, Graph Theory, and Computing, 1978, 607–618.
6. T. X. Yao, Reid's conjecture on score sets in tournaments, *Kexue Tongbao* **33** (1988) 481–484 (in Chinese).
7. T. X. Yao, On Reid's conjecture of score sets for tournaments, *Chinese Sci. Bull.* **34** (1989) 804–808.