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On the Asymptotic Behavior of Solutions of a Nonlinear Difference Equation with Bounded Multiple Delay

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Abstract. In this paper, we study the asymptotic behavior of solutions of a nonlinear difference equation with bounded multiple delay

$$x_{n+1} = \lambda_n x_n + \sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}).$$

We give conditions implying that this equation has solutions which are oscillatory, bounded, convergence or periodic.

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Introduction

The asymptotic behavior of solutions of delay nonlinear difference equations has been studied extensively in recent years; see for example [1-9]. Our main motivation in studying asymptotic behavior of solutions of delay nonlinear difference equation

$$x_{n+1} = \lambda_n x_n + \sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i})$$

is the oscillation of solutions of the difference equation

$$x_{n+1} - x_n + p_n x_{n-m} = 0, \quad n \in \mathbb{N}$$

in [1]; the positive periodic solutions of nonlinear delay functional difference equations

$$x_{n+1} = a_n x_n \pm \beta h_n f(x_{n-\tau_n})$$

in [8] and the extinction, persistence, global stability and nontrivial periodicity in the model

$$x_{n+1} = \lambda x_n + F(x_{n-m})$$

of population growth in [2, 3]. In this paper, to investigate the oscillation, we let $\lambda_n = 1$ for all $n \in \mathbb{N}$ and to study the boundedness, the convergence, the periodicity we will give some restrictions on the function F, the sequences $\{\lambda_n\}_n, \{\alpha_i(n)\}_n$ or the delays m_i. Our results can be considered as the generalization of some earlier results in [1, 2].

1. The Oscillation

Consider the difference equation

$$x_{n+1} = x_n + \sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}) \tag{1.1}$$

for $n \in \mathbb{N}, n \geqslant a$ for some $a \in \mathbb{N}$, where $r, m_i \geqslant 1, \quad 1 \leqslant i \leqslant r$ are fixed positive integers, the functions $\alpha_i(n)$ are defined on $\mathbb{N}$ and the function F is defined on $\mathbb{R}$. Recall that, a solution $\{x_n\}_{n \geqslant a}$ of (1.1) is called *oscillatory* if for any $n_1 \geqslant a$ there exists $n_2 \geqslant n_1$ such that $x_{n_2} x_{n_2+1} \leqslant 0$. The difference equation (1.1) is called oscillatory if all its solutions are oscillatory. The following theorem and its corollary give some sufficient conditions for the oscillation of the solutions of (1.1).

Theorem 1. *Assume that*

$$xF(x) < 0, \quad x \neq 0 \text{ and } \liminf_{x \to 0} \frac{F(x)}{x} = M < 0.$$

Then, (1.1) *is oscillatory if the following holds*

$$\sum_{i=1}^{r} (\liminf_{n \to \infty} \alpha_i(n)) \cdot M \cdot \frac{(m_i + 1)^{m_i + 1}}{m_i^{m_i}} < -1 \tag{1.2}$$

where $\alpha_i(n) \geqslant 0, n \in \mathbb{N}, 1 \leqslant i \leqslant r$.

Proof. We first prove that the inequality

$$x_{n+1} - x_n - \sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}) \leqslant 0, \quad n \in \mathbb{N} \tag{1.3}$$

has no eventually positive solution. Indeed, assume the contrary and let $\{x_n\}_n$ be a solution of (1.3) with $x_n > 0$ for all $n \geqslant n_1, n_1 \in \mathbb{N}$. Setting $v_n = \frac{x_n}{x_{n+1}}$ and dividing this inequality by x_n, we obtain

$$\frac{1}{v_n} \leqslant 1 + \Big[\sum_{i=1}^{r} \alpha_i(n) \frac{F(x_{n-m_i})}{x_{n-m_i}} v_{n-m_i} \cdots v_{n-1}\Big], \tag{1.4}$$

where $n \geqslant n_1 + m, \quad m = \max_{1 \leqslant i \leqslant r} m_i$.

Clearly, x_n is nonincreasing with $n \geqslant n_1+m$, and so $v_n \geqslant 1$ for all $n \geqslant n_1+m$. Also, v_n is bounded above because otherwise (1.2) and (1.4) imply that $v_n < 0$ for arbitrarily large n. Put $\liminf_{n\to\infty} v_n = \beta$. Notice that $F(x_{n-m_i}) < 0, \quad \forall i = 1, 2, \cdots r$, we get

$$\limsup_{n\to\infty} \frac{1}{v_n} = \frac{1}{\beta} \leqslant 1 - \liminf_{n\to\infty} \Big\{\sum_{i=1}^{r} \alpha_i(n) \Big[-\frac{F(x_{n-m_i})}{x_{n-m_i}}\Big] v_{n-m_i} \cdots v_{n-1}\Big\}$$

$$\leqslant 1 + \Big[\sum_{i=1}^{r} (\liminf_{n\to\infty} \alpha_i(n)) \cdot M \cdot \beta^{m_i}\Big],$$

$$1 \geqslant \Big[\sum_{i=1}^{r} (\liminf_{n\to\infty} \alpha_i(n)) \cdot M \cdot \frac{\beta^{m_i+1}}{1-\beta}\Big].$$

But

$$\frac{\beta^{m_i+1}}{1-\beta} \leqslant -\frac{(m_i+1)^{m_i+1}}{m_i^{m_i}}, \quad i = 1, 2, \cdots r,$$

so

$$\sum_{i=1}^{r} (\liminf_{n\to\infty} \alpha_i(n)) \cdot M \cdot \frac{(m_i+1)^{m_i+1}}{m_i^{m_i}} \geqslant -1.$$

This contradicts our assumption, hence (1.3) has no eventually positive solution.

Similarly, we can prove that the inequality

$$x_{n+1} - x_n - \sum_{i=1}^{r} \alpha_i(n) F(x_{n-m_i}) \geqslant 0, \quad n \in \mathbb{N}$$

has no eventually negative solution. So, the proof is complete. ∎

Corollary. *Assume that* $\alpha_i(n) \geqslant 0, n \in \mathbb{N}, 1 \leqslant i \leqslant r$,

$$xF(x) < 0, \quad x \neq 0 \ \textit{and} \ \liminf_{x\to 0} \frac{F(x)}{x} = M < 0.$$

Then, (1.1) *is oscillatory if either of the following holds*

$$M \cdot \sum_{i=1}^{r} (\liminf_{n\to\infty} \alpha_i(n)) \cdot \frac{(\tilde{m}+1)^{\tilde{m}+1}}{\tilde{m}^{\tilde{m}}} < -1,$$

or

$$M \cdot r \Big[\prod_{i=1}^{r} (\liminf_{n\to\infty} \alpha_i(n))\Big]^{\frac{1}{r}} \cdot \frac{(\hat{m}+1)^{\hat{m}+1}}{\hat{m}^{\hat{m}}} < -1,$$

where $\tilde{m} = \min_{1 \leqslant i \leqslant r} m_i, \quad \hat{m} = \frac{1}{r} \sum_{i=1}^{r} m_i.$

The proofs of the following theorems can be obtained similarly as the proofs of Theorem 6.20.1, Theorem 6.20.2 in [1], so we omit them here.

Theorem 2. *Assume that*

$$xF(x) < 0, \quad x \neq 0 \text{ and } \liminf_{x \to 0} \frac{F(x)}{x} = M < 0.$$

Suppose further that, F is nonincreasing on $\mathbb{R}$ and

$$\limsup_{n \to \infty} \sum_{i=1}^{r} \sum_{\ell = n - m^*}^{n} \alpha_i(\ell) > -\frac{1}{M},$$

where $m^* = \max_{1 \leqslant i \leqslant r} m_i$ *and* $\alpha_i(n) \geqslant 0, n \in \mathbb{N}, 1 \leqslant i \leqslant r$. *Then,* (1.1) *is oscillatory.*

Theorem 3. *Assume that*

$$xF(x) < 0, -F(x) \geqslant x, \quad x \neq 0$$

and F is nonincreasing on $\mathbb{R}$. Then, (1.1) *is oscillatory if the following holds*

$$\liminf_{n \to \infty} \sum_{i=1}^{r} \alpha_i(n) > 0 \text{ and } \limsup_{n \to \infty} \sum_{i=1}^{r} \alpha_i(n) > 1 - \liminf_{n \to \infty} \sum_{i=1}^{r} \alpha_i(n).$$

2. Convergence, Boundedness and Periodicity

Consider the difference equation

$$x_{n+1} = \lambda_n x_n + \sum_{i=1}^{m} \alpha_i F(x_{n-i}) \tag{2.1}$$

for $n = 0, 1, 2, \cdots$, where $\alpha_i \geqslant 0, \quad i = 1, 2, \cdots, m; \sum_{i=1}^{m} \alpha_i = 1$; $F : [0, \infty) \to [0, \infty)$ is a continuous function, and $m > 0$ is a fixed integer. The positive initial values $x_{-m}, x_{-m+1}, \cdots, x_0$ are given. The following theorem gives a sufficient condition for the convergence to zero of the solutions of (2.1).

Theorem 4. *Assume that* $\lambda_n \in (0, 1)$ *and there exists* $\lambda^* \in (0, 1)$ *such that* $\lambda_n \leqslant \lambda^*$ *for all* $n \in \mathbb{N}$. *Then, every solution of* (2.1) *converges to* 0 *if* $F(u) < (1 - \lambda^*)u$ *for all* $u > 0$.

Proof. First assume that $F(u) < (1 - \lambda^*)u$ for all $u > 0$. Let $\{x_n\}_n$ be a positive solution of (2.1) and $M = \max_{-m \leqslant j \leqslant 0} x_j$. We prove that $x_n \leqslant M$ for all n. Indeed using induction assume that $x_k \leqslant M$ for all $k \leqslant n$. Then by the difference equation (2.1)

$$\begin{aligned} x_{n+1} &= \lambda_n x_n + \sum_{i=1}^{m} \alpha_i F(x_{n-i}) \\ &\leqslant \lambda^* M + \sum_{i=1}^{m} \alpha_i (1-\lambda^*) M = M. \end{aligned}$$

Therefore $x_n \leqslant M$ for all n. Let $M_1 = \limsup_{n\to\infty} x_n$. It is clear that $0 \leqslant M_1 < \infty$. Moreover, there is a subsequence $\{n_k\}$ such that

$$M_1 = \lim_{k\to\infty} x_{n_k}.$$

Let $\epsilon > 0$ be given. There exists a positive integer $N = N(\epsilon)$ such that for all $n_k > N$ we have $M_1 - \epsilon \leqslant x_{n_k} \leqslant M_1 + \epsilon$ and for all $n > N - m - 1 : x_n \leqslant M_1 + \epsilon$. On the other hand,

$$x_{n_k} = \lambda_{n_k} x_{n_k - 1} + \sum_{i=1}^{m} \alpha_i F(x_{n_k - 1 - i}).$$

By the mean value theorem, we can choose a sequence $\{y_{n_k}\}$ such that

$$\sum_{i=1}^{m} \alpha_i F(x_{n_k-1-i}) = F(y_{n_k}),$$

where

$$y_{n_k} \in [\min\{x_{n_k-2}, \cdots, x_{n_k-1-m}\}, \max\{x_{n_k-2}, \cdots, x_{n_k-1-m}\}].$$

So we obtain

$$\begin{aligned} x_{n_k} &= \lambda_{n_k} x_{n_k-1} + F(y_{n_k}), \\ x_{n_k} &\leqslant \lambda^* x_{n_k-1} + F(y_{n_k}) < \lambda^* x_{n_k-1} + (1-\lambda^*) y_{n_k}, \\ \lambda^* x_{n_k-1} &> x_{n_k} - (1-\lambda^*) y_{n_k} \geqslant M_1 - \epsilon - (1-\lambda^*)(M_1+\epsilon), \\ x_{n_k-1} &\geqslant M_1 - \frac{2-\lambda^*}{\lambda^*}\epsilon. \end{aligned}$$

Thus

$$M_1 - \frac{2-\lambda^*}{\lambda^*}\epsilon \leqslant x_{n_k-1} \leqslant M_1 + \epsilon$$

and

$$\lim_{k\to\infty} x_{n_k-1} = M_1.$$

Similarly, we get

$$M_1 - \frac{1+\lambda^*}{\lambda^*}\epsilon \leqslant y_{n_k} \leqslant M_1 + \epsilon$$

and

$$\lim_{k\to\infty} y_{n_k} = M_1.$$

Because F is continuous, M_1 is a solution of equation $u = \lambda^* u + F(u)$. But, by assumption $F(u) < (1-\lambda^*)u$ for all $u > 0$ we have $M_1 = 0$ i.e the sequence $\{x_n\}_n$ converges to 0.

Remark.
(i) If $\lambda_n = \lambda^* \in (0,1), \quad \forall n \in \mathbb{N}$, then the converse of Theorem 4 is true.
(ii) If $\{\lambda_n\}_n$ is a monotone sequence where $\lambda_n \in (0,1)$ for all $n \in \mathbb{N}$, then the conclusion of Theorem 4 is also true. Namely, in the case $\{\lambda_n\}_n$ is increasing, we can choose $\lambda^* = \lim_{n\to\infty} \lambda_n$; in the case $\{\lambda_n\}_n$ is decreasing, we can choose $\lambda^* = \lambda_0$.
(iii) Due to the proof of Theorem 4, one can easily see that if either $m = \{m_n\}_n$ is bounded or unbounded but $\lim_{n\to\infty}(n - m_n) = \infty$ holds and $\sum_{i=1}^{m_n} \alpha_i = 1$ then Theorem 4 remains true.

The following theorem gives a sufficient condition for the boundedness of every solution of (2.1).

Theorem 5. *Assume that the sequence $\{\lambda_n\}_n$ is as in Theorem 4 and $F(x) = H(x,x)$, where $H : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function, increasing in x but decreasing in y and $H(x,y) > 0$ if $x, y > 0$. Suppose further that*

$$\limsup_{x,y\to\infty} \frac{H(x,y)}{x} < 1 - \lambda^*, \tag{2.2}$$

Then every solution $\{x_n\}_{n=-m}^{\infty}$ of (2.1) is bounded.

Proof. The proof follows from applying the mean value theorem and the proof of Theorem 2 in [2].

To study the periodicity in the equation (2.1) we assume that $\{\lambda_n\}_n$ is $p-$periodic for p is an integer with $p \geq 1$ and $0 < \lambda_n < 1$ for all $n \in [0, p-1]$. Let

$$G(n,u) = \frac{\prod_{s=u+1}^{n+p-1} \lambda_s}{1 - \prod_{s=n}^{n+p-1} \lambda_s}, \quad \eta = \frac{G(n,n)}{G(n, n+p-1)},$$

$$A = \max_{n\in[0,p-1]} \sum_{u=0}^{p-1} G(n,u), \quad B = \min_{n\in[0,p-1]} \sum_{u=0}^{p-1} G(n,u),$$

$$\ell_1 = \lim_{x\to\infty} \frac{F(x)}{x} \in (0,\infty), \quad \ell_2 = \lim_{x\to 0} \frac{F(x)}{x} \in (0,\infty).$$

The following therems can be proved similarly as Theorem 2.3, Theorem 2.4 in [8].

Theorem 6. *If one of two following conditions is satisfied*

$$\ell_1 > \frac{1}{\eta B} \text{ and } \ell_2 < \frac{1}{A} \tag{2.3}$$

or

$$\ell_1 < \frac{1}{A} \text{ and } \ell_2 > \frac{1}{\eta B} \tag{2.4}$$

then (2.1) has at least one positive periodic solution.

Theorem 7. *If either*

$$\ell_1 = 0 \text{ and } \ell_2 = \infty \tag{2.5}$$

or

$$\ell_1 = \infty \text{ and } \ell_2 = 0 \tag{2.6}$$

then (2.1) *has at least one positive periodic solution.*

Now, to apply the results of Theorem 6 and Theorem 7 we will construct some examples. In [8], although the author obtained sufficient conditions for the existence of multiple positive periodic solutions of the nonlinear delay functional difference equation

$$x_{n+1} = \lambda_n x_n \pm \beta h_n f(x_{n-\tau_n}),$$

but he did not give any illustrative examples.

Example. Consider the following nonlinear difference equations with bounded delay

$$x_{n+1} = \lambda_n x_n + \sum_{i=1}^{m} \alpha_i \gamma_1 x_{n-i} \frac{1 + x_{n-i}}{1 + c x_{n-i}}, \quad \gamma_1 > 0, c \geq 1, \tag{2.7}$$

$$x_{n+1} = \lambda_n x_n + \sum_{i=1}^{m} \alpha_i \frac{\gamma_2 x_{n-i}}{\delta + \sigma e^{-\chi x_{n-i}}}, \tag{2.8}$$

where $\{\lambda_n\}_n$ is $2-$ periodic. We have

$$\lambda_n = \frac{1}{2}\left[(\alpha+\beta) + (\alpha-\beta)(-1)^n\right], \quad \lambda_0 = \alpha, \lambda_1 = \beta, \alpha, \beta \in (0,1).$$

It is easy to check that

$$A = \max_{n\in[0,1]} \sum_{u=0}^{1} G(n,u) = \frac{\beta+1}{1-\alpha\beta},$$

$$B = \min_{n\in[0,1]} \sum_{u=0}^{1} G(n,u) = \frac{\alpha(\beta+1)}{1-\alpha\beta},$$

$$\eta = \begin{cases} \beta, & \text{if } \ n = 2k, k \in \mathbb{N} \\ \alpha, & \text{if } \ n = 2k+1, k \in \mathbb{N}. \end{cases}$$

For (2.7), since

$$\ell_1 = \lim_{x\to\infty} \frac{F(x)}{x} = \frac{\gamma_1}{c} \in (0,\infty), \quad \ell_2 = \lim_{x\to 0} \frac{F(x)}{x} = \gamma_1 \in (0,\infty),$$

the condition (2.4) in Theorem 6 becomes $\frac{1-\alpha\beta}{\beta+1} < \gamma_1 < c$. We can choose $\alpha, \beta \in (0,1)$ and $c \geqslant 1$ satisfying this inequality so by Theorem 6, (2.7) has at least one positive periodic solution.

For (2.8), since

$$\ell_1 = \lim_{x\to\infty} \frac{F(x)}{x} = \frac{\gamma_2}{\delta} \in (0,\infty), \quad \ell_2 = \lim_{x\to 0} \frac{F(x)}{x} = \frac{\gamma_2}{\delta+\sigma} \in (0,\infty),$$

the condition (2.3) in Theorem 6 implies $\frac{1}{\delta} > \frac{1}{\delta+\sigma}$. Hence, the condition (2.3) in Theorem 6 is satisfied for all $\gamma_2, \delta, \sigma, \chi \in (0,\infty)$. This means that (2.8) has at least one positive periodic solution.

Note that, if we choose $\{\lambda_n\}_n$ as above and $F(x) =$ constant or $F(x) = \frac{a}{x}$, $a > 0, x \in (0,+\infty)$ then Theorem 7 is applied, i.e. (2.7), (2.8) always have at least one positive periodic solution.

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